CSE439, Fall 2025 Problem Set 7 Due Tue. 12/2, 11:59pm

Reading: For next week, read the mentioned materials on the Singular Value Decomposition: Section 2.8 on the SVD of Professor Knepley's notes and MIT Courseware npted on the SVD:
https://cse.buffalo.edu/ knepley/classes/cse439/ClassNotes.pdf
https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

The homework is roughly parallel to the first half of last year's Assignment 6, though actually its first problem was given before our Prelim II. I've made it due on Tue. 12/2 so that questions can be asked in lecture or office hours that day. Assignment 8 will have SVD problems a-la the second half, and will be due on Monday, 12/8.

——————-*Assignment 7, due Tue. 12/2 "midnight stretchy" on CSE Autolab*——————-

(1) Compute the unitary matrices $e^{i\pi\mathbf{X}}$ and $e^{i\frac{\pi}{2}\mathbf{X}}$. Note that $\pi\mathbf{X}$ is no longer unitary but remains Hermitian, and that the π multiplies the eigenvalues but not the eigenvectors. ($9 + 12 = 21$ pts.)

(2) Let Γ stand for the quantum state of the graph-state circuit of the three-node triangle graph G (on input $|000\rangle$ by default) *before* applying the second Hadamard transform. In the notation of the "quantum assembly language" of the C++ simulator demo'ed for Shor's Algorithm in class, the state results from $|000\rangle$ by `H 1 H 2 H 3 CZ 1 2 CZ 2 3 CZ 1 3`. We skip the final Hadamard transform to minimize pencil-pushing and to use our intuition from the "maze diagrams" of the signs of the terms $|000\rangle$ through $|111\rangle$ of the resulting state. (If you did simply `E 1 2 E 2 3 E 1 3` using our **E** gates, you would have to follow with `H 1 H 2 H 3` in order to *undo* the final Hadamard transform. It doesn't matter structurally so much because single-qubit gates can never change entanglements.)

(a) Show the 8×8 density matrix $\rho_\Gamma = |\Gamma\rangle\langle\Gamma|$. (It's quicker and nicer than you expect if you've grokked the ideas of the maze diagrams. 6 pts.) **Then we give an option to cheat to make a more-elegant problem: Interchange the middle two entries of Γ so that its self-outer-product ρ has four quadrants of the same sign.** (This would result legitimately from applying a 3-qubit gate "$(-1)^{x=\bar{y}\wedge y=z}$," which flips the sign of $xyz = 011$ and $xyz = 100$ but leaves the other strings alone.) **Doing both options is up to 15 pts. extra credit.**

(b) Label the nodes A, B, C (it doesn't matter which node is which letter since the graph is symmetrical, but let C correspond to the third qubit, which is the low-end bit in our big-endian notation). Then compute the 4×4 density matrix ρ' that results from tracing out C. **If you took the elegant option, show that it represents the pure, unentangled state $|-+\rangle$. Else, say why it doesn't—and isn't even a pure state at all.** (Either way, note that this is different from the state in problem 4(a) of Prelim II for the one-edge graph left over when you delete vertex C and its edges. 9 pts.)

For the continuation of this problem, we refer to the following six 4×4 matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two represent measuring just the first qubit in the standard basis. The latter four represent measuring both qubits in the **Bell basis**, respectively $\Phi^+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $\Phi^- = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $\Psi^+ = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, and $\Psi^- = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Note that both the first two matrices and the last four matrices sum to the identity and that each stays itself when squared, so the first two matrices satisfy the formal definition of a measurement and so do the last four.

(c) Calling the first matrix M_0, calculate the trace of $M_0\rho' M_0$ to give the probability of getting $|0\rangle$ for the first qubit. Also show what the resulting state is, if that outcome happens. **Under the "cheat" option, show the resulting state is $|0\rangle \otimes |+\rangle$; under the correct Γ, show that you get $|0\rangle \otimes$ the completely-mixed state instead. Parts (d)–(g) remain valid as worded.** (6+3 = 9 pts.)

(d) Using the third matrix, show that if ρ' (that is, the pure state $|-+\rangle$) is measured in the Bell basis, then outcome Φ^+ *cannot happen*. (3 pts.)

(e) What about if you multiplied (on both sides) by the all-1s matrix J (divided by 4) instead? Note that $J = |++\rangle\langle ++|$ and hence represents the projector for the outcome $|++\rangle$ in the Hadamard-transformed basis we've seen for several problems (which should not be confused with the Bell basis). (3 pts.)

(f) Using the fourth matrix, calculate the probability of outcome Φ^- when measuring ρ' in the Bell basis. (6 pts.)

(g) *Finally*, we add one more vertex labeled D to the graph G, connecting it to B and C, but not to A, to make a new graph G''. Write down the sixteen signs of the terms $|0000\rangle$ through $|1111\rangle$ in the corresponding graph state Γ'' (again, *before* the final Hadamard transform). You should have 8 + and 8 − signs because G'' is also "net-zero." Our $64,000 question is whether tracing out D leaves the triangle graph G and its state Γ back again. You should be able to answer this without the torture of writing out the whole 16×16 density matrix $|\Gamma''\rangle\langle\Gamma''|$ and doing the whole traceout—just do enough of the lower-left (or upper right) corner to tell. (6 + 6 = 48 pts. on the whole problem and 69 on the set)

If you're curious, you can do the whole 8×8 density matrix ρ'' that results from tracing out node D, and try to tell: is ρ'' a mixed or pure state, and either way, is ρ'' entangled? The lecture on Thursday Dec. 4 will continue this example while showing how to use the SVD to find separable states that most closely approximate entangled ones, among other applications. If you're even more curious, look at my "GLL" blog article https://rjlipton.com/2022/01/05/quantum-graph-theory/, which shows how the frontier of real research—including some by Professor Chunming Qiao and others at UB—is not too far beyond these examples and topics.