CSE491/596 Lecture Wed. 11/4: NP Completeness By Component Design III

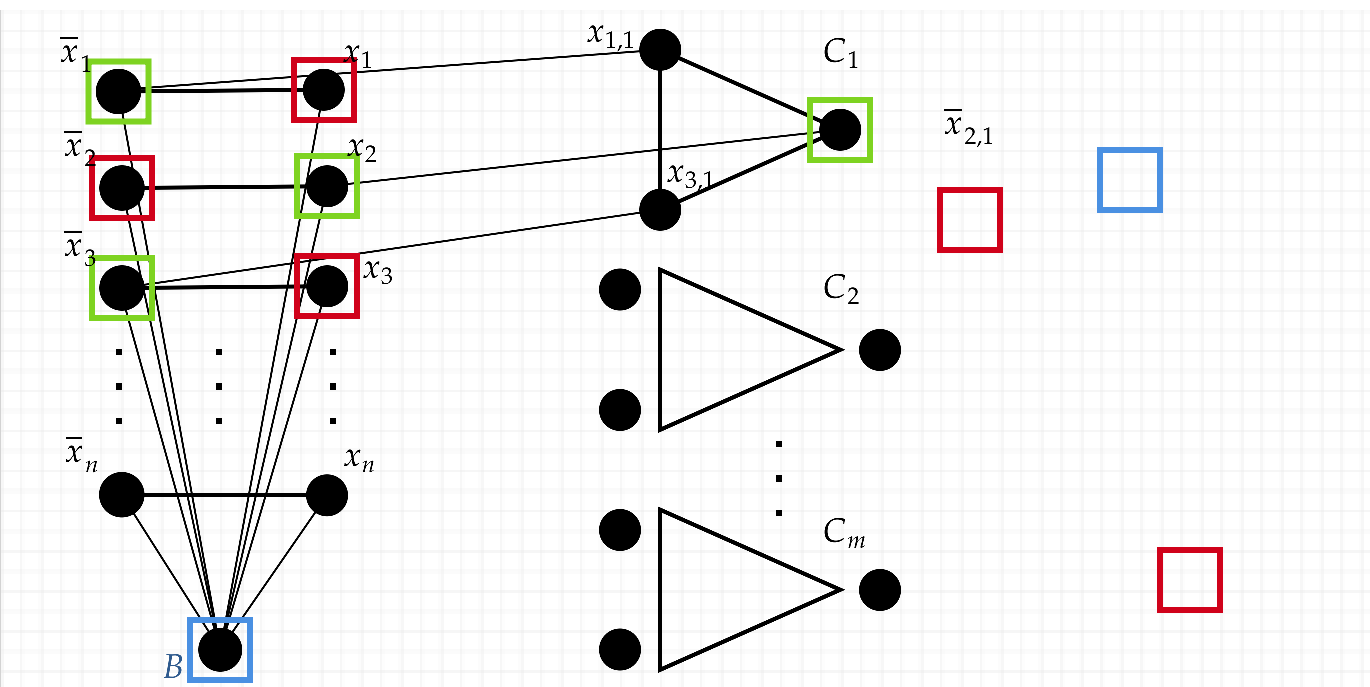
Now we consider the Graph 3-Coloring Problem.

G3C

Instance: Just an undirected graph $G = (V, E)$ (no "k").
Question: Is there a map $\chi : V \to \{R, G, B\}$ such that for all $(u, v) \in E$, $\chi(u) \neq \chi(v)$?

The Greek chi is for "chromo-" meaning "color". The language of 3-colorable graphs is clearly in NP: we just guess the coloring, which is a string in $\{R, G, B\}^n$, and verify the coloring on each of $m \leq \binom{n}{2} = O(n^2)$ edges. To show it is NP-complete, we use the same basic rungs-and-gadgets layout, but with one or two twists.

The first thing to think about is how to establish a correspondence between colorings and truth assignments to begin with, before thinking about "good" colorings (i.e., those that meet the "such that" property of having no monochrome edges) *vis-à-vis* satisfying assignments. The natural idea is to give each rung an edge so that each x_i and $\bar{x}_i$ pair must be given different colors so that one color stands for true and the other for false. Well, we have to limit that to two colors for each rung, so we do so by connecting *all* $2n$ rung nodes to a special node called B for the intent to color it blue. So on the ladder side, we have [2023 Note: As in the previous lecture, I did more on early diagrams rather than jump to as new diagram each time I made some changes.