CSE 491/596 Lecture Fri 9/9 Fall 2022 ①

Definition of the concatenation of two languages A, B:

$$A \cdot B = \{x \cdot y : x \in A \ \&\ y \in B\}$$

For a simple example, if $A = \{$"1", "2", ..., "9", "10", "11", "12"$\}$ and $B = \{$"am", "pm"$\}$, then $A \cdot B$ gives "1am" thru "12pm".

Now an example that tests one's understanding:

Does $A \cdot A = \underbrace{\{x \cdot x : x \in A\}}$? NO

Try $A = \{0, 01, 10\}$. This $= \{0 \cdot 0, 01 \cdot 01, 10 \cdot 10\}$

$A \cdot A$ really $=$ ~~$\{0 \cdot 0, 0 \cdot 01, 0 \cdot 10, 01 \cdot 0, 01 \cdot 01, 01 \cdot 10,$~~

$= \{00, 001, 010, 0101, 0110, 100, 1001, 1010\}$ $|A \cdot A| = 8$

Whereas the Cartesian Product $A \times A$ has 9 member pairs.

Powering: $A \cdot A \cdot A = (A \cdot A) \cdot A = \{x \cdot y \cdot z : x, y, z \in A\} = A \cdot (A \cdot A)$

Concatenation is associative, so we can simply write A^3 for this.

$A^3 = A \cdot A \cdot A$, $A^2 = A \cdot A$, $A^1 = A$. What does A^0 equal? $\{\varepsilon\}$

Fact: For any language B, $B \cdot \{\varepsilon\} = \{\varepsilon\} \cdot B = B$.

Second Rule: For any B and integers $i, j \geq 0$.

$$A^i \cdot A^j = A^{i+j} \qquad \text{Powering Rule.}$$

What if $j=0$? The RHS is A^i. So A^j needs to $= \{\epsilon\}$.

Whereas, $A \cdot \emptyset = \{xy : x \in A \ \&\ \underbrace{y \in \emptyset}_{\text{always false!}}\} = \emptyset$.

Definition: $A^* = A^0 \cup A^1 \cup A^2 \cup A^3 \cup \cdots$

$$= \bigcup_{i=0}^{\infty} A^i \qquad \text{Kleene star}$$

read "zero or more"

(Stephen CLAY-nee)

= {all strings formed by concatenating zero or more strings in A}

Example Q: Does $\{0, 01, 10\}^*$ include all (or most) strings? No.

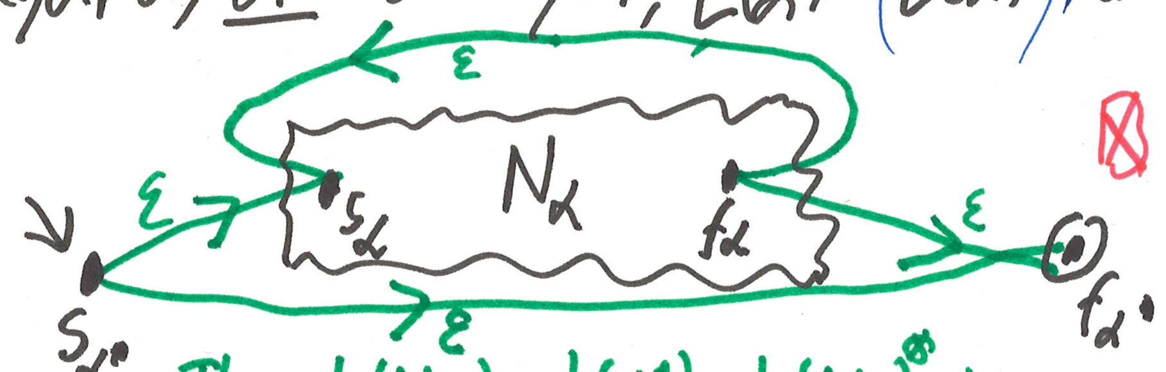
$A^+ = A^1 \cup A^2 \cup A^3 \cdots$ (excludes ϵ if $\epsilon \notin A$.)

