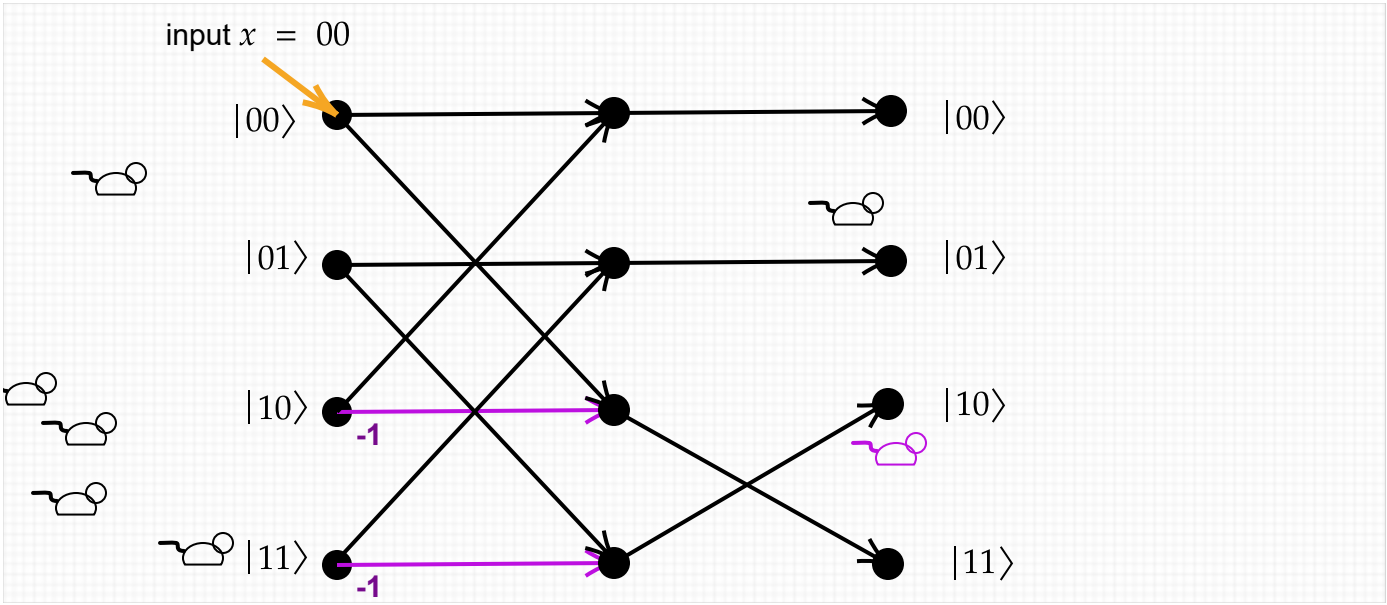
CSE439 Fall 2025 Week 6: Reckoning and Visualizing Circuits and Measurements (ch. 7)

There are basically three ways to "reckon" a quantum circuit computation on q total qubits, $Q = 2^q$:

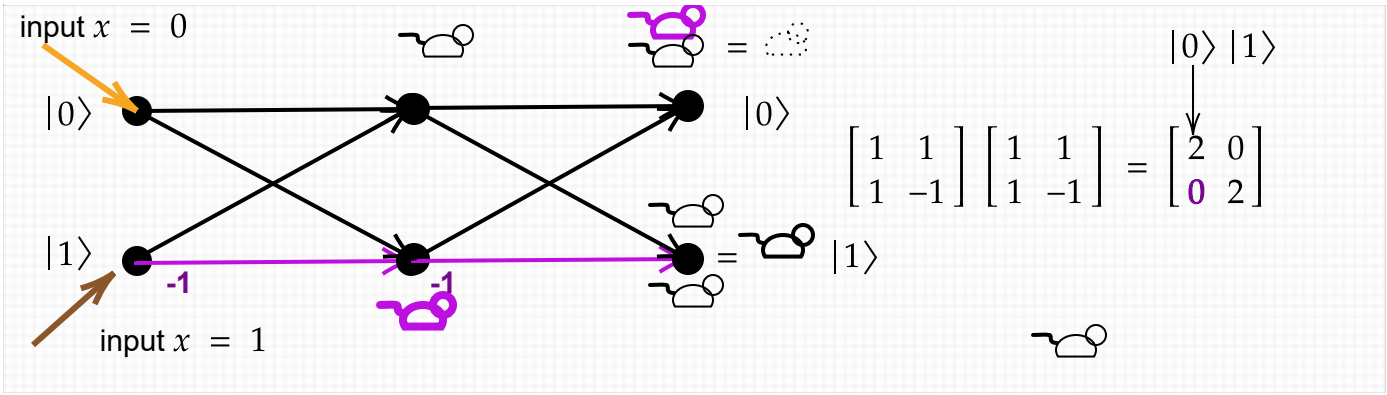
1. Multiply the $Q \times Q$ matrices together---using *sparse-matrix techniques* as far as possible. If BQP $\neq$ P and you try this on a problem in the difference then the sparse-matrix techniques must blow up at some (early) point. The downside is that the exponential blowup is paid early; the upside is that once you pay it, the matrix multiplications don't get any worse, no matter how more complex the gates become. This is often called a "Schrödinger-style" simulation.
2. Any product of s-many $Q \times Q$ matrices can be written as a single big sum of s-fold products. For instance, if A, B, C, D are four such matrices and u is a length-Q vector, then
$$ABCDu[i] = \sum_{j,k,l,m=0}^{Q-1} A[i,j] \cdot B[j,k] \cdot C[k,l] \cdot D[l,m] \cdot u[m].$$
Every (*nonzero*) product of this form can be called a (*legal*) **path** through the system. [As hinted before, in a quantum circuit, u will be at left---on an input x, it will be the basis vector $\mathbf{e}_{x0^{r+m}} = |x0^{r+m}\rangle$ under the convention that 0s are used to initialize the output and ancilla lines---and D will be the first matrix from gate(s) in the circuit as you read left-to-right. Thus the output will come out of A, which is why it is best to visualize the path as coming in from the top of the column vector u, going out at some row m (where u_m is nonzero---for a standard basis vector, there is only one such m), then coming in at column m of D, choosing some row l to exit (where the entry $D[l, m]$ is nonzero), then coming in at column l of C, and so on until exiting at the designated row i of A. This is the discrete form of Richard Feynman's **sum-over-paths** formalism which he originally used to represent integrals over quantum fields (often with respect to infinite-dimensional Hilbert spaces). The upside is that each individual path has size $O(s)$ which is linear not exponential in the circuit size. The downside is that the number of nonzero terms in the sum can be far worse than Q and doubles each time a Hadamard gate (or other nondeterministic gate) is added to the circuit.
3. Find a way to formulate the matrix product so that the answer comes out of symbolic linear algebra---if possible!

For the textbook, I devised a way to combine the *downsides* of 1 and 2 by making an exponential-sized "maze diagram" up-front but evaluating it Feynman-style. Well, the book only uses it for $1 \leq Q \leq 3$ and I found that the brilliant Dorit Aharonov had the same idea. All the basic gate matrices have the property that all nonzero entries have the same magnitude---and when normalizing factors like $\frac{1}{\sqrt{2}}$ are collected and set aside, the Hadamard, **CNOT**, Toffoli, and Pauli gates (ignoring the global i factor in $\mathbf{Y}$) give just entries $+1$ or -1, which become the only possible values of any path. That makes it easier to sum the results of paths in a way that highlights the properties of **amplification** and **interference** in the "wave" view of what's going on. The index values $m, l, k, j, i, \ldots$ become "locations" in the wavefront as it flows for time s, and since it is discrete, the text pictures packs of...well...spectral lab mice running through the maze.

One nice thing is that you can read the mazes left-to-right, same as the circuits. Here is the **H + CNOT** entangling circuit example: [Note: The mice are sometimes left in final positions, sometimes in a startup or midway position, for what I demonstrated in lecture.]

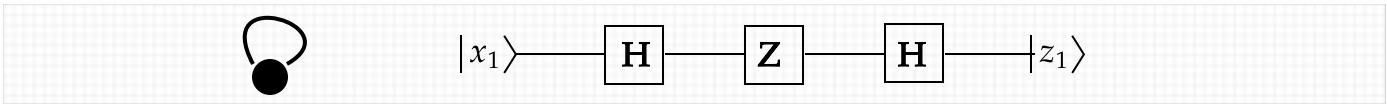


No interference or amplification is involved here---the point is that if you enter at $|00\rangle$, then $|00\rangle$ and $|11\rangle$ are the only places you can come out---and they have equal weight. To see interference, you can string the "maze gadgets" for two Hadamard gates together:

