CSE439, Fall 2024 **Final Exam** **Dec. 17, 2024**

Open book, open notes, closed neighbors, 170 minutes after a 10-minute read-through period. The exam totals 267 pts., subdivided as shown. Do *all seven problems* in the exam booklets provided. em Always show your work—this may help for partial credit. Use of a quantum circuit simulation app—whether online or downloaded—or matrix calculator is forbidden. Some parts have helps on steps where such an app might be used. You may use—and cite—theorems and facts from lectures and homeworks without further justification.

Notation: You may freely mix "standard linear algebra notation" and "Dirac *bra-ket* notation" for quantum states. For example, e_{10} means the same thing as $|10\rangle$ and is represented (in big-endian notation) by the unit column vector $[0, 0, 1, 0]^T$. For matrix outer-products the *ket-bra* form typified by $|+\rangle\langle -|$ **is** standard. The notation $\langle u \mid v\rangle$ is the same as $\langle u, v\rangle$ for inner product. Quantum state vectors use big-endian notation by default; if you switch to little-endian you must say so. The conjugate transpose of a matrix U is denoted by U^*.

(1) (3 × 15 = 45 pts.) *Short-answer questions.* A single pithy sentence may suffice; general expectation is a paragraph of 2–4 sentences.

(a) Find a case where **u** and **v** are unit vectors, the vector **w** defined as $\frac{1}{\sqrt{2}}(\mathbf{u} + \mathbf{v})$ is also a unit vector, the vectors **u** and **v** are each separable, but the vector **w** is entangled. You must explain your answer; it suffices to consider two qubits.

(b) For the Deutsch-Jozsa problem, where we are promised that a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ is either *constant* or *balanced* (the latter meaning that 2^{n-1} of its values are 0, the other half of them are 1), we can design a *classical* algorithm A that works as follows: Guess n arguments $x_1, \dots, x_n \in \{0, 1\}^n$ at random. If the values $f(x_1), \dots, f(x_n)$ are all the same, say "constant"; else, say "balanced." Give and explain two respects in which the quantum algorithm for this problem is considered to perform better than A does—at least in theory.

(c) Let $M = pq$ be the product of two odd primes and let $a < M$ be relatively prime to M. The "period r of a" means the period of the function $f(x) = a^x \pmod{M}$ as in Shor's algorithm. If the period r of a is even, what is the period of a^2? Again you must explain your answer.

(2) (9+6+15 = 30 pts. total)

Make a quantum circuit of two qubits—both initialized to $|0\rangle$—with the following gates:

1. A Hadamard on line 1 only.
2. Then a CNOT gate with control on line 1 and target on line 2.
3. After the CNOT gate, a Hadamard gate on line 2 only.
4. After the second Hadamard, a CNOT gate with control on line 2 and target on line 1.

(a) Calculate the resulting two-qubit state ϕ. (Writing $|\phi\rangle$ is optional here and below.)

(b) Is ϕ entangled? Prove your answer.

(c) Now replace the second CNOT gate by a **CZ** gate on the same two qubits. Calculate the final state ϕ' now and again say whether it is entangled.

(3) (9 + 15 + 18 = 42 pts. total)

Let G_0 be a graph of 4 nodes t, u, v, w with the following edges: a self-loop at t, and edges (t, u), (u, v), (v, w), and (w, t) that form a cycle.

(a) Draw the corresponding graph-state circuit C_0 of four qubits, making t the first qubit.

(b) Show that $\langle 0^4 \mid C_0 \mid 0^4\rangle = 0$ *without* resorting to 16×16 matrix multiplications. Using a "maze diagram" is OK, but you can slice the work down even more: The self-loop effectively makes all the rows from 1000 to 1111 start off negative. The rows that begin with 0 can only flip if the middle two bits are both 1 or the last two bits are both 1. Only the last eight rows are more tedious. [And a perfect answer to part (c) can cover for this part anyway.]

Now let G be an *arbitrary* graph of n nodes. Make a new graph G' of $n + 2$ nodes by first selecting an edge (u, w) of G. Then add a new node t with a self-loop at itself and edges (t, u) and (t, w) to u and w. Finally insert a new vertex v into the original edge (u, w), thus replacing (u, w) by two edges (u, v) and (v, w). This "implants" the graph G_0 in place of the original edge (u, w). Let C' be the corresponding graph-state circuit. Note that if G consists of nothing more than the single edge (u, w), then what you get is exactly the graph G_0.

(c) Prove that $\langle 0^{n+2} \mid C' \mid 0^{n+2}\rangle = 0$ regardless of what G is. Do so by considering three cases: (i) u and w are both white, (ii) u and w are both black, and (iii) one of u, w is white, the other black. Show in each case that two of the colorings of t and v produce an odd number of black-black edges while the other two colorings preserve the parity, so that the global effect is a complete cancellation. The self-loop counts as a B-B edge when t is black.

