CSE491/596 Lecture 9/22/23: Myhill-Nerode Theorem, continued

We pick up with more examples of the "proof script."

Example 3. $L = \left\{x \in \{a,b\}^* : x^R = x\right\}$, where x^R means x reversed, e.g., $abbab^R = babba$. [What is ϵ^R?] That is, L is the language of strings that are **palindromes** and has the standard name PAL.

Take $S =$ __ __. Clearly S is infinite.

Let any $x, y \in S$ $(x \neq y)$ be given. Then we can write $x =$ __ __ and $y =$ __ __ where ________ and m.

Take $z =$ __ __.

Then $L(xz) \neq L(yz)$ because ___ $\in$ PAL but which is not in PAL because $m \neq n$ and the single b prevents any other possible way of "parsing" yz as a palindrome___.

Because x, y are an arbitrary pair of strings in S, this shows that S is PD for L, and since S is infinite, it follows that L is nonregular by the Myhill-Nerode Theorem.

We did not need the "wlog." provision this time---but you can always take it even if you don't need it. We also could have started with $S = a^*$ and made the b the first char in z. Why did I put the b "up front" in S? It is to emphasize its importance and help avoid a common mistake of forgetting it altogether. The mistake (in this case---it pops up in others too) is to think that $a^m \cdot a^n$ is not a palindrome whenever $m \neq n$. That may be true with your breakdown but there could be others. E.g. $a^3a^5 = a^4a^4$ which is now clearly a palindrome. Indeed, a^{m+n} is *always* in PAL.

Example 4. $L = \left\{x \in \{(,)\}^* : x \textit{ is balanced}\right\}$. What does "balanced" mean? [Discuss if time allows, but this will be more important when we cover **pushdown automata** as a special case of Turing machines.] This language is often called BAL. It is in fact "isomorphic to" the language of the "unlimited spears" and dragons game *when you win only if you leave the dungeon with zero spears*. E.g., if you are holding 5 spears, you need 5 "closing dragons" to balance out. With this understood, we can **re-use** the proof of Example 2.

The Full MNT

We have proved only one direction. The whole theorem says:

Theorem: A language L is regular $\iff$ all PD sets for L are finite.

We've proved that if L has an infinite PD set, then L is not regular. This is the $\Longrightarrow$ direction, though it may sound like the reverse. It is the contrapositive of the $\Longrightarrow$ direction. To complete the equivalence, we need to prove the $\Longleftarrow$ direction.

Proof: All PD sets for L are finite is the same as saying the equivalence relation $\sim_L$ has only finitely many equivalence classes. Take Q to be the set of equivalence classes. For any string $x \in \Sigma^*$ (where Σ is understood to be the alphabet that L is "over"), there is exactly one equivalence class R_x to which it belongs. Note that R_ϵ is an equivalence class, thus a member of Q, and it will serve as the start state s of the DFA M we are building. Next define

$$F = \{R_x : x \in L\}.$$

Note that even though L may be infinite, F can be finite because R_x and R_y can coincide---indeed, will coincide whenever $x \sim_L y$. Indeed, F must be finite, because F is a subset of Q which is finite by the premise of $\Longleftarrow$. Finally, we define δ by the rule

$$\delta(R_x, c) = R_{xc} .$$

For this to be "well defined" we need to show that it depends only on the equivalence class, not on any x that happens to represent it. So suppose $y \sim_L x$, i.e., that y also belongs to R_x, so that $R_y = R_x$. We need to show that $\delta(R_y, c) = R_{xc}$ too. This follows if R_{yc} is the same as R_{xc}. And justifying this is left as a study guide. Then $M = (Q, \Sigma, \delta, s, F)$ is a legal DFA. And $L(M) = L$ because M hits its accepting states exactly on the strings x that belong to L. Thus L is regular. ☒

Corollary: In the $\Leftarrow$ direction of MNT, the DFA you get not only has the least possible number of states, it is unique. Hence, every regular language has a *unique minimum-size DFA*. ☒

Putting a checkbox in the corollary statement signifies that we've already essentially proved it. The notes by Debray prove instead that every DFA can be reduced to a unique minimum one via the *DFA minimization algorithm*. The algorithm is interesting for its own sake---it is IMHO the easiest example of "dynamic programming"---but for us it is just a "skim". The reasoning of both halves of MNT helps us recognize minimum-size DFA cases, even extreme ones.