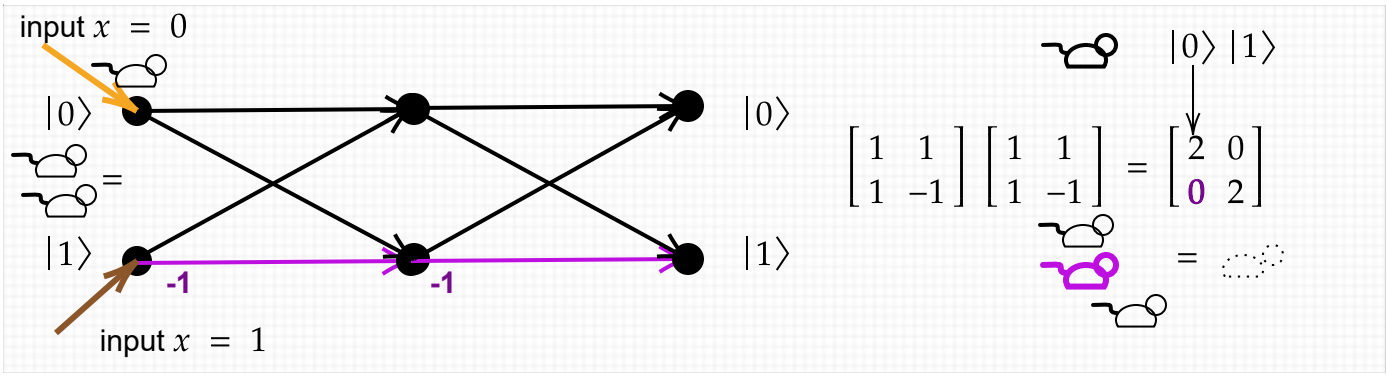


In linear-algebra terms, all that happened at lower right was $1 \cdot 1 + -1 \cdot 1$ giving 0. But the wave interference being described that way is a real physical phenomenon. Even more, according to Deutsch the two serial Hadamard gates branch into 4 universes, each with its own "Phil the mouse" (which can be a photon after going through a beam-splitter). One of those universes has "Anti-Phil", who attacks a "Phil" that tries to occupy the same location (coming from a different universe) and they fight to mutual annihilation.

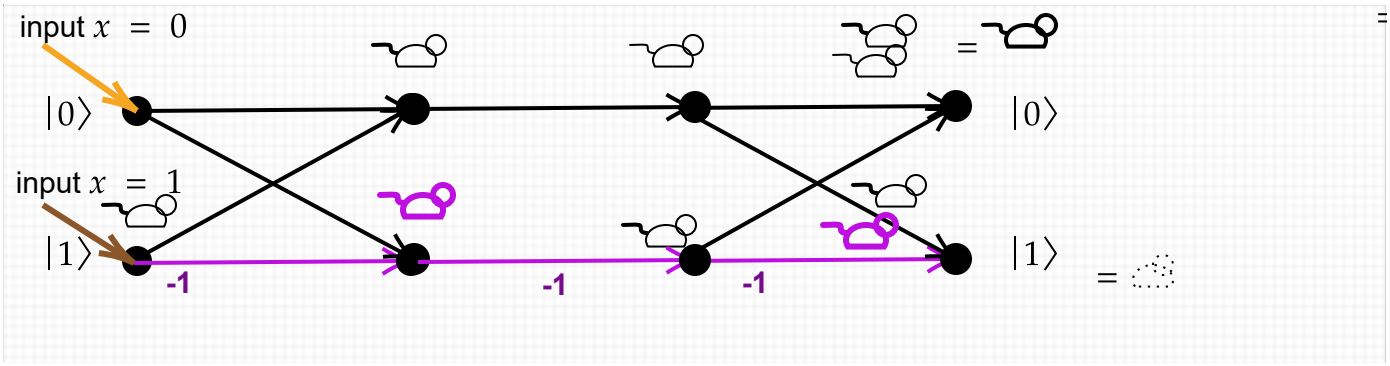
Can we build any interesting things with just a few qubits? Yes, in fact. Even the simplest **graph state circuit**---for a graph of just one node with a self-loop---is instructive to visualize.



We have seen the equation $\mathbf{HZH} = \mathbf{X}$. How is this reflected when we visualize the quantum properties? There is only one change from the "maze" for two $\mathbf{H}$-gates canceling, which was:



The change is to insert a stage that again has a -1 on the $|1\rangle$ basis value but no "crossover":



This time, when "Phil" starts running from $|0\rangle$ at left, the "mice" cancel at $z = |0\rangle$ and amplify at $z = |1\rangle$. And on input $x = |1\rangle$ they output the basis state $|0\rangle$. The result is Boolean NOT, i.e., $\mathbf{X}$.

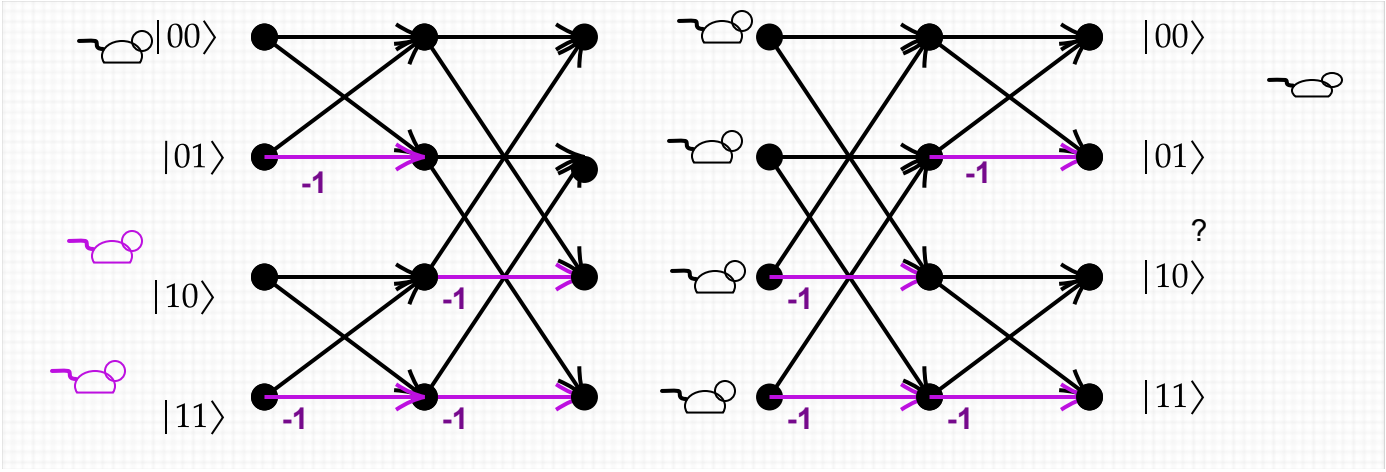
[Footnote: A basic outcome $|z\rangle$ for the circuit C on input x has amplitude $\langle z|U_C|x\rangle$, not $\langle x|U_C|z\rangle$ as I've once been guilty of writing. Perhaps the diagrams should write the bra-form, $\langle 0|$ and $\langle 1|$ and so on, for z at right to emphasize this. But we've identified the ket-form with the notion of "outcome"; this is the form that would be given as input to a further piece of the circuit. This dilemma is another reason why Lipton and I first tried for a "handedness-free" approach.]

Phenomena of interest (tracing the "mice" is analogous to propagating a waveform):

1. Superposition
2. Amplification
3. Phase changes
4. Interference.

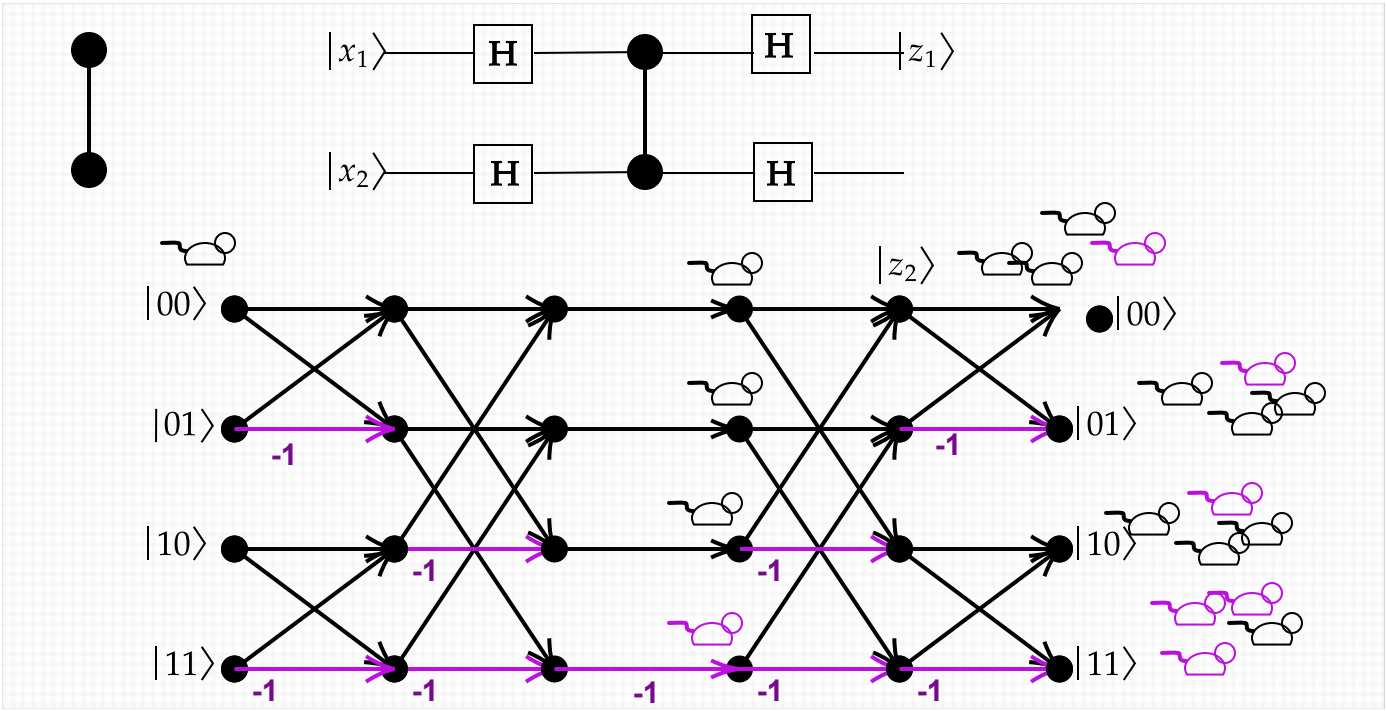
For graph state circuits of 2 nodes we need 2 qubits. The Hadamard transform of two qubits is

diagrammed as at left and right. It does not matter what order the two $\mathbf{H}$ gates go in.