(4) (18 + 9 + 9 = 36 pts.)

Let A be the Hermitian PSD matrix $\begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$.

(a) Find its eigenvalues λ_1 and λ_2 and find associated eigenvectors u_1 and u_2.

(b) Normalize u_1 and u_2 to be unit vectors and verify the spectral decomposition $A = \lambda_1 |u_1\rangle \langle u_1| + \lambda_2 |u_2\rangle \langle u_2|$.

(c) Calculate a matrix B such that $B^2 = A$. Does it have integer entries?

(5) (9 + 3 + 30 = 42 pts.)

Let $\boldsymbol{\phi}$ be the quantum state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle$. In our adopted big-endian notation, its state vector is $[\frac{1}{\sqrt{2}}, \frac{1}{2}, 0, \frac{1}{2}]^T$.

(a) Show that $\boldsymbol{\phi}$ is entangled.

(b) If you didn't already reshape the state vector into a 2×2 matrix A_ϕ in your answer to (a), do so now.

(c) Use the SVD truncation technique to approximate $\boldsymbol{\phi}$ by a separable state $\mathbf{u} \otimes \mathbf{v}$. Start by calculating $B = A^T A$, then find the eigenvalues. Here is a helpful point: If the rows of B have the same sum, then $[1,1]^T$ (which you can normalize appropriately) is always an eigenvector. Which eigenvalue does it belong to—the bigger one or the smaller one? By orthogonality you can then get the other eigenvector—though for truncation you might not need it. As for $\mathbf{u}$, you can skip the detail of normalizing it if your answer is clear enough. You should be able to estimate the singular values numerically close enough to give a rough opinion on whether the approximation is a "good" one—even without a calculator and even without multiplying out $\mathbf{u} \otimes \mathbf{v}$ to see how close the resulting 4-vector is to $\boldsymbol{\phi}$.

(6) (36 pts. total)

Design a quantum circuit C of 3 qubits that on input $|000\rangle$ prepares the pure state

$$\frac{1}{2}\left(|001\rangle - |010\rangle + |011\rangle - |111\rangle\right).$$

(7) (6 × 6 = 36 pts. total) *Multiple choice*—no justifications are required but may help for partial credit. Each question has a unique *best answer*.

1. If a matrix A equals the tensor product $B \otimes C$ of two smaller matrices, then:

   (a) $A^* = B^* \otimes C^*$.

   (b) $A^* = C^* \otimes B^*$.

   (c) Part (a) holds only if A is a square matrix.

   (d) Part (b) holds only if A is a square matrix.

2. A four-qubit quantum circuit C with the property that C on input $|0000\rangle$ produces $|0000\rangle$, C on input $|0100\rangle$ produces $|0101\rangle$, C on input $|1000\rangle$ produces $|1010\rangle$, and C on input $|1100\rangle$ produces $|1111\rangle$:

   (a) Can be built simply by placing a CNOT gate with control on line 1 and target on lines 3, followed by a CNOT gate with control on line 2 and target on line 4.

(b) Always produces an entangled state whenever the first two qubits are initialized to $|1\rangle$ rather than $|0\rangle$.

(c) Is impossible, as shown by the No-Cloning Theorem.

(d) Is impossible, by Holevo's theorem that n qubits can yield only n bits of classical information.

3. If U is a 4×4 unitary matrix, then

(a) $U + U^*$ is always invertible.

(b) $U + U^*$ is always a tensor product of two 2×2 unitary matrices.

(c) $U + U^*$ is always Hermitian.

(d) $U + U^*$ has all of its entries being real numbers.

4. Simon's Algorithm is significant because:

(a) It introduces the idea of using a quantum subroutine repeatedly to do sampling within an otherwise-classical algorithm.

(b) Unlike with the Deutsch-Jozsa task, its author proved that a classical randomized algorithm that generates values $f(x)$ for random x cannot succeed in expected polynomial time.

(c) Its use of periodicity suggested the idea of Shor's Algorithm.

(d) All of the above.

5. If a two-qubit pure state $\boldsymbol{\phi}$ is entangled, then:

(a) It can be the tensor product of two mixed states.

(b) Any single-qubit unitary operation on qubit 1 cannot change the state.

(c) The 4×4 self-outer-product matrix $|\boldsymbol{\phi}\rangle \langle\boldsymbol{\phi}|$ has rank at least 2.

(d) The Hadamard transform $\mathbf{H}^{\otimes 2}\boldsymbol{\phi}$ is also entangled.

6. The quantum Fourier transform of n qubits:

(a) Is computable exactly by a quantum circuit that has only Hadamard and CNOT gates.

(b) Is represented by an $n \times n$ unitary matrix in which every entry is nonzero.

(c) Is represented by a $2^n \times 2^n$ unitary matrix in which every entry is nonzero.

(d) Has all the integers from 1 to 2^n as eigenvalues.

End of Exam