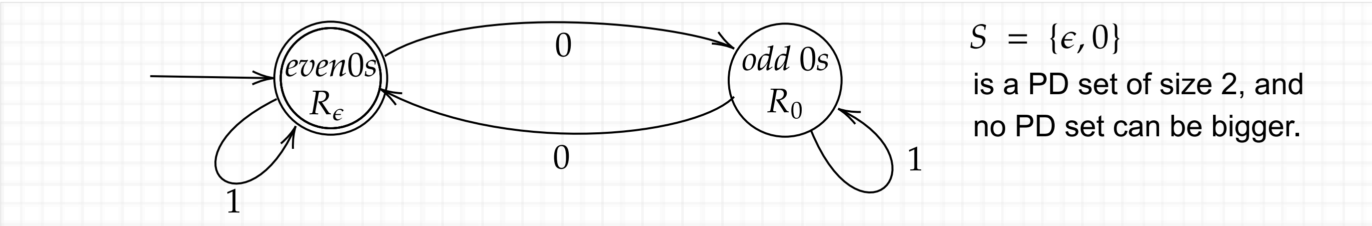
Proposition: For each $k \geq 1$, the unique minimum DFA for $L_k = (0+1)^* 1 (0+1)^{k-1}$ has 2^k states.

Proof: Take $S = \{0,1\}^k$. Then S has size 2^k. We claim that S is PD for L_k: Let any $x, y \in S$, $x \neq y$, be given. By $x \neq y$, there is some position i (let's number from 1) in which they differ. Take $z = 0^{i-1}$. Then xz and yz differ in position k from the end, so $L_k(xz) \neq L_k(yz)$. This proves the claim, so the consequence is that any DFA M_k such that $L(M_k) = L_k$ needs at least 2^k states. Well, we can build a correct M_k of that size by having one state q_w for each possible combination w of last k bits read (treating an initial small string like 10 as if it had $k-2$ leading 0s) and defining $\delta(q_{bv}, c) = q_{vc}$. The final states are q_w for those w that begin with 1: since $|w| = k$, this 1 is in the kth position from the right. So M_k is the unique minimum DFA for L_k. ☒

Note that the NFA N_k from an earlier lecture only needs $k+1$ states. Thus this also demonstrates cases where the NFA-to-DFA construction has an *unavoidable* "exponential explosion." Furthermore, the regular expression for L_k in the proposition statement (call it r_k) needs only $12 + \log_2(k)$ symbols, the log part for the bits in the number $k-1$. This is an exponential step **down** in size. The upshot is that NFAs can sometimes be exponentially more **succinct** than DFAs, and regular expressions (with numerical powering) can be even more succinct in some cases.

Using MNT For Design Hints (as time allows)

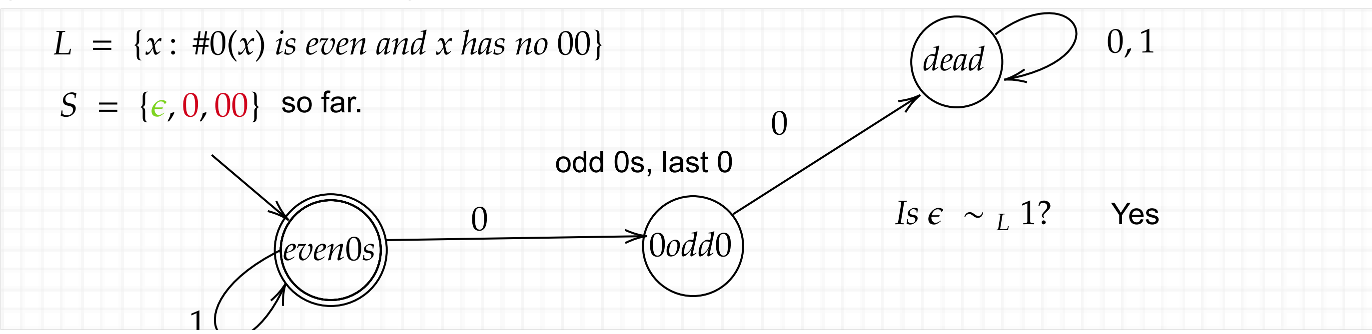
We can use this $\Longleftarrow$ direction to help us understand regular languages and build DFAs for them. Let's revisit the example $L = \{x \in \Sigma^* : \#0(x) \text{ is even}\}$. Then $x \sim_L y$ iff the numbers of 0s in x and y are both even or both odd. Hence the relation $\sim_L$ has just two equivalence classes. Here is the DFA:



Now let's try a trickier example by conjoining "even 0s" with another condition of not having 00 as a substring:

$$L = \{x : \#0(x) \text{ is even and } x \text{ has no } 00\} \tag{1}$$

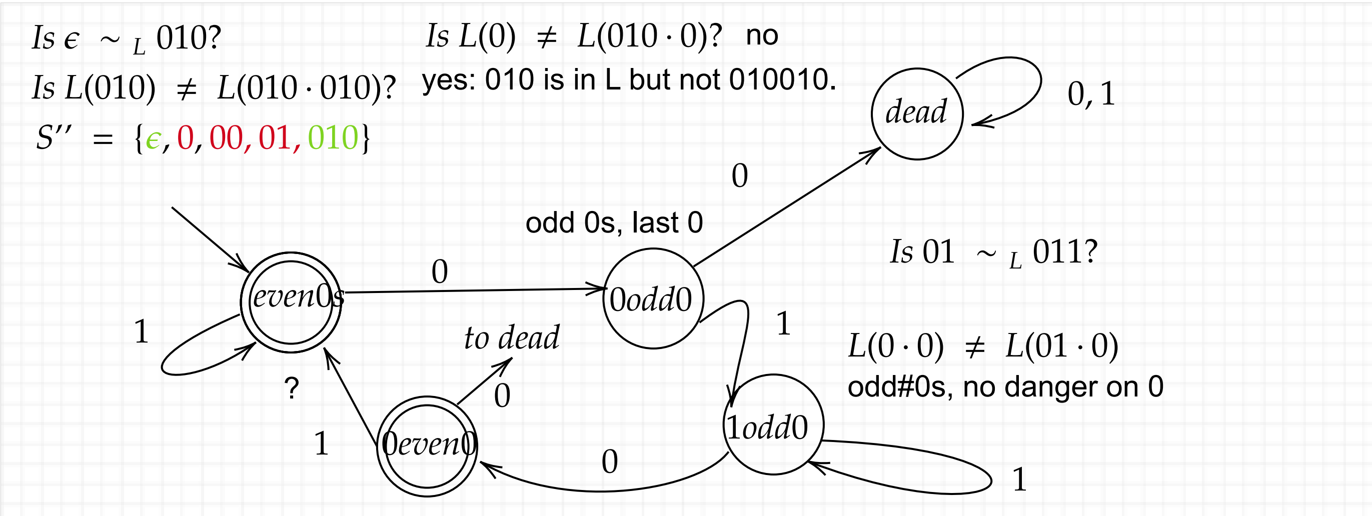
[In regular expression terms, L equals $\left(1^*01^*0\right)^* 1^* \setminus (0+1)^*00(0+1)^*$ but set-minus $\setminus$ is not a native regular operator so that doesn't help us even build an NFA, let alone a DFA, to accept L. So let's ignore this attempt and try using (1) to build a DFA M by "MNT-enlightened trial and error."] We know that $\epsilon \in L$, so the start state will be accepting, and that 0 and 00 are both not in L. Indeed, 00 causes a "dead condition" because no string beginning with 00 can possibly belong to L, so it should go to a dead state. That gives us part of the machine:



How about the string 1? It can still be a loop at the start state. At the left end of a string it makes no difference to having a possible 00, so $1 \in R_\epsilon$. But what about the loop on 1 which we had at the "odd" state? Can we still direct it back to that state? It is equivalent to ask whether $0 \sim_L 01$. To see why not, consider $x = 0$ and $y = 01$. Take $z = 0$. Then $xz = 00$ is not in L but $yz = 010$ is indeed in L, because the 1 helped us avoid a 00. For the same reason, $01 \nsim_L 00$, and clearly $01 \nsim \epsilon$ because ϵ is in L and 01 is not (technically, they are distinguished by $z = \epsilon$). Thus $S' = \{\epsilon, 0, 00, 01\}$ is a PD set of size 4, and so we need a fourth state to process it to. Now, what about that string 010? It is in L, but is it in R_ϵ?

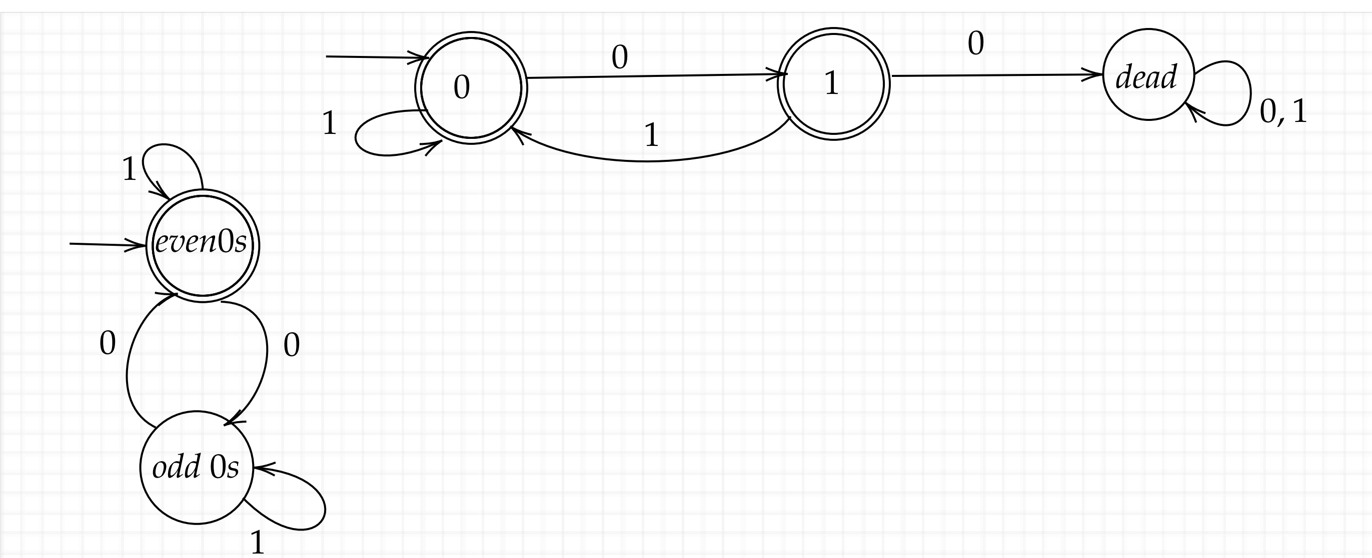
It is not, but finding a string z such that $L(\epsilon \cdot z) \neq L(010 \cdot z)$ is not so fast. We need to activate the "no 00" condition by making z begin with 0, but then we need another 0-

--but not right away---to make $z \in L$. Thus $z = 010$ is the shortest distinguishing string. This gives us:



So we wound up needing 5 states. Is that enough? Well, can we complete the machine with arcs from the "even 0s, last char 0" state? Clearly 0 goes to $dead$, and 1 must go to an accepting state. If 1 can go to the start state, then we're done. Can it? Yes---by similar reasoning to putting a loop on 1 at the start state. So M is done and S'' is a largest possible PD set.

There is another kind of reasoning we could have done. L is the $\cap$ of two languages represented by the 2-state DFA above and the following simple 3-state DFA for the "no substring 00" condition:



Doing the Cartesian Product construction seems to suggest the final DFA will have $2 \times 3 = 6$ states. But the operation is intersection, so the "dead" condition in the upper DFA knocks-on to make the whole third column dead in the product machine. Since you don't need two separate dead states, the number goes down to 5 after all. It is a good exercise to carry out the construction and verify that you get the same 5-state DFA as above.

[On tap Wednesday: Turing Machines.]