Note that the mouse running from $|00\rangle$ encounters no phase change, nor mice ending at $|00\rangle$ regardless of origin. This simply expresses that the Hadamard transform (and the QFT too) have every entry $+1$ (divided by the normalizing constant $R = \sqrt{2^n}$) in the row and column for $|00\rangle$. We will focus on the amplitude of getting $|00\rangle$ as output given $|00\rangle$ as input. If G is the graph, C_G the graph-state circuit, and U_G the unitary operator it computes, then the amplitude we want is $\langle 00|U_G|00\rangle$.

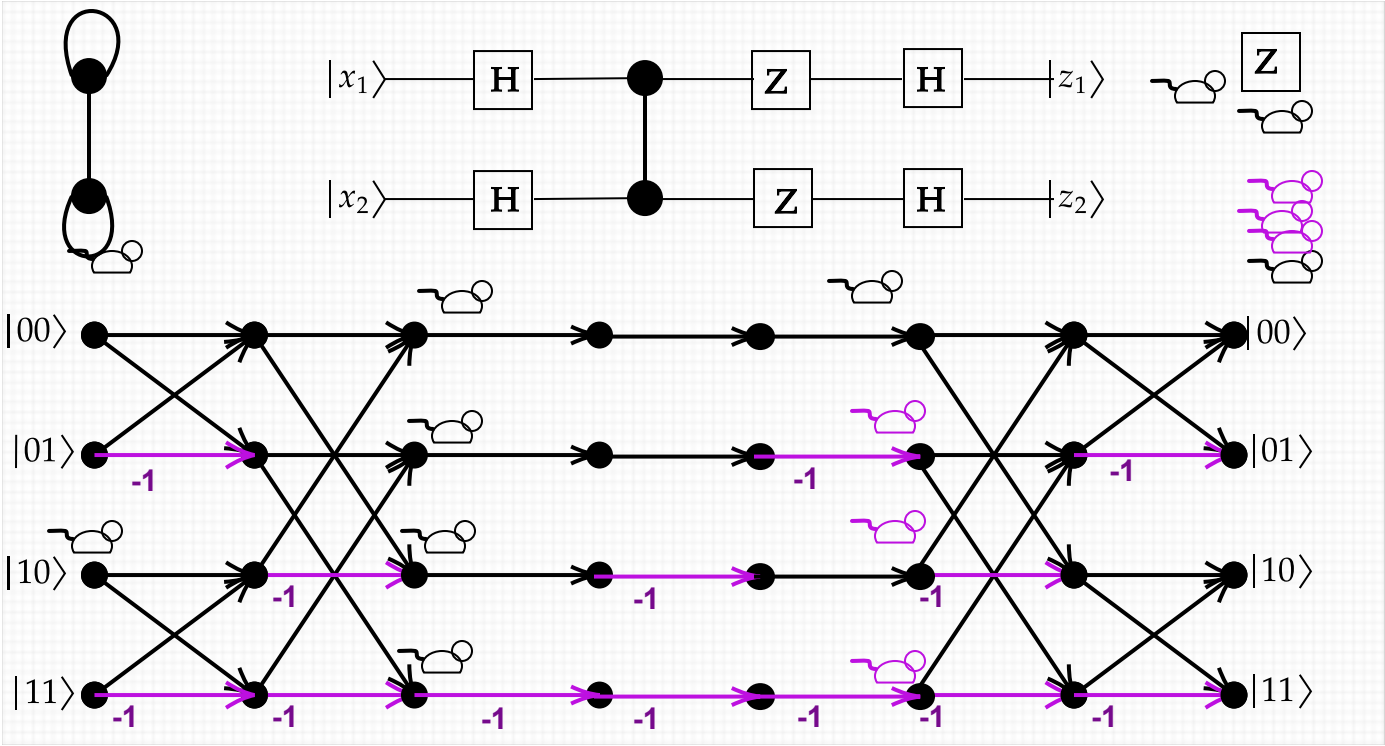
The simplest two-node G has a single edge connecting the two nodes. This introduces a single $\mathbf{CZ}$ gate between the qubits standing for the nodes.



If we take the two Hadamard gates away from line 1, then we have $\mathbf{H}\,2\,\mathbf{CZ}\,1\,2\,\mathbf{H}\,2$, which is equivalent to $\mathbf{CNOT}$. But with them, we get equal superpositions once again. Most in particular, the

amplitude of $\langle 00|U_G|11\rangle$ (= $\langle 11|U_G|00\rangle$) is nonzero. [The lecture also noted how $\frac{1}{2}[1,1,1,-1]^T$ is a fixed point of $\mathbf{H}^{\otimes 2}$, ignoring multiplication by the unit scalar -1.]

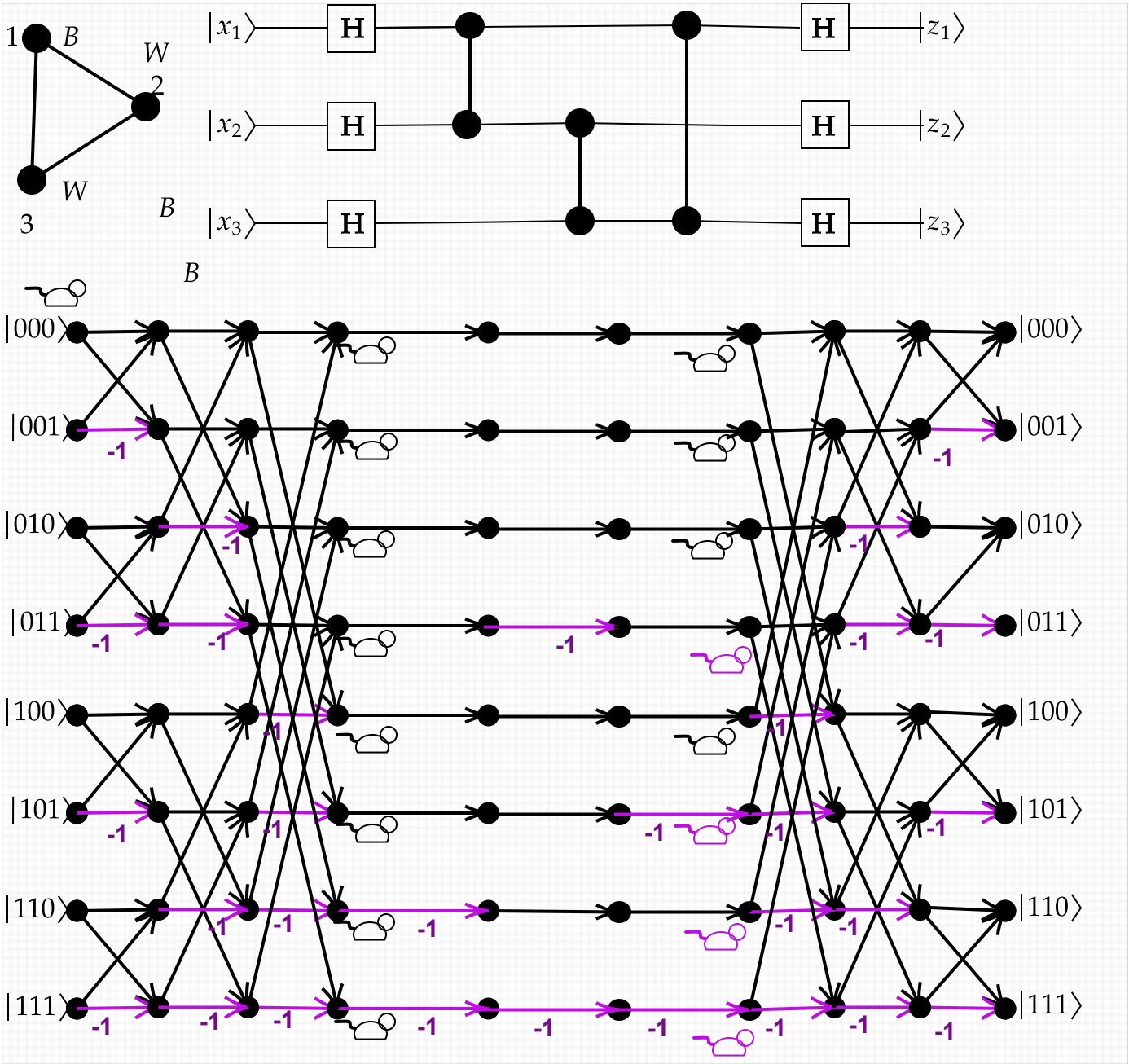
Now let's try a graph that adds a loop at each node. We can call it the "Q-Tip" graph:



The -1 phase shifts for the $\mathbf{Z}$ gates go on the basis states that have a 1 on line 1 or 2, respectively. Now the amplitude value $\langle 00|U_G|00\rangle$ is negative. Its sign does not affect the probability and the state still gives an equal superposition.

