Name and St.ID#:____

CSE439, Fall 2024 — Prelim I — Oct. 10, 2024

Open book, open notes, closed neighbors, 75 minutes. The exam totals 100 pts., subdivided as shown. Do *all five problems* on these exam sheets—there is no "choice" option. *Show your work*—this may help for partial credit. [Problem (4) has possible extra credit.]

Use of a quantum circuit simulation app—whether online or downloaded—or matrix calculator is forbidden. You may use—and cite—theorems and facts from lectures and homeworks without further justification.

Notation: You may freely mix "standard linear algebra notation" and "Dirac *bra-ket* notation" for quantum states. For example, e_{10} means the same thing as $|10\rangle$ and is represented (in big-endian notation) by the unit column vector $[0,0,1,0]^T$. For matrix outer-products the *ket-bra* form typified by $|+\rangle\langle -|$ **is** standard. Quantum state vectors use big-endian notation by default; if you switch to little-endian you must say so.

(1) (3+3+3+6+4 = 19 pts.) *Unit(ary) vectors and matrices.*

For each of the following vectors and matrices (a)–(d), answer yes/no: is it a unit vector or unitary matrix? In part (d), *showing work is required.* In part (e), fill in missing entries to create a unitary matrix.

(a) Vector $e_{10} \otimes e_{01}$.

(b) Vector $\frac{1}{\sqrt{2}}[1+i, 1-i]^T$.

(c) Matrix $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

(d) Matrix $\frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ 1+i & -1-i \end{bmatrix}$.

(e) Fill in the five '__' entries to make U into a unitary matrix: $U = \frac{1}{3}\begin{bmatrix} 2 & \_\_ & \_\_ \\ 2 & 2 & \_\_ \\ \_\_ & \_\_ & 2 \end{bmatrix}$.

(a) _______ (b) _______ (c) _______ (d) _______ (e) _________________

(2) ($5 \times 3 = 15$ pts.) *True/False.*

Please write out the words `true` and/or `false` in full, in the blanks below or clearly next to the statements. No justifications are needed, but could help for partial credit.

(a) If $f(n) = n\sqrt{n}$ and $g(n) = n^2$, then $f(n) = o(g(n))$.

(b) The tensor product of two unit vectors is always a unit vector.

(c) The average of two unit vectors (meaning $\frac{\mathbf{u_1}+\mathbf{u_2}}{2}$) is always a unit vector.

(d) If C is a two-qubit quantum circuit, then the value of C on the state $|{++}\rangle$ is twice the average of its values on the standard basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

(e) For every two-qubit quantum state $\mathbf{u}$, the outer product $|\mathbf{u}\rangle\langle\mathbf{u}|$ is a 4×4 unitary matrix.

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(a) _________    (b) _________    (c) _________    (d) _________    (e) _________
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(3) (24 pts. total)

For each of the following two-qubit quantum state vectors, answer whether it is entangled. If you say yes, prove your answer by equations; if no, write it as the tensor product of two single-qubit vectors.

(a) $\frac{1}{5}[2, 1, 4, 2]^T$.

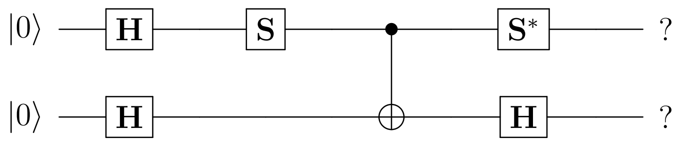
(b) $\frac{1}{5}[1, 2, 4, 2]^T$.

(c) $\frac{1}{3}[1, 1-i, 1-i, -2i]^T$.

(scratchwork page)

(4) (21 pts. total)

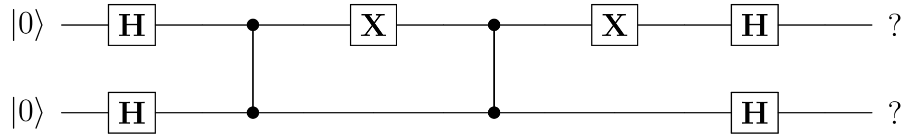
Calculate the output state of the following two qubit quantum circuit, on input e_{00} as shown. Recall that $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ and $\mathbf{S}^*$ is its inverse.



Also answer: is the resulting state entangled? Justify your answer. [For 3 pts. of **extra credit**, show how to solve all this without doing any computations involving length-4 vectors or 4×4 matrices, not even with the matrix of **CNOT**.]

(5) (21 pts. total—end of exam)

Consider the following two-qubit quantum circuit C, on input e_{00} as shown. It is a graph-state circuit except for the presence of the two **X** gates. (Note the two **CZ** gates flanking the first **X** gate.)



(a) Draw the middle portion of the "maze diagram" for C—as on homework, you may skip the initial and final sections for the pairs of Hadamard gates. Work out what you think **X** on line 1 should do from its effects on the basis states.

(b) Compute $\langle 00 \mid C \mid 00 \rangle$. Is it zero? (If you cannot get the maze diagram, then you can recover most of the credit by doing this with 4×4 matrix computations or some other way of figuring the quantum state at each vertical stage of the circuit.)