CSE439, Fall 2024 Problem Set 3 Due Thu. 10/3, 11:59pm

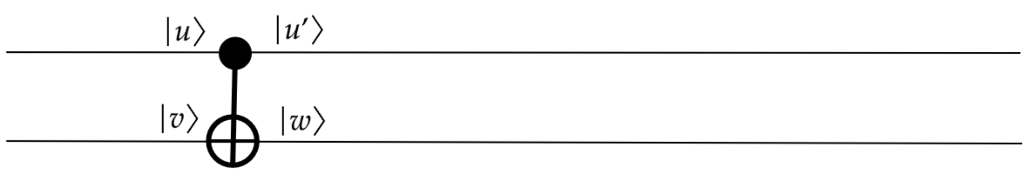
Reading: For next week, read Chapter 8. This is the last chapter that will be covered by the **First Prelim Exam**, which will be held *in class period* on **Thursday, October 10**. The doman of the exam will be homeworks through this one and factual material such as found on multiple-choice or true/false-type questions.

For a note on notation, I find it convenient to interchange general linear-algebra notation such as e_0 and e_1 for standard-basis vectors and u for a general vector, with the equivalent Dirac notation forms $|0\rangle$, $|1\rangle$, and $|u\rangle$. Bolding the former, as $\mathbf{e_0}$, $\mathbf{e_1}$, and generally $\mathbf{u}$, emphasizes when they are unit vectors. Of course the Dirac *ket* forms can only be unit vectors. Similarly, a general matrix may be written A, while $\mathbf{A}$ emphasizes that the matrix is unitary. Or bold can be used for a non-unitary Hermitian matrix like the outer-product $\mathbf{J}_n = |+^n\rangle\langle +^n|$—we will see much later a sense in which Hermitian and unitary matrices correspond to each other. There are three contexts where I find Dirac notation best:

- To represent any conceptual attribute, like a binary string x or the spade suit in a deck of playing cards, as a basis vector: $|x\rangle$, $|\spadesuit\rangle$.
- For outer products $|u\rangle\langle v|$, so you can combine with inner products elsewhere, etc.
- For functional-superposition states $\sum_x |x\rangle |f(x)\rangle$.

——————-*Assignment 3, due Thu. 10/3 "midnight stretchy" on CSE Autolab*——————-

(1) This question is about cases where the states on the two qubit lines before and after a **CNOT** gate are both separable. Note that we have seen cases where the state coming in (after one Hadamard gate on line 1) is $|+\rangle \otimes |0\rangle$, which is separable, but the state going out is *entangled*. So for the exit state to be also separable in the form $|u'\rangle \otimes |w\rangle$, as shown in the following diagram, is already fairly special.



We have seen cases where $|u\rangle$ is the standard basis state e_0. Then $|w\rangle = |v\rangle$ so the gate is a no-op, and also $|u'\rangle = |u\rangle = e_0$ back on line 1. And if $|u\rangle$ is the standard basis state e_1, then any $|v\rangle = ce_0 + de_1$ gets flipped around to $|w\rangle = de_0 + ce_1$.

(a) Find the only other case where $|u'\rangle = |u\rangle$. Note that if you write $|u\rangle = ae_0 + be_1$, then "other case" (meaning apart from $|u\rangle = e_0$ and $|u\rangle = e_1$) entails that both a and b are nonzero. What happens to $|v\rangle$ and $|w\rangle$ in that case? (Show your scratchwork. 12 pts.)

(b) Now try $|v\rangle = |-\rangle = \frac{1}{\sqrt{2}}[1, -1]^T$. Do you get $|u'\rangle = |u\rangle$ in this case? Also take an initial stab at trying to answer: Does this exhaust the possible cases in which the states before and after the **CNOT** are both separable? (6+3+3=12 pts.)

(c) A vector x is an **eigenvector** of a general matrix A, with associated **eigenvalue** λ, if $Ax = \lambda x$. Use all the above discussion and figuring to produce four eigenvectors of the **CNOT** matrix, and their eigenvalues, such that the four vectors are linearly independent and orthogonal to each other. (12 pts.)

For some help on (c), consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $x = [1,1]^T$ is an eigenvector with eigenvalue 1. Also $y = [1,-1]^T$ is an eigenvector with eigenvalue -1, since $Ay = [-1,1]^T = -y$. Dividing x and y by any constants, such as by $\sqrt{2}$ to make them unit vectors, still leaves them as eigenvectors with the *same* eigenvalue. Since x and y are orthogonal, the division by $\sqrt{2}$ produces an **orthonormal eigenbasis**.

(2) Text, problems 3.13–3.14 in Chapter 3, *and then also* problem 6.9 in chapter 6. (15+9+6 = 30 pts.)[1]

(3) (a) Draw the graph-state quantum circuit C_G for the graph $G = (V, E)$ with $V = \{1, 2, 3\}$ and $E = \{(1, 2), (2, 3), (1, 3), (1, 1)\}$. That is, G is the undirected triangle graph on three nodes (thus far as shown in lecture) but with an extra self-loop on node 1. You may use a snip from a simulator such as Davy Wybiral's or *Quirk* or *Qiskit* showing the circuit instead.

(b) Use a "maze diagram" to compute $\langle 000| C_G |000\rangle$—at least to tell whether this amplitude is zero. Recall from lecture that the "wavefront" after the initial stage $\mathbf{H}^{\otimes 3}$ will have all-positive "mice," and the final $\mathbf{H}^{\otimes 3}$ stage going back up to $|000\rangle$ will not change the signs of any mice after they get there. So you only need to draw the middle section of the diagram for the three **CZ** gates and the one **Z** gate (which can be in any order) then count how many mice end up still positive and how many end up negative.

(c) Now make G' by adding a self-loop at node 3 too. What happens to $\langle 000| C_{G'} |000\rangle$? Tweak your trace from part (b) to show the answer. (18 pts. total, for 84 on the problem set)

[1] I am actually working on a version of problem 6.10 where the object is just to simulate the **T** gate on one qubit line (identity or whatever on the other line). *This can be done to any desired degree of approximation, but I suspect it cannot be done exactly.*