It does not matter whether we put the $\mathbf{Z}$ gates "before" or "after" the $\mathbf{CZ}$. The diagonal matrices all commute, and this is clear from how the paths go straight across without branching. We could simply make the whole graph into one diagonal gate with phase shifts that multiply the -1 factors along each row. A related thing to note is that if we repeat an edge or loop, then the two cancel completely. It's as if we have a graph with edges defined by even-odd parity rather than number.

Now let's try a three-node graph, the triangle:



For computing the amplitude $\langle 000|U_G|000\rangle$ it is not necessary to follow the "mice" through the Hadamard parts of the "maze". The mice entering the graph part from $x = |000\rangle$ are all positive, and the mice going to $z = |000\rangle$ will not change color once they leave the graph. So we need only track the middle portion and count how many mice are + and how many are −. For the triangle graph, the answer is: four of each. They **cancel**. So $\langle 000|U_G|000\rangle = 0$.

This leads us to more insight and a strategy for determining this amplitude for a general n-node graph $G = (V, E)$:

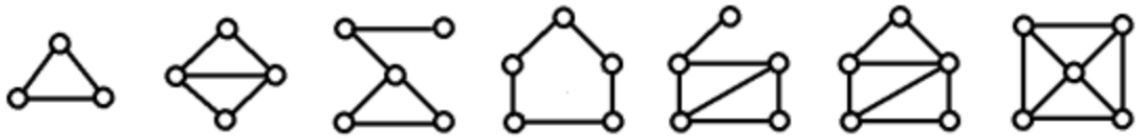
- Every basis state $|x\rangle$ with $x \in \{0,1\}^n$ corresponds to a **2-coloring** χ_x of the vertices. Say a node u is **black (B)** if $x_u = 1$, white (W) if $x_u = 0$. (The Greek letter χ (*chi*) looks like an X and

indeed X is its capital form, but the Greek letter that sounds like English X is ξ (*xi*) with capital Ξ. The χ gives the *ch* in *chromatic*. Well, we can say that the binary string x "is" the coloring χ.)

- For any edge $(u, v) \in E$, the edge contributes a -1 in its $\mathbf{CZ}$ gate if both u and v are colored **B**. Call it a **B-B** edge.
- Therefore, a coloring gives a $\mathbf{-1}$ net contribution if it gives G an odd number of B-B edges.
- The amplitude value $\langle 0^n | U_G | 0^n \rangle$ is positive if fewer than 2^{n-1} (i.e., half) the colorings create an odd number of B-B edges, zero if exactly half do, negative if more.

Whether *one* amplitude is positive or negative does not matter so much in quantum up to equivalence under scalar multiplication. (My lecture demo'ed some examples.) But patterns of signs between different amplitudes $a_z|z\rangle$ of possible outcomes z may have further significance.

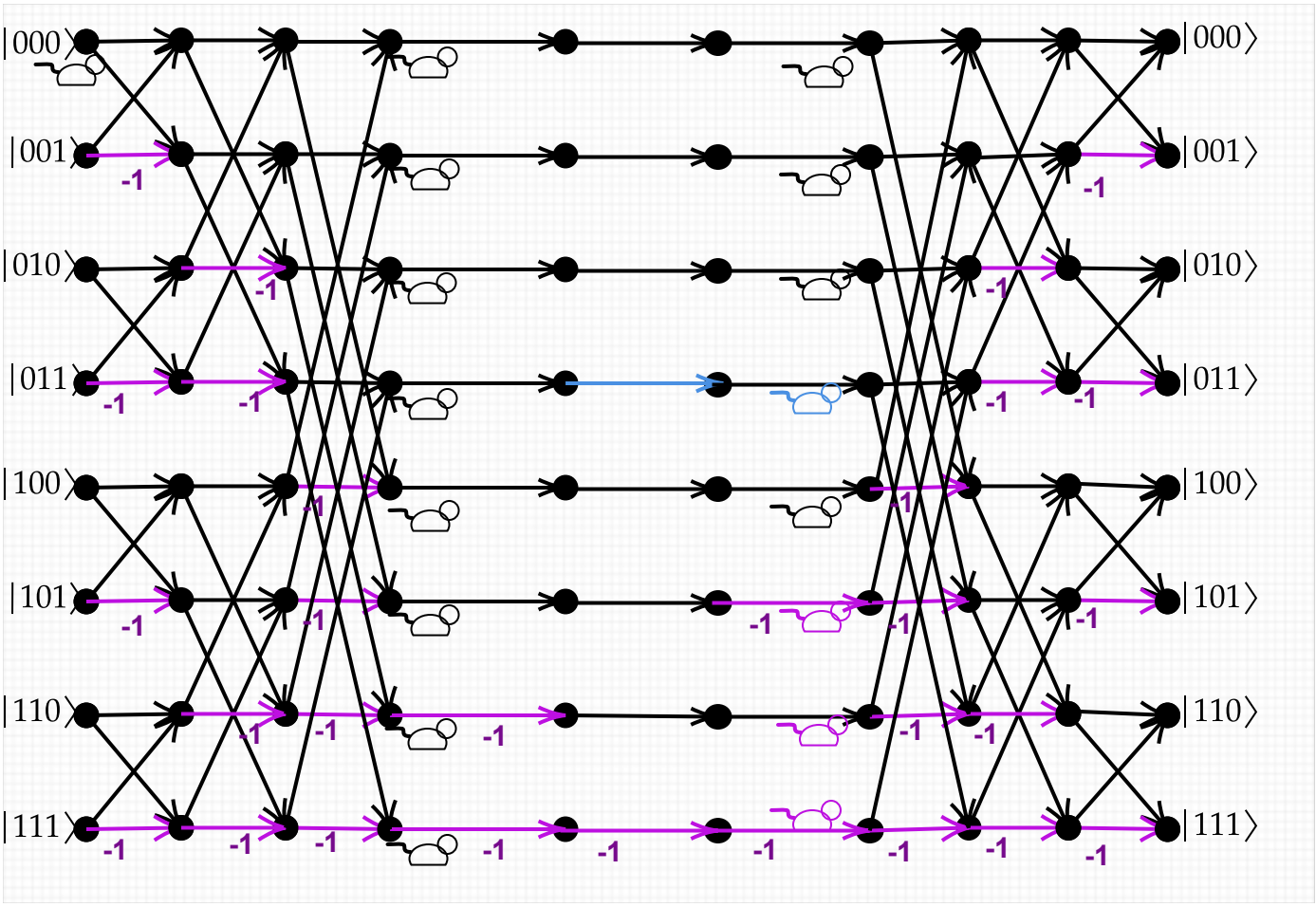
Whether the amplitude is *zero*, however, is absolute. I call a graph G "**net-zero**" if $\langle 0^n | U_G | 0^n \rangle = 0$. Above we first observed that the single-node loop graph is net-zero. The smallest simple undirected graph (meaning no loops or multiple edges) that is net-zero is the triangle. Here are all such graphs up to five vertices:



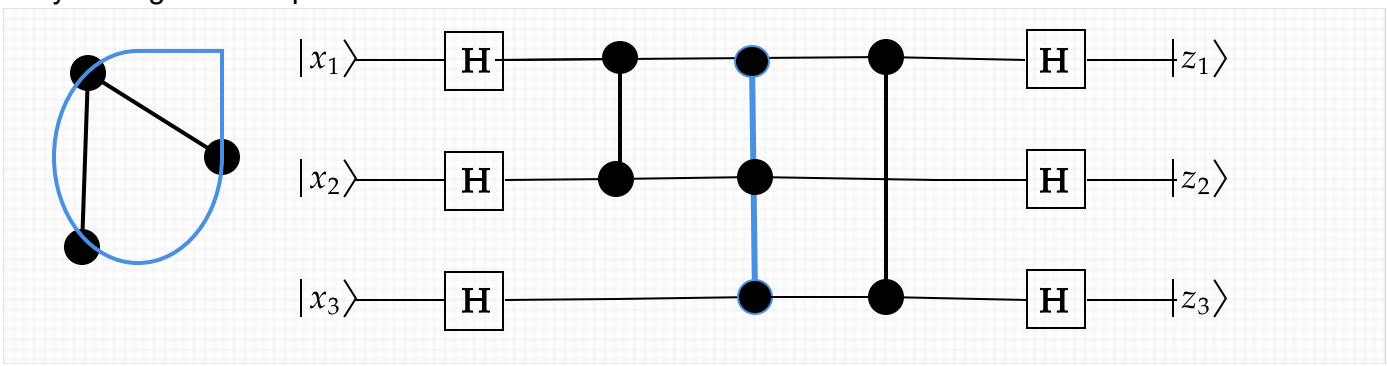
I do not see any simple way to tell "visually" whether a graph is net-zero. My recent PhD graduate Chaowen Guan and I improved the known running time to decide this algorithmically from $O(n^3)$ to whatever the time to multiply two $n \times n$ matrices is (currently $< O(n^{2.37286})$). The algorithm works by converting the graph-state circuit into a quadratic equation of a kind that converts into a linear equation in $O(n)$ variables, whose solutions can be counted in yea-much time. But a simple, more-direct criterion for a graph to be net-zero could give a practically much better algorithm. Guan and I wrote about this on the GLL blog at

https://rjlipton.wpcomstaging.com/2019/06/10/net-zero-graphs/

Some generalizations of graph-state circuits can be handled with equal efficiency. We can simulate $\mathbf{CNOT}$ gates since $\mathbf{CNOT}\ i\ j$ is equivalent to $\mathbf{H}\ j\ \mathbf{CZ}\ i\ j\ \mathbf{H}\ j$. The extra $\mathbf{H}$ gates take things outside the realm of graph-state circuits as strictly defined, but keeps them within the class of so-called **stabilizer circuits**, or equivalently, **Clifford circuits**, to which the same $< O(n^{2.37286})$ runtime applies (for getting any one amplitude, that is). The gates allowed in these circuits are $\mathbf{H}$, $\mathbf{CNOT}$, $\mathbf{S}$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, $\mathbf{CZ}$, but notably not $\mathbf{Tof}$, $\mathbf{T}$, or $\mathbf{CS}$. Or $\mathbf{CCZ}$ for that matter. But there are other tweaks that seem to be easy to bring within our framework, yet yield hard problems. Consider:

