Ah hah! Algorithms

Recursion and iteration

Asymptotic analysis

The repeated squaring trick

Much ado about Fibonacci numbers

Agenda

- The worst algorithm in the history of humanity
- Asymptotic notations: Big-O, Big-Omega, Theta
- An iterative solution
- A better iterative solution
- The repeated squaring trick

And the worst algorithm in the history of humanity

FIBONACCI SEQUENCE

Fibonacci sequence

- 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

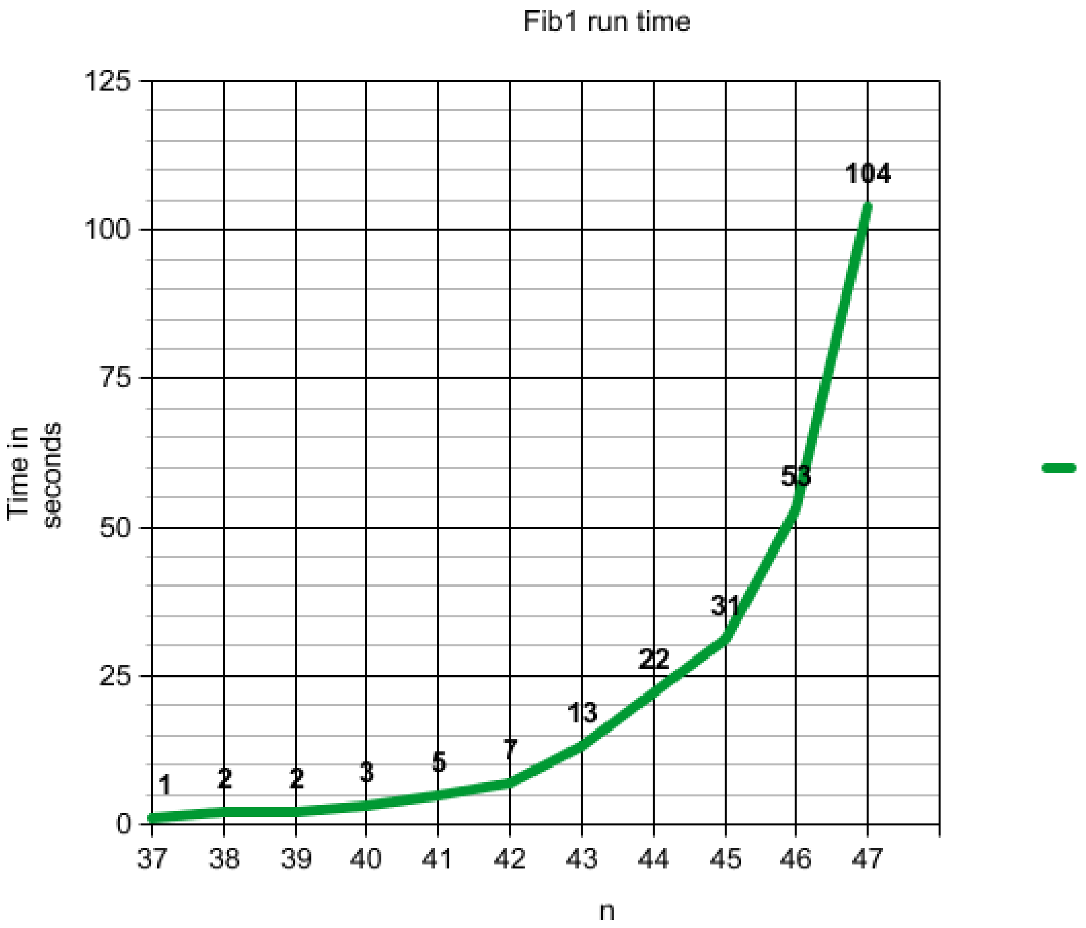
- F[0] = 0
- F[1] = 1
- F[2] = F[1] + F[0] = 1
- F[3] = F[2] + F[1] = 2
- F[4] = F[3] + F[2] = 3
- F[n] = F[n-1] + F[n-2]

Recursion – fib1()

```
/**
 *---------------------------------------------------
 *  the most straightforward algorithm to compute F[n]
 *---------------------------------------------------
 */
unsigned long long fib1(unsigned long n) {
   if (n <= 1) return n;
   return fib1(n-1) + fib1(n-2);
}
```

Run time on my laptop

2.53GHz Intel Core 2 Duo, 4 GB DDR3



On large numbers

- Looks like the run time is doubled for each n++
- We won't be able to compute F[120] if the trend continues
- The age of the universe is 15 billion years < 2^{60} sec
- The function looks … exponential
  – Is there a theoretical justification for this?

