CSE439, Fall 2025 **Problem Set 1** **Due Thu. 9/11, 11:59pm**

Reading: Next week's lectures will focus on the rest of Chapter 3 and also Chapter 4. I may also reference the larger Hadamard and Fourier matrices in sections 5.1–5.2 of Chapter 5 as examples, and also (while covering chapter 4), mention the fact in Section 5.3 that a Toffoli gate can simulate a NAND gate, so please peek ahead to them.

———————-*Assignment 1, due Thu. 9/11 "midnight stretchy" on CSE Autolab*———————-

(1) For each pair of growth rate functions $f(n)$ and $g(n)$, say whether (i) $f(n) = o(g(n))$, (ii) $f(n) = \Theta(g(n))$, or (iii) $g(n) = o(f(n))$. If you answer (ii), then further say whether $f(n) \sim g(n)$, meaning that the limit of $f(n)/g(n)$ exists and equals 1. [If you say "$f(n) = O(g(n))$" without clarifying whether (i) or (ii) applies, you get half credit. All logarithms are to base 2. $6 \times 2 + 6 = 18$ pts.]

(a) $f(n) = (999n + 777)^2$, $g(n) = (0.0001n - 888)^3$.

(b) $f(n) = \frac{(2n-1)^3}{(n+1)^2}$, $g(n) = 8n$.

(c) $f(n) = n^n$, $g(n) = n^{n-1}$.

(d) $f(n) = 2^{3n}$, $g(n) = (2^n)^3$.

(e) $f(n) = (2^n)^3$, $g(n) = 2^{n^3}$.

(f) $f(n) = \log(n^3)$, $g(n) = \log(n^{999})$.

(2) For each pair of vectors $\mathbf{a}$ and $\mathbf{b}$, give:

(i) the inner product $\langle \mathbf{a}, \mathbf{b} \rangle$,

(ii) the tensor product $\mathbf{a} \otimes \mathbf{b}$, and

(iii) the outer product $|\mathbf{a}\rangle \langle \mathbf{b}|$.

Note that the superscript ^{T} means that column vectors are given. Also say which of the tensor products is a unit vector. For part (d), recall the definition of the e_x, where x is a binary string, as a standard basis vector in N-dimensional Hilbert space, where $N = 2^n$ and n is the length $|x|$ of the string x. ($4 \times 3 \times 3 + 4 = 40$ pts.)

(a) $\mathbf{a} = [0.7, 0.1, 0.7, 0.1]^T$, $\mathbf{b} = [0.6, 0.8]^T$;

(b) $\mathbf{a} = [1, 0, -1]^T$, $\mathbf{b} = [1, 2, 3]^T$;

(c) $\mathbf{a} = [\frac{1-i}{2}, \frac{1+i}{2}]^T$, $\mathbf{b} = \frac{1}{2}[i, -i]^T$;

(d) $\mathbf{a} = \frac{1}{\sqrt{3}}(e_{00111} + e_{01001} + e_{10110})$, $\mathbf{b} = \frac{1}{\sqrt{2}}(e_{01000} + e_{01001})$.

(3) For each of the following vectors $\mathbf{c}$ in $\mathbb{C}^4$, say whether it is **separable** or **entangled**. In the former case, find vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^2$ such that $\mathbf{c} = \mathbf{a} \otimes \mathbf{b}$. In the latter case, prove that such $\mathbf{a}$ and $\mathbf{b}$ cannot exist. ($4 \times 6 = 24$ pts.)

(a) $[0.7, 0.1, 0.7, 0.1]$,

(b) $\frac{1}{2}[1, -1, 1, 1]$,

(c) $\frac{1}{2}[1 + i, 0, 0, 1 + i]$,

(d) $\frac{1}{\sqrt{50}}[1, 2, 3, 6]$,

(4) Challenge Question with Bonus: This question is worth 6 regular-credit points (making 88 total on the set) and up to 12 more extra-credit points.

Let us consider *nested vectors*—i.e., vectors whose "entries" may be other vectors. We actually did that with the baseball-pitch example in the Tue. 9/2 lecture, first forming the nested vector $((x, y, z), (a, b))$ before *flattening* it to make the simple length-5 vector (x, y, z, a, b). We can define the flattening $fl(v)$ of a vector v rigorously in general by saying to erase all internal parentheses, which leaves a single sequence of comma-separated elements.

Now we define the "generalized" tensor product $\mathbf{u} \otimes \mathbf{v}$ for nested vectors by applying the rule from lecture (say in its "big-endian" form) but when we have to multiply an internal entry that's a vector, we use our *tensor* product again, recursively. The base case is OK because the "tensor product" of two simple numbers c and d is the same as their ordinary product.

For example, to compute $((x, y, z), (a, b)) \otimes (c, (d, e))$, we follow the rule "multiply each entry of the first vector by a copy of the second" to make $((x, y, z) \otimes (c, (d, e)), (a, b) \otimes (c, (d, e)))$. Expanding the inner tensor products recursively then gives

$$((x \otimes (c, (d, e)), y \otimes (c, (d, e)), z \otimes (c, (d, e))), (a \otimes (c, (d, e)), b \otimes (c, (d, e)))).$$

Now we use the second part of the rule which says to multiply inside the copy of the second vector to get:

$$(x \otimes c, x \otimes (d, e), y \otimes c, y \otimes (d, e), z \otimes c, z \otimes (d, e), a \otimes c, a \otimes (d, e), b \otimes c, b \otimes (d, e)).$$

Doing this once more, and dropping the "$\otimes$" part between scalars leaves the final answer

$$(xc, xd, xe, yc, yd, ye, zc, zd, ze, ac, ad, ae, bc, bd, be).$$

Now, this happens to be the same as the ordinary tensor product $(x, y, z, a, b) \otimes (c, d, e)$ of the flattenings of these two vectors. Was that just the luck of this example, or is it a general rule? Put generally, the question is:

Is the generalized tensor product of two nested vectors $\mathbf{u}$ and $\mathbf{v}$ always equal to the ordinary tensor product $fl(\mathbf{u}) \otimes fl(\mathbf{v})$ of their flattenings?

If you say *yes*, you must *give a proof* to earn the 12 extra-credit points. If you say *no*, then a *concrete counterexample* is needed for those 12 points. Any reasonable effort to comprehend and start on the question, showing some scratchwork, brings the regular 6 pts. (Just giving a one-word answer will not bring any credit.)