read "one or more"

$$\boxed{A^+ = A \cdot A^*}$$

Finishing the defn of regular expressions and equivalent NFA$_\epsilon$ s:

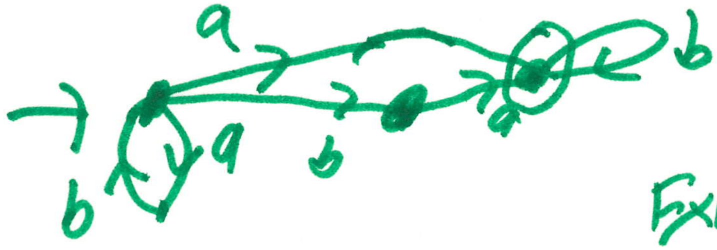
(I3): For any regexp α, $\underline{\alpha^*}$ is a regexp, $L(\alpha^*) = (L(\alpha))^*$, and given an NFA N_α such that $L(N_\alpha) = L(\alpha)$, build N_{α^*} =



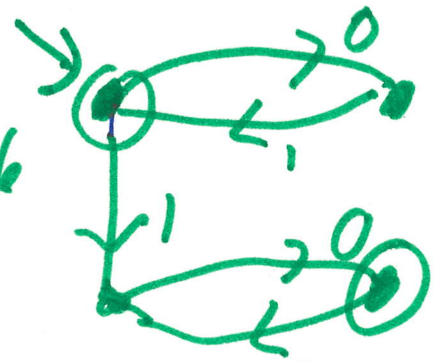
Then $L(N_{\alpha^*}) = L(\alpha^*) = L(N_\alpha)^*$ because...

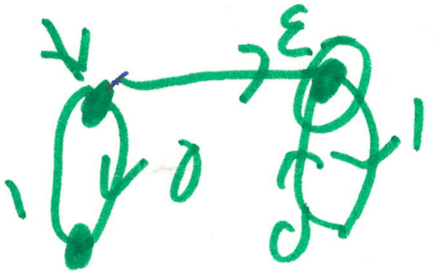
Example of regexp-to-NFA: You need not follow this construction literally.

$\beta = (ab)^* \cdot (a+ba) \cdot b^*$



Example N





$\equiv$ $L(N) = (01)^*(10)^*$

[lecture ended here]

<u>Extra Material</u>: The formal definition of a relation $R \subseteq A \times B$ being a <u>function</u> is:

$(\forall a \in A)(\exists! b \in B)\ (a,b) \in R \equiv (\forall a \in A)(\exists b \in B)\underline{(\forall c \in B)}\ (a,b) \in R \wedge [(a,c) \in R \rightarrow c = b]$

"there exists a unique b" $\equiv (\forall a \in A)\underline{(\forall c \in B)}(\exists b \in B)\ (a,b) \in R \wedge [(a,c) \in R \rightarrow c = b]$.

Now for a question: If $A = \emptyset$ and $B = \emptyset$, then is $R = \emptyset$ a function from A to B?

<u>Yes</u>—because of the rule that a $\forall$ quantifier over an empty domain defaults to <u>true</u>.

Now B^A stands for the set of functions <u>from</u> A <u>to</u> B. E.g. $\{0,1\}^{\{1,2,3,4,5\}}$ is the set of binary functions on 5 elements. There are $2^5 = 32$ such functions, same as the number of binary strings of length 5. Generally $|B^A| = |B|^{|A|}$. <u>So</u>: $|\emptyset^\emptyset| = 1 = |\emptyset|^{|\emptyset|} = 0^0 = 1$.

$Q_{\alpha^*} = \{Q_\alpha \cup \{s_{\alpha^*}, f_{\alpha^*}\}$ The construction in formal math:

(This was written after lecture and will be continued on Piazza:)

$\delta_{\alpha^*} = \delta_\alpha \cup \{(s_{\alpha^*}, \varepsilon, s_\alpha), (f_\alpha, \varepsilon, s_\alpha), (f_\alpha, \varepsilon, f_{\alpha^*}), (s_{\alpha^*}, \varepsilon, f_{\alpha^*})\}$