A Note on "Functions"

- Sometimes we mean a C++ function
- Sometimes we mean a mathematical function like F[n]
- A C++ function can be used to compute a mathematical function
  - But not always! There are un-computable functions
  - Google for "busy Beaver numbers" and the "halting problem", for typical examples.
- What we mean should be clear from context

Guess and induct strategy

Thinking about the main body

ANALYSIS OF FIB1()

Guess and induct

- For n > 1, suppose it takes c mili-sec in fib1(n) not counting the recursive calls
- For n=0, 1, suppose it takes d mili-sec
- Let T[n] be the time fib1(n) takes
- T[0] = T[1] = d
- T[n] = c + T[n-1] + T[n-2] when n > 1

- To estimate T[n], we can
  - Guess a formula for it
  - Prove by induction that it works

The guess

- Bottom-up iteration
  - T[0] = T[1] = d
  - T[2] = c + 2d
  - T[3] = 2c + 3d
  - T[4] = 4c + 5d
  - T[5] = 7c + 8d
  - T[6] = 12c + 13d

- Can you guess a formula for T[n]?
  - T[n] = (F[n+1] – 1)c + F[n+1]d

The Proof

- The base cases: $n=0,1$
- The hypothesis: suppose
  - $T[m] = (F[m+1] – 1)*c + F[m+1]*d$ *for all* $m < n$
- The induction step:
  - $T[n] = c + T[n-1] + T[n-2]$

$$= c + (F[n] - 1)*c + F[n]*d$$
$$+ (F[n-1] - 1)*c + F[n-1]*d$$
$$= (F[n+1] - 1)*c + F[n]*d$$

How does this help?

$$F[n] = \frac{\phi^n - (-1/\phi)^n}{\sqrt{5}}$$

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.6$$

The golden ratio

So, there are constants C, D such that

$$C\phi^n \leq T[n] \leq D\phi^n$$

This explains the exponential-curve we saw

- Back of the envelope time/space estimation
- Independent of whether our computer is fast
- Big-o, big-omega, theta

ASYMPTOTIC ANALYSIS

From intuition to formality

- Suppose fib1() runs on a computer with $C = 10^{-9}$:

$$10^{-9}(1.6)^{140} \geq 3.77 \cdot 10^{19} > 100 \cdot \text{age of univ.}$$

- We need a formal way to state that $(1.6)^n$ is the "correct" measure of fib1()'s runtime
  - How fast the target computer runs shouldn't concern us

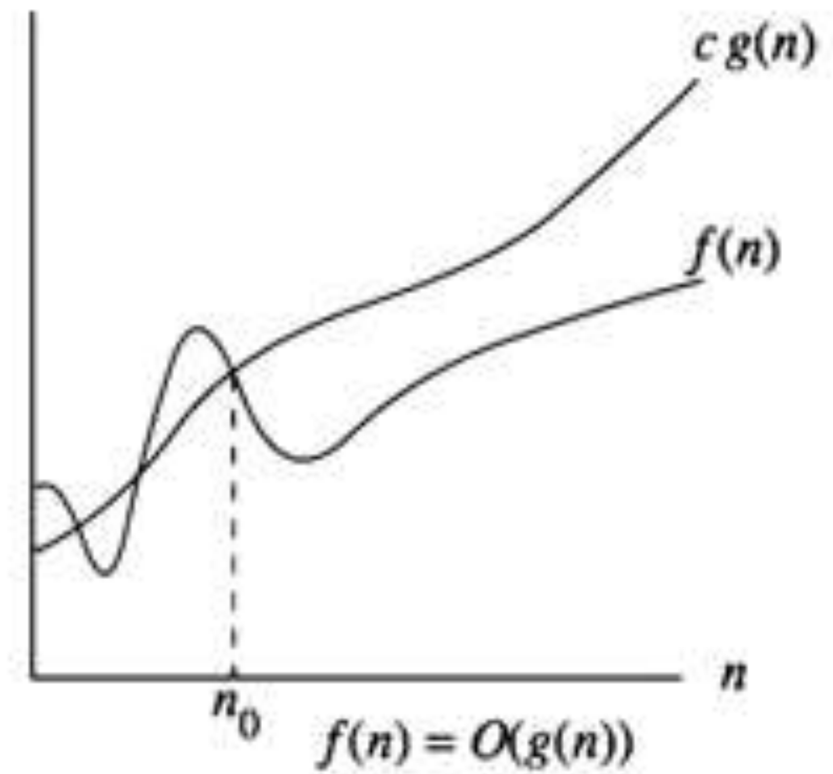
Big-O

$$f, g : \mathbb{N} \rightarrow \mathbb{R}^{+}$$

$$f(n) = O(g(n)) \text{ iff } \exists \text{ constants } C, n_0 > 0$$

$$\text{such that } f(n) \leq C g(n), \forall n \geq n_0$$

Intuition



in our case $T[n] = O(\phi^n)$

In English

- *f(n) = O(g(n))* means: for n sufficiently large, *f(n)* is bounded above by a constant scaling of *g(n)*
  - Does the "English translation" make things worse?