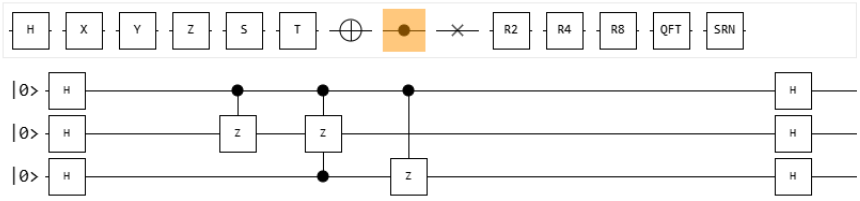


The only change was in the middle column, removing the -1 from the row for $|011\rangle$. The middle column now "fires" only when all 3 bits are 1, i.e., for the component of $|111\rangle$ in any state. This is the action of the double-controlled Z-gate, **CCZ** (which is really a triple control of a 180° phase shift). It is easy to diagram in a quantum circuit:



In graph-theoretic terms, this has replaced the edge $(2, 3)$ by the **hyper-edge** $(1, 2, 3)$, thus creating a **hypergraph**. The effect of changing only the color of the mouse in row 4 (for $|011\rangle$) may seem small, but it has a wild effect on the state vector. Now $z = |000\rangle$ has 5 positive paths from $x = |000\rangle$ instead of 4, so its amplitude is $\frac{5-3}{8} = \frac{1}{4}$. Six other components have amplitude $\frac{1}{4}$, and they collectively have $\frac{7}{16}$ of the probability. The other one, for $|100\rangle$, has 7 positive paths to 1 negative, and so amplitude $\frac{7-1}{8} = \frac{3}{4}$ which squares to $\frac{9}{16}$. Note that the previous amplitude was $\frac{6-2}{8} = \frac{1}{2}$ which squares to just $\frac{1}{4}$,

so flipping just one path of eight made a $\frac{5}{16}$ difference to the probability, more than one might expect. The CCZ gate could likewise be in any order---the gates commute so there is no element of time sequencing until the final bank of H gates. The middle part is "instantaneous." Here is a snip as shown in lecture from the Wybiral simulator, which does allow you to do a CCZ gate by placing two controls:



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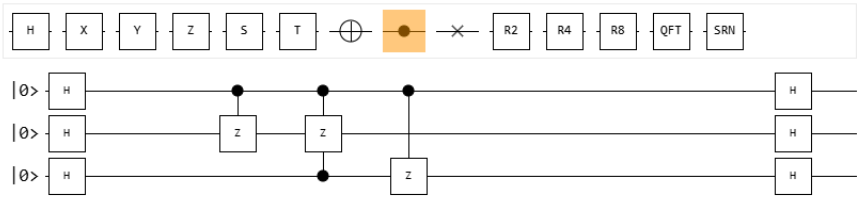
0.25000000+0.00000000i |001>	6.2500%
0.25000000+0.00000000i |010>	6.2500%
0.25000000+0.00000000i |011>	6.2500%
0.75000000+0.00000000i |100>	56.2500%
-0.25000000+0.00000000i |101>	6.2500%
-0.25000000+0.00000000i |110>	6.2500%
-0.25000000+0.00000000i |111>	6.2500%

This little illustration of wildness (amplitudes and probabilities being suddenly imbalanced after placing the CCZ gate) sits over a more general point. The equation resulting from having the CCZ gate changes from quadratic to *cubic*. Counting solutions to this kind of cubic equation is NP-**hard**. In fact, sandwiching the CCZ gate between two H gates (on any one qubit line) gives the Toffoli gate (with target on that line). So CCZ likewise goes outside the Clifford ambit and gives a universal gate set.

What About Measurement?

Let's say we measure qubit 1 (big-endian) of the above imbalanced state. There is a 1/4 chance of getting the result 0 and 3/4 chance of getting 1. If we measured all the qubits, we would see a 9/16 chance of getting 100, 1/16 each for 101, 110, and 111. But when we measure just one qubit, the rest of the state stays superposed. Which part is "the rest of the state" depends on the outcome of the measurement. In this case:



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0.25000000+0.00000000i |000>	6.2500%
0.25000000+0.00000000i |001>	6.2500%
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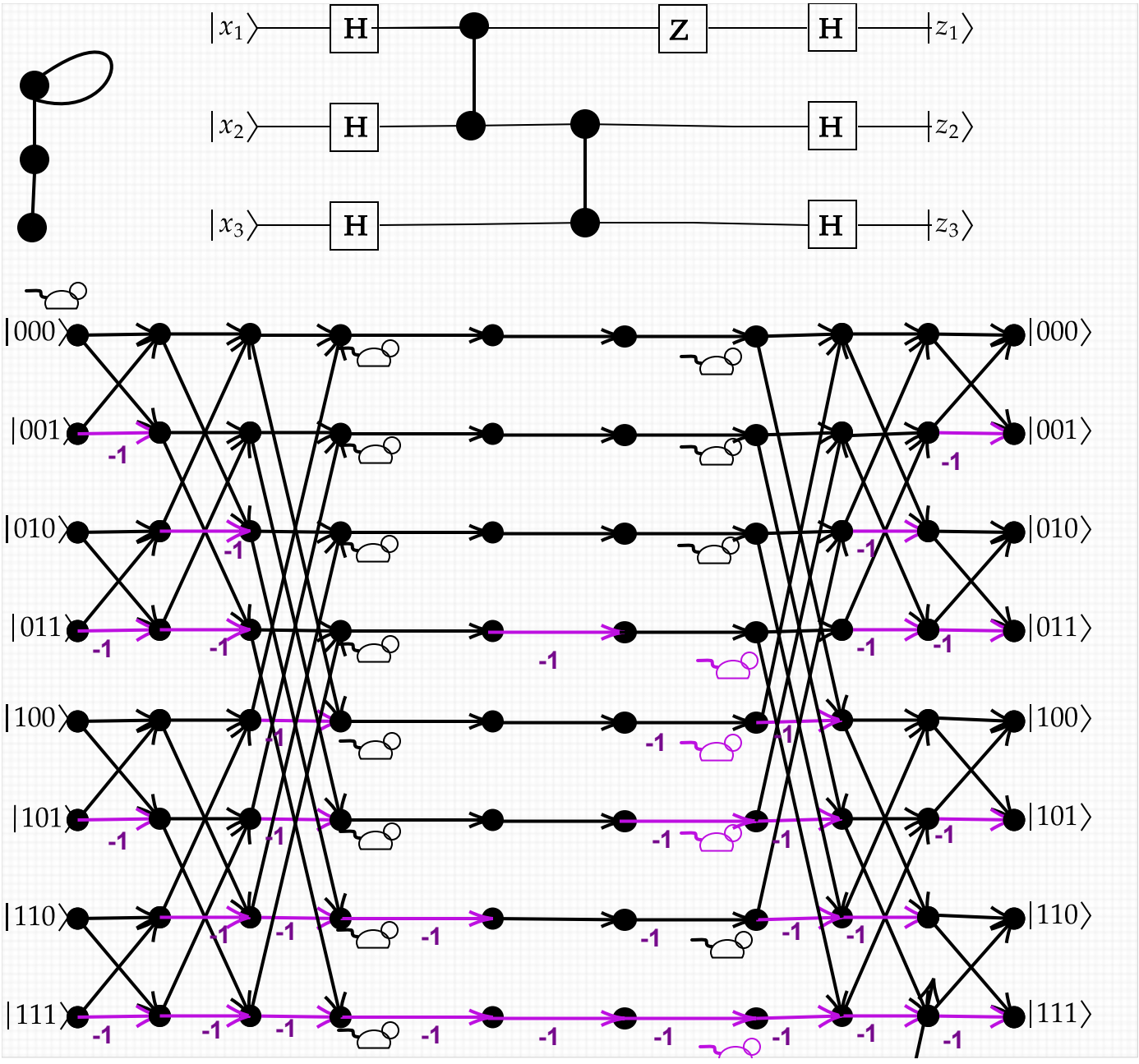
-0.25000000+0.00000000i |101>	6.2500%
-0.25000000+0.00000000i |110>	6.2500%
-0.25000000+0.00000000i |111>	6.2500%

- If the outcome is 0, the new state on qubits 2 and 3 is $\frac{1}{2}[1,1,1,1]^T$. Equal weight superposition with positive signs
- If the outcome is 1, then preserving the relative amplitudes the gives $[3,-1,-1,-1]^T$. (Or $[-3,1,1,1]^T$, which has the same ratios of amplitudes .) To renormalize this, divide by the square root of 12, which is twice the square root of 3. The state also equals $\frac{1}{\sqrt{3}}[1.5,0.5,0.5,0.5]^T$.

