CSE439, Fall 2024 **Problem Set 1** **Due Sun. 9/15, 11:59pm**

Reading: For next week, read Chapter 4 *and* sections 5.1–5.3 of Chapter 5. The lectures will motivate quantum circuits in regard to classical Boolean circuits, as an alternative to the text's flow starting from classical *machine* computations. Section 5.3 shows how the quantum Toffoli gate can simulate all Boolean operations—at the cost of using extra *ancilla* (Latin for "helper") qubits. Note also that much of section 5.1 has already been covered, while section 5.2 will furnish the quantum circuit example that appears in the solved problem on page 58.

——————-*Assignment 1, due Sun. 9/15 "midnight stretchy" on CSE Autolab*——————-

(1) For each pair of growth rate functions $f(n)$ and $g(n)$, say whether (i) $f(n) = o(g(n))$, (ii) $f(n) = \Theta(g(n))$, or (iii) $g(n) = o(f(n))$. If you answer (ii), then further say whether $f(n) \sim g(n)$, meaning that the limit of $f(n)/g(n)$ exists and equals 1. [If you say "$f(n) = O(g(n))$" without clarifying whether (i) or (ii) applies, you get half credit. All logarithms are to base 2. $6 \times 2 + 6 = 18$ pts.]

(a) $f(n) = 3\log n$, $g(n) = 4\log n$.

(b) $f(n) = 2^{3\log n}$, $g(n) = 2^{4\log n}$.

(c) $f(n) = 3 + \log(n)$, $g(n) = 4 + \log n$.

(d) $f(n) = 2^{3+\log n}$, $g(n) = 2^{4+\log n}$.

(e) $f(n) = n^{1/3}$, $g(n) = n^{1/4}$.

(f) $f(n) = 2^{n/3}$, $g(n) = 2^{n/4}$.

(2) For each pair of vectors $\mathbf{a}$ and $\mathbf{b}$, give:

(i) the inner product $\langle \mathbf{a}, \mathbf{b} \rangle$,

(ii) the tensor product $\mathbf{a} \otimes \mathbf{b}$, and

(iii) the outer product $|\mathbf{a}\rangle \langle \mathbf{b}|$.

Note that the superscript ^{T} means that column vectors are given. Also say which of the tensor products is a unit vector. For part (d), recall the definition of the e_x, where x is a binary string, as a standard basis vector in N-dimensional Hilbert space, where $N = 2^n$ and n is the length $|x|$ of the string x. ($4 \times 3 \times 3 + 4 = 40$ pts.)

(a) $\mathbf{a} = [0.8, 0.6]^T$, $\mathbf{b} = [0.6, 0.8]^T$;

(b) $\mathbf{a} = [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}]^T$, $\mathbf{b} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{2}]^T$;

(c) $\mathbf{a} = [i+i, 1-i]^T$, $\mathbf{b} = [i, -i]^T$;

(d) $\mathbf{a} = \frac{1}{\sqrt{2}}(e_{00111} + e_{10110})$, $\mathbf{b} = \frac{1}{\sqrt{2}}(e_{01000} + e_{01001})$.

(3) For each of the following 2-qubit quantum state vectors $\mathbf{c}$, say whether it is **separable** or **entangled**. In the former case, find 1-qubit state vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{c} = \mathbf{a} \otimes \mathbf{b}$. In the latter case, prove that such $\mathbf{a}$ and $\mathbf{b}$ cannot exist. ($4 \times 6 = 24$ pts.)

(a) $\frac{1}{2}[1, i, 1, i]$,

(b) $\frac{1}{2}[0, i, i, 0]$,

(c) $\frac{1}{2}[1, i, i, 1]$,

(d) $\frac{1}{2}[1, 1, 1, i]$,

(4) Design a 4×4 unitary matrix U such that $U\left|{++}\right\rangle$ equals the state $|\phi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |10\rangle + |11\rangle)$. For a hint, first create U' such that $U'|00\rangle = |phi\rangle$. Then "pre-compose" U' with a transformation H that we know carries $\left|{++}\right\rangle$ to $|00\rangle$. If you prefer to avoid Dirac notation, the requirement is to design U such that

$$U\frac{1}{2}[1, 1, 1, 1]^T = \frac{1}{\sqrt{3}}[1, 0, 1, 1]^T,$$

and the hint is first to make U' such that

$$U'e_{00} = U'[1, 0, 0, 0]^T = \frac{1}{\sqrt{3}}[1, 0, 1, 1]^T.$$

Indeed, you can make U' real and symmetric (hence Hermitian) as well as unitary. But is the matrix $U = U'H$ that you get also Hermitian? Is the product of two Hermitian matrices always Hermitian? (15 pts. for U' plus 3 points for these extra two questions, making 18 points total on the problem and 100 on the set.)