- An algorithm with runtime *f(n)* is at least as good as an algorithm with runtime *g(n)*, asymptotically

Examples

$$n^2 = O(n^2)$$

$$n^2 = O(n^2/10^6)$$

$$n = O(n^2)$$

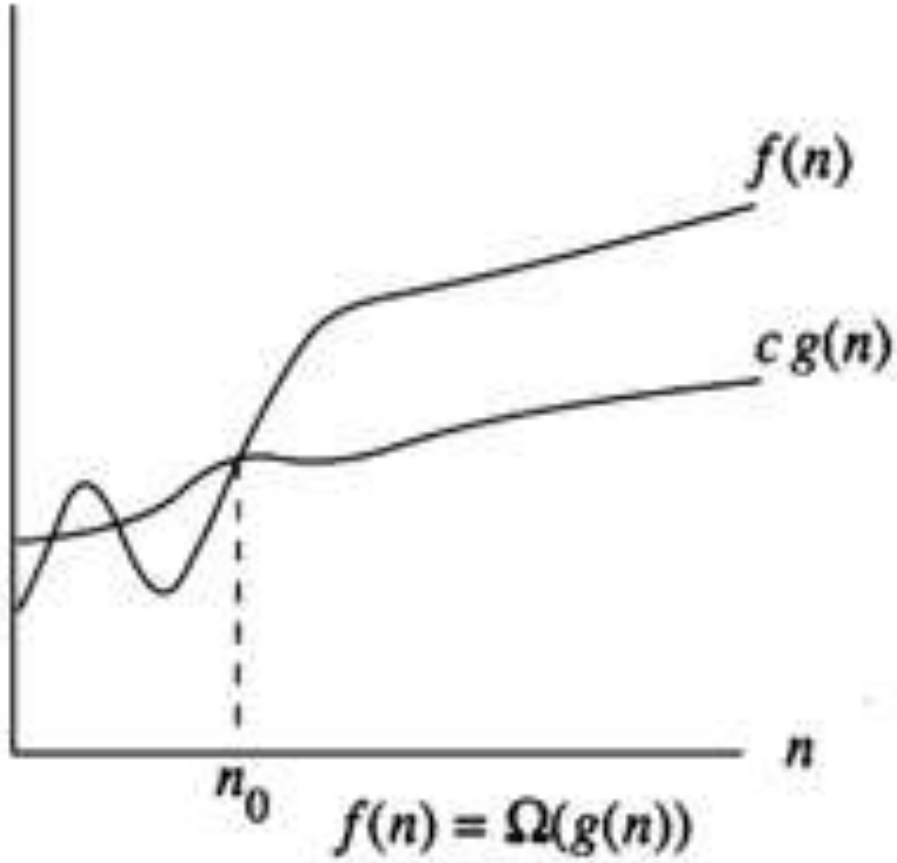
Big-Omega

$$f, g : \mathbb{N} \to \mathbb{R}^+$$

$$f(n) = \Omega(g(n)) \text{ iff } \exists \text{ constants } C, n_0 > 0$$

$$\text{such that} f(n) \geq C g(n), \forall n \geq n_0$$

In picture



Examples

$$n \log n = \Omega(n)$$

$$2^n/10^6 = \Omega(n^{100})$$

Equivalence

$$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$$

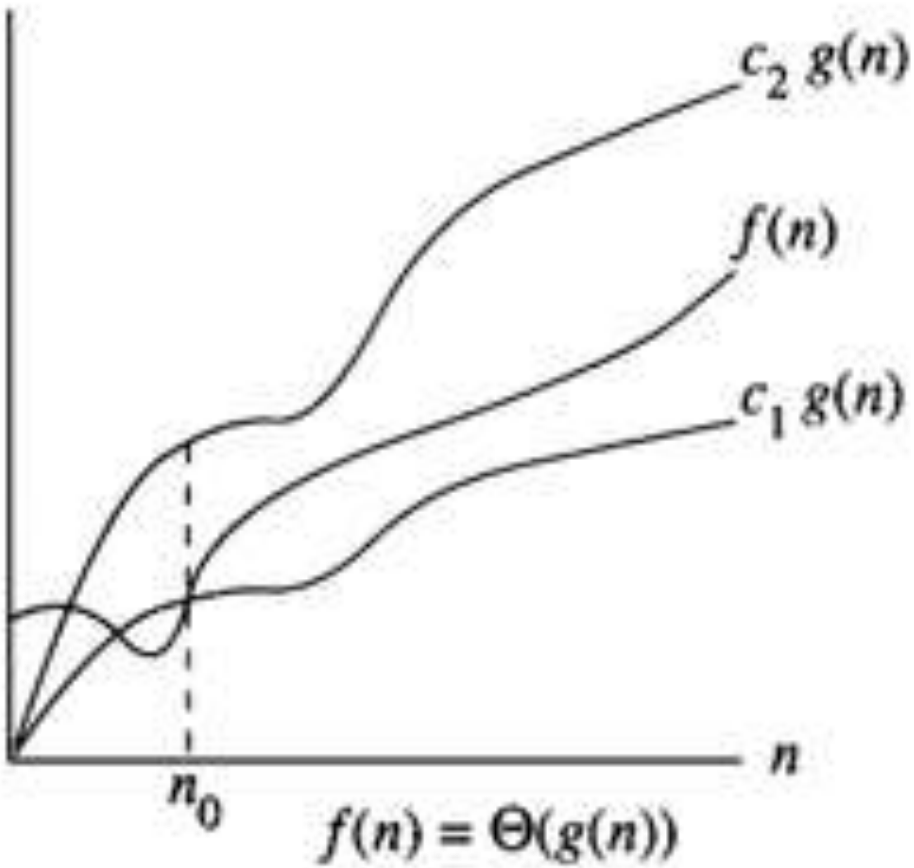
Theta

$$f(n) = \Theta(g(n)) \Leftrightarrow f(n) = O(g(n) \text{ and } g(n) = O(f(n))$$

We say they "have the same growth rate"

$$\text{in fib1() example: } T[n] = \Theta(\phi^n)$$

In picture



- A Linear time algorithm using vectors
- A linear time algorithm using arrays
- A linear time algorithm with constant space

BETTER ALGORITHMS FOR COMPUTING F[N]

An algorithm using vector

```
unsigned long long fib2(unsigned long n) {
    // this is one implementation option
    if (n <= 1) return n;
    vector<unsigned long long> A;
    A.push_back(0); A.push_back(1);
    for (unsigned long i=2; i<=n; i++) {
        A.push_back(A[i-1]+A[i-2]);
    }
    return A[n];
}
```

Guess how large an n we can handle this time?

Data

n	10^6	10^7	10^8	10^9
# seconds	1	1	9	Eats up all my CPU/RAM

How about an array?

```
unsigned long long fib2(unsigned long n) {
    if (n <= 1) return n;
    unsigned long long* A = new unsigned long long[n];
    A[0] = 0; A[1] = 1;
    for (unsigned long i=2; i<=n; i++) {
        A[i] = A[i-1]+A[i-2];
    }
    unsigned long long ret = A[n];
    delete[] A;
    return ret;
}
```

Guess how large an n we can handle this time?

Data

n	10^6	10^7	10^8	10^9
# seconds	1	1	1	Segmentation fault

Data structure matters a great deal!

Some assumptions we made are false if too much space is involved: computer has to use hard-drive as memory