Heres's a **challenge** : Can we get this state using just the graph-state gates on two qubits? We will also allow you CNOT and Pauli X and Y and even the **phase gate** S, but not T or CS . And not CCZ or Toffoli since only two qubits *without ancillae*. If not, can we prove not?

There is a more exact rule for computing the new state, predicated on the result of the measurement. Since we have adopted the principle of deferred measurement, we can defer it to ch. 14 in November . But we can see the results in *Quirk* by applying its postselection operators. Note that they are outerproducts. In any event, this shows special effects one can do with non-Clifford gates like CCZ.

Another Graph State Circuit Example:



Small-Scale Applications (chapter 8)

At last we get into some famous instances of quantum applications. The first one was cooked up just to show that quantum computations could meet a goal that classical algorithms cannot. Whether the goal is compared *fairly* is open to debate, however. Here is some back-story:

David Deutsch, drawing on two papers by Feynman and other sources, introduced quantum computing while I was a graduate student and he was a postdoc at Oxford in the mid-1980s. At first, he claimed quantum computers could solve the Halting Problem in finite time. Fellows of Oxford's Mathematical Institute refuted the claim. But it was not crazy: a year ago it was proved that a binary quantum system of "interactive provers" ***can*** (kind-of-)solve the Halting Problem in finite time. (My review of the paper is at https://rjlipton.wordpress.com/2020/01/15/halting-is-poly-time-quantum-provable/) Per my memory of observing some meetings about it, the gap in Deutsch's argument had to do with properties of probability measures based on infinite binary sequences.

So Deutsch fell back on something less ambitious: demonstrating that there was a "very finite" task that quantum computers can do and classical ones cannot. (Well, unless the playing field is leveled for them...but before we argue about it, let's see the task.)

Deutsch's Algorithm

The task is a **learning problem**, a kind of interaction we haven't covered yet. Instead of "input x, compute $y = f(x)$", a learning problem is to determine facts about an initially-unknown entity f that you can **query**.

1. **Oracle Turing machines** give a classic way to define this kind of problem. For oracle functions f or languages A drawn from a limited class---such as subclasses of the regular languages---can we design an OTM M that on input 0^n (for large enough n) can distinguish what A is in time (say) polynomial in n? The computation $M^A(0^n)$ can learn about A by making queries y on selected strings y and observing the answers $A(y)$.
2. One can also define **oracle circuits** that have special **oracle gates** with some number m of input wires and enough output wires to give the answer $f(y)$ on any $y \in \{0,1\}^m$.
3. An ordinary electrical test kit behaves that way. It is a circuit with a place(s) for you to insert one or more (possibly-defective) electrical components A. The test results should diagnose electrical facts about A.
4. Quantum circuits for all of the Deutsch, Deutsch-Jozsa, Simon, Shor, and Grover algorithms work this way. They involve an **oracle function** $f : \{0,1\}^n \to \{0,1\}^r$ given in **reversible form** as the function $F : \{0,1\}^{n+r} \to \{0,1\}^{n+r}$ defined by:

$$F(x,z) = (x, f(x) \oplus z).$$

Usually z is 0^r and the comma is just concatenation (i.e., tensor product) so the output is just $xf(x)$. In the simplest case $n = r = 1$, F is a two-(qu)bit function. Some examples:

- If f is the identity function, $f(x) = x$, then $F(x,z) = (x, x \oplus z) = \mathbf{CNOT}(x,z)$.
- If $f(x) = \neg x$, then $F(x,z) = (x, x \leftrightarrow z)$: $F(00) = 01$, $F(01) = 00$, $F(10) = 10$, $F(11) = 11$.
- If f is always false, i.e., $f(x) = 0$, then F is the two-qubit identity function.
- If $f(x) = 1$, then $F(x,z) = (x, \neg z)$, so $F(00) = 01$, $F(01) = 00$, $F(10) = 11$, $F(11) = 10$.

These are all deterministic as functions of two-qubit basis states, so they permute the quantum coordinates $0 = 00$, $1 = 01$, $2 = 10$, and $3 = 11$. Recall that **CNOT** gives the permutation that swaps the coordinates 2 and 3, that is, **CNOT** $= (2\ 3)$ in swap notation. In full, we have:

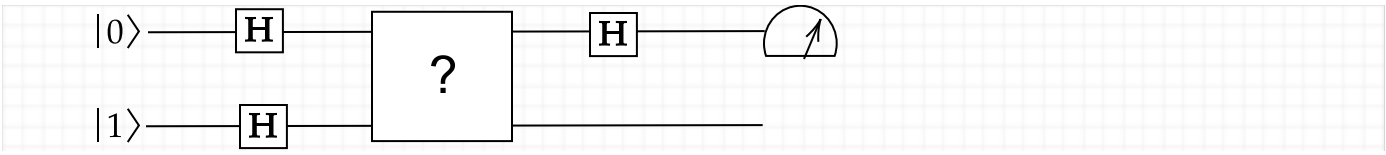
$$F_{id} = (2\ 3), \quad F_{\neg} = (0\ 1), \quad F_0 = (\ ), \quad F_1 = (0\ 1)(2\ 3).$$

The functions $f(x) = 0$ and $f(x) = 1$ are *constant*. The identity and ¬ functions have one true and one false value each, so they *balance* values of 0 and 1. The question posed by Deutsch is:

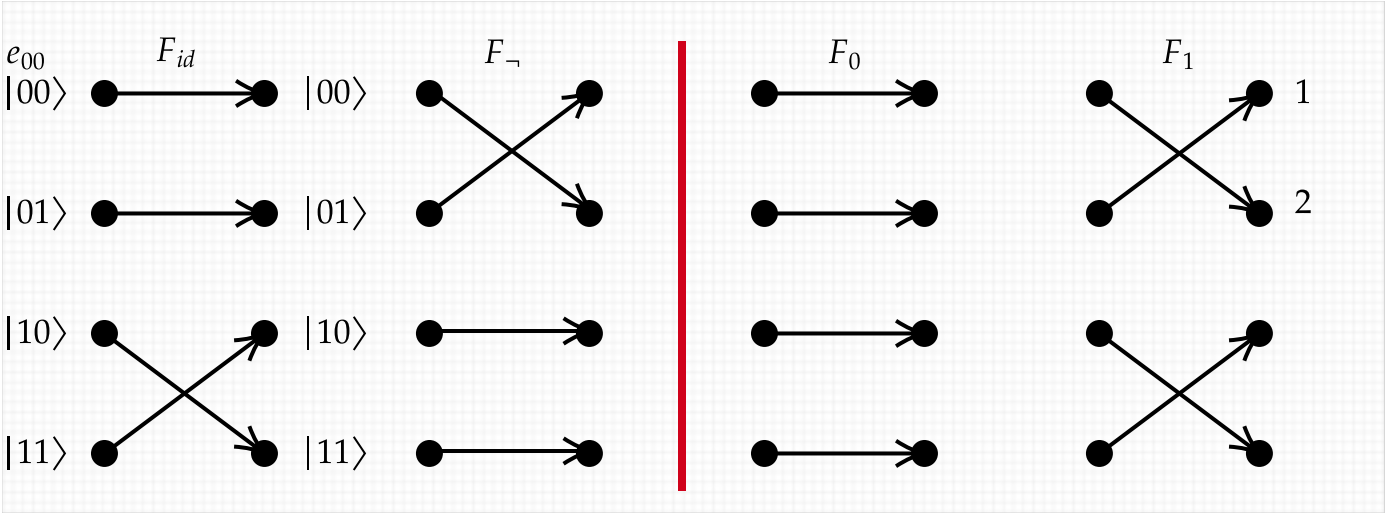
How many queries are needed to tell whether f is constant from whether f is balanced?

If we just think of f, suppose we try the query $y = 0$ and ask for $f(y)$. If we get the answer "$f(0) = 0$" then it f could be constant-false, but f could also be the balanced identity function. The answer $f(0) = 1$ would leave both constant-true and negation as possibilities. Likewise if we try $y = 1$. The first point is that this impossibility of hitting things with one query carries forward to the way we have to modify the problem for quantum:

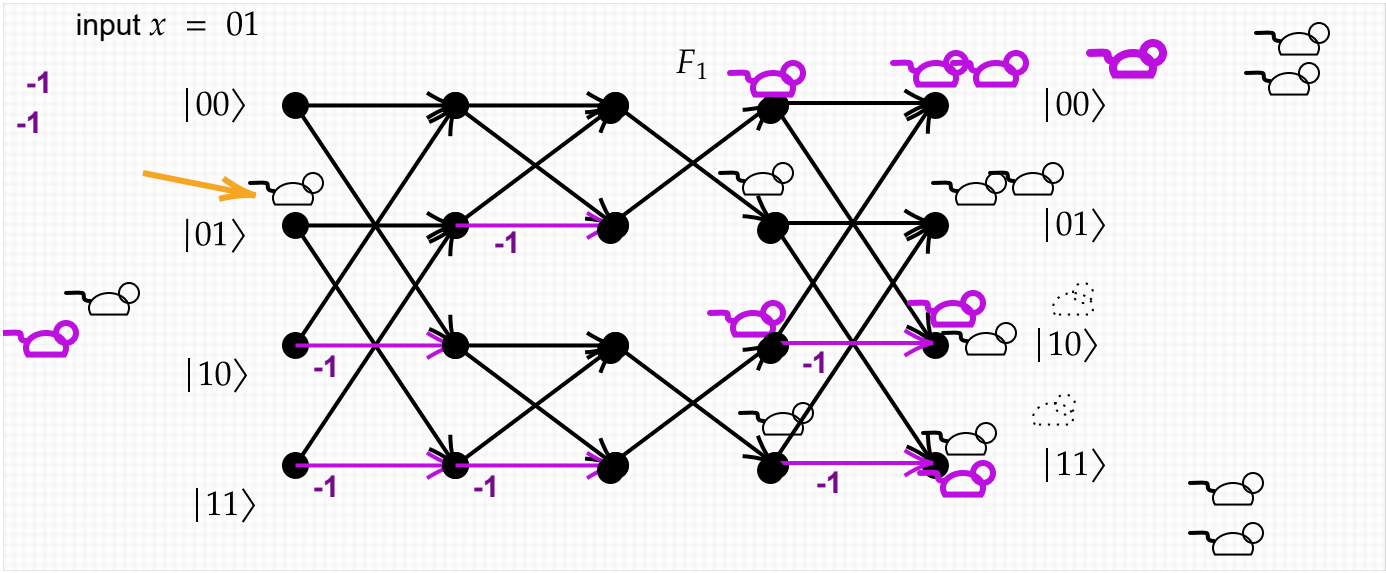
How many queries are needed to tell (F_{id} *or* $F_{\neg}$) apart from (F_0 *or* F_1)?



It seems like we have more of a chance because now we can query two things: 00, 01, 10, or 11. Or in the permutation view, we can query $y = 0, 1, 2$, or 3. The problem is that the range of answers we can get is too limited for this to help. $F(0)$ and $F(1)$ can only be 0 or 1; $F(2)$ and $F(3)$ can only be 2 or 3. So suppose you query $y = 2$ and get the answer 3. Then F could be F_{id} or F could be F_1. The basic problem for a classical algorithm is that every quadrant of the following diagram has both a straight and a cross:



A quantum circuit, however, can make one query to an oracle gate for any of these four functions, and can distinguish a member of the first pair from a member of the second pair by the answer to one qubit after a measurement. The input is not $|00\rangle$ but instead $|01\rangle$; that is, the ancilla is initialized to 1, not to 0. Here is the wavefront ("maze") diagram of how it works:



Interlude: Is the comparison fair?

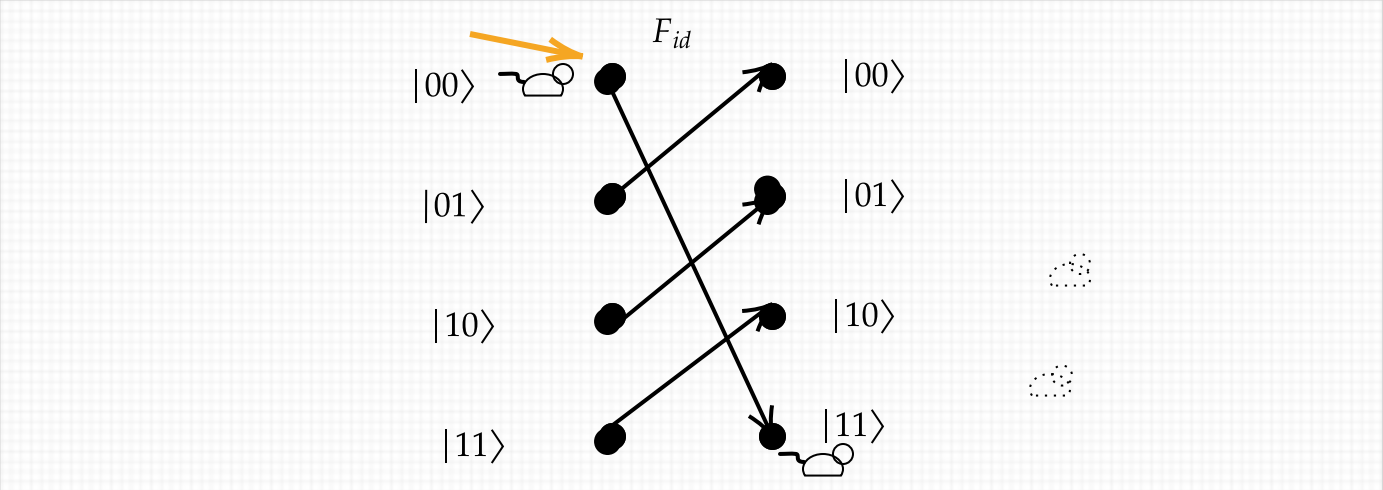
The unfair aspect (IMHO) is that the classical algorithm is being allowed to evaluate the oracle only at basis vectors. The quantum algorithm gets to evaluate it at a linear combination. We can represent this state using the Dirac notation from Chapter 14 as

$$|+-\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$

Well, suppose we allow evaluating ordinary Boolean functions at linear combinations of 0 and 1, such as **0.25**. We are really talking about the algebraic equivalents f' of these functions:

- If f is the identity function, $f(x) = x$, then $f'(x) = x$ too, but as algebra.
- If $f(x) = \neg x$, then $f'(x) = 1 - x$. Maybe the only non-obvious choice?
- If f is always false, i.e., $f(x) = 0$, then $f'(x) = 0$. i.e., f' is always zero too.
- If $f(x) = 1$, then $f'(x) = 1$ too.

If we evaluate the unknown f at 0.25, then we get four different answers that distinguish the four possibilities entirely, not just telling "balanced" apart from "constant." So the classical algorithm does even better---and still with just one "query."
FYI: https://rjlipton.wordpress.com/2011/10/26/quantum-chocolate-boxes/



Superdense Coding

It is easy to rig cases F_0, F_1, F_2, F_3 where you can distinguish them exactly by asking one query and measuring both qubits. Just define $F_i(00) = i$, for instance, where i ranges over $\{00, 01, 10, 11\}$---or if you prefer, i ranges over the permutation elements $1, 2, 3, 4$ as used above---and have the other values $F(01), F(10), F(11)$ go in cycle after that. See the above diagram for $F(11)$.

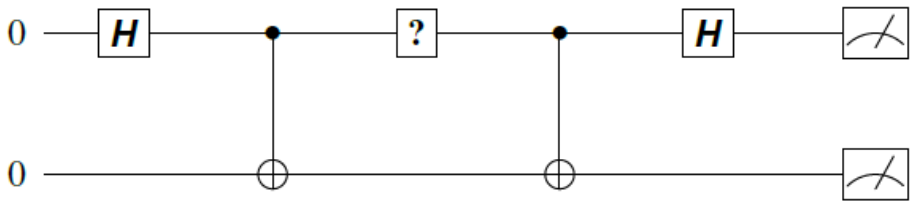
"Superdense coding" is a case where the rigging has a bit of surprise because it appears to convey 2 bits of information with just 1 qubit of communication *after* a certain point in time. This is impossible by the following theorem:

Holevo's Theorem: It is not possible to extract more than q bits of classical information from any q-qubit quantum state.

The most important case where this "bites" IMHO is with graph states: You can input $\sim \frac{1}{2}n^2$ bits of information by choosing the $\mathbf{CZ}$ gates for edges of an undirected n-vertex graph G in a graph-state circuit C_G on n qubits, one for each vertex. But you can only get n bits of information out by measuring. Hence graph-state encoding is majorly *lossy* and is often used only for special classes of graphs that

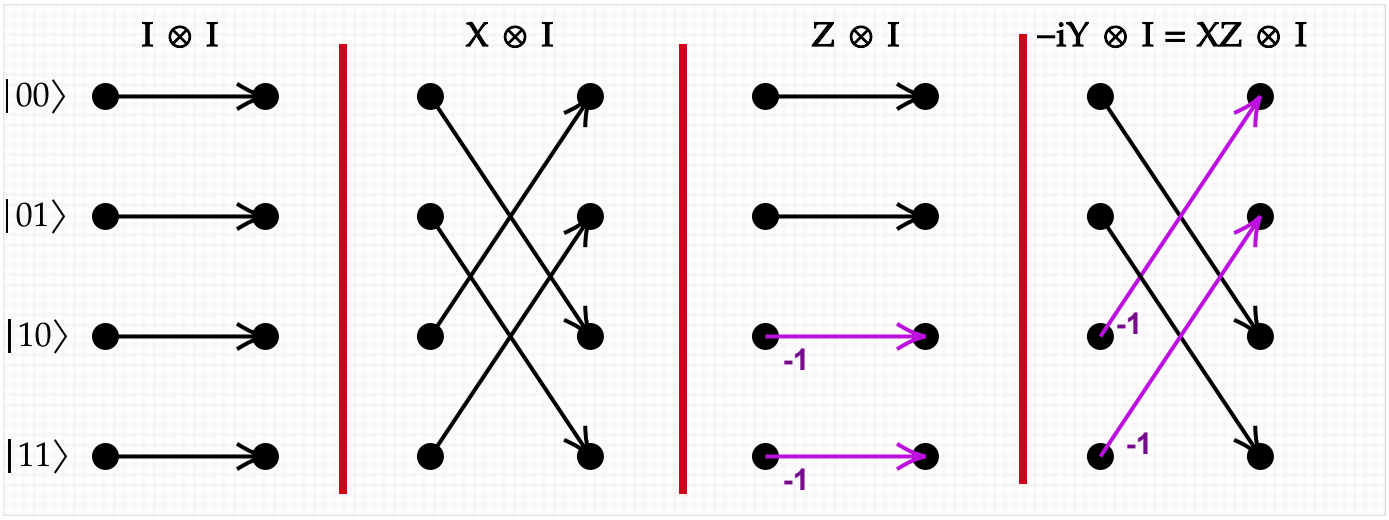
already have low information content, such as "grid graphs."

The "cheat" in superdense coding is that the communicating parties "Alice" and "Bob" exchange 1 bit of information beforehand in order to set them up with an entangled qubit pair. Here is their circuit:

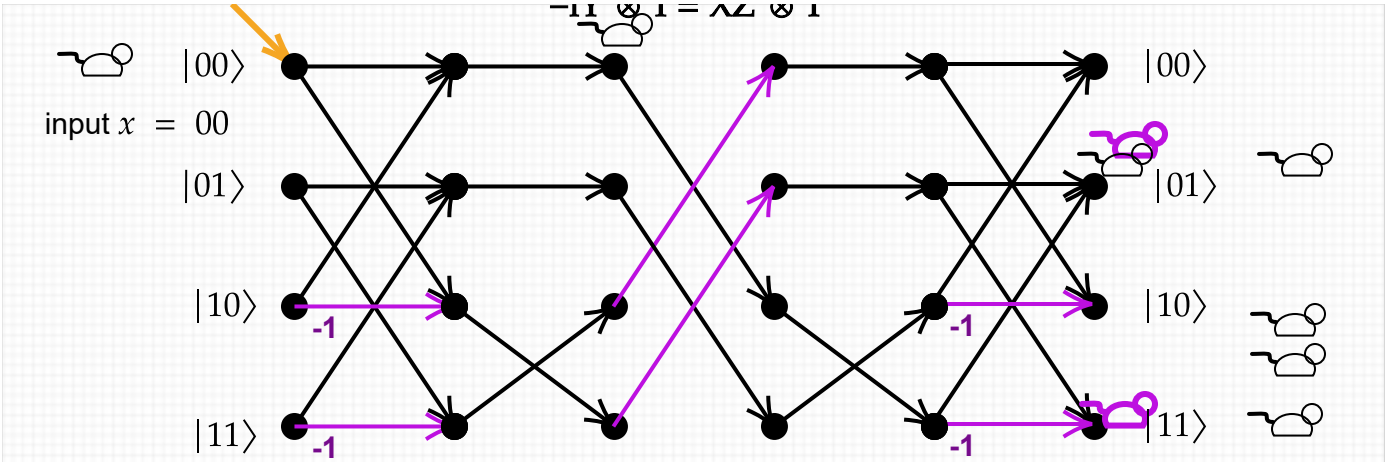


The opening Hadamard and CNOT set up the entangled pair. Alice then chooses one of the four Pauli operators for the unknown operation in the middle. After the second CNOT, she applies Hadamard to her qubit, measures, and sends the result to Bob. Bob then measures his qubit, and is able to infer which of the four operators Alice used. Well, he got a qubit from Alice to begin with, and even though it was before Alice made her 2-bit choice at the "?", it counts as 2 bits of "contact" anyway.

Even after the "magic" is explained away, this remains a nice illustration of a Deutsch-style learning problem using the four Pauli matrices. We want to identify one of the following four possibilities **exactly** by the results of **two** qubits.



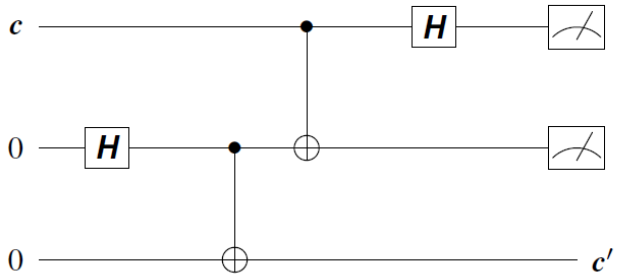
This time the input is $|00\rangle$. To work it out via wavefronts (the figure below is left with $XZ \otimes I$ in the middle, but all four will be exemplified):



Example: Quantum Teleportation

Quantum teleportation involves three qubits, two initially owned by Alice and one by Bob. Alice and Bob share entangled qubits as before, whereas Alice's other qubit is in an arbitrary (pure) state $\boldsymbol{c} = a\boldsymbol{e}_0 + b\boldsymbol{e}_1$. Alice has no knowledge of this state and hence cannot tell Bob how to prepare it, yet entirely by means of operations on her side of the lake she can ensure that Bob can possess a qubit in the identical state.

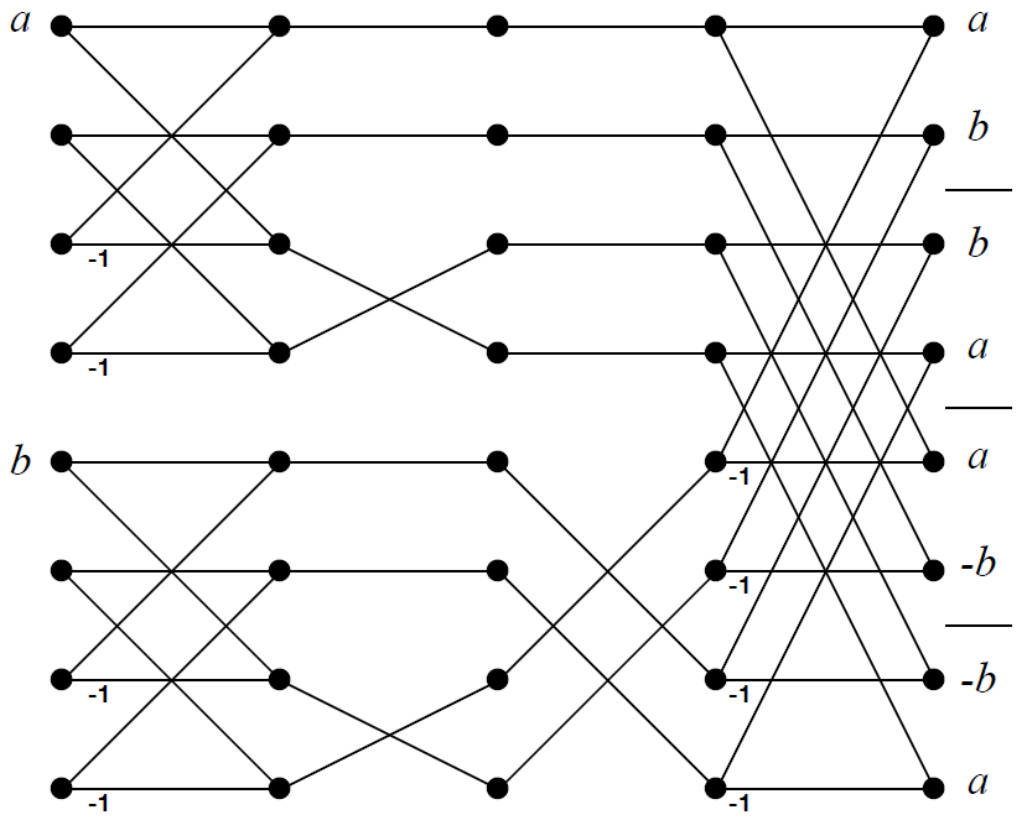
The following quantum circuit shows the operations, with $\boldsymbol{c}$ in the first quantum coordinate, Alice's entangled qubit second, and Bob's last. The circuit includes the Hadamard and ***CNOT*** gates used to entangle the latter two qubits.



With this indexing, the start state is $\boldsymbol{c} \otimes \boldsymbol{e}_{00}$, which equals $a\boldsymbol{e}_{000} + b\boldsymbol{e}_{100}$. After the first two gates, the state is

$$\boldsymbol{c} \otimes \frac{1}{\sqrt{2}} (\boldsymbol{e}_{00} + \boldsymbol{e}_{11}),$$

with Alice still in possession of the first coordinate of the entangled basis vectors. The point is that the rest of the circuit involves operations by Alice alone, including the measurements, all done on her side of the lake. This is different from using a two-qubit swap gate to switch the $\boldsymbol{c}$ part to Bob, which would cross the lake. No quantum interference is involved, so a maze diagram helps visualize the results even with "arbitrary-phase Phils" lined up at the first and fifth rows shown in figure 8.5, which are the entrances for $\boldsymbol{e}_{000}$ and $\boldsymbol{e}_{100}$.



Because Bob's qubit is the rightmost index, the measurement of Alice's two qubits selects one of the four pairs of values divided off by the bars at the right. Each pair superposes to yield the value of Bob's qubit *after* the two measurements "collapse" Alice's part of the system. The final step is that Alice sends two *classical* bits across the lake to tell Bob what results she got, that is, which quadrant was selected by nature. The rest is in some sense the inverse of Alice's step in the superdense coding: Bob uses the two bits to select one of the Pauli operations $\boldsymbol{I}, \boldsymbol{X}, \boldsymbol{Z}, i\boldsymbol{Y}$, respectively, and applies it to his qubit $\boldsymbol{c}'$ to restore it to Alice's original value $\boldsymbol{c}$.