Dynamic programming!

```
unsigned long long fib3(unsigned long n) {
    if (n <= 1) return n;
    unsigned long long a=0, b=1, temp;
    unsigned long i;
    for (unsigned long i=2; i<= n; i++) {
        temp = a + b; // F[i] = F[i-2] + F[i-1]
        a = b;        // a = F[i-1]
        b = temp;     // b = F[i]
    }
    return temp;
}
```

Guess how large an n we can handle this time?

Data

n	10^8	10^9	10^{10}	10^{11}
# seconds	1	3	35	359

The answers are incorrect because $F[10^8]$ is greater than the largest integer representable by unsigned long long

But that's ok. We want to know the runtime

- The repeated squaring trick

AN EVEN FASTER ALGORITHM

Math helps!

- We can re-formulate the problem a little:

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}F[n-1]\\F[n-2]\end{bmatrix} = \begin{bmatrix}F[n]\\F[n-1]\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}F[n+1]\\F[n]\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^n\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

How to we compute A^n quickly?

- Want

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

- But can we even compute 3^n quickly?

First algorithm

```
unsigned long long power1(unsigned long n) {
    unsigned long i;
    unsigned long long ret=1;
    for (unsigned long i=0; i<n; i++)
        ret *= base;
    return ret;
}
```

When n = 10^{10} it took 44 seconds

Second algorithm

```
unsigned long long power2(unsigned long n) {
    unsigned long long ret;
    if (n == 0) return 1;
    if (n % 2 == 0) {
        ret = power2(n/2);
        return ret * ret;
    } else {
        ret = power2((n-1)/2);
        return base * ret * ret;
    }
}
```

When n = 10^{19} it took < 1 second
Couldn't test n = 10^{20} because that's > sizeof(unsigned long)

Runtime analysis

- First algorithm O(n)
- Second algorithm O(log n)
- We can apply the second algorithm to the Fibonacci problem: fib4() has the following data

n	10^8	10^9	10^{10}	10^{19}
# seconds	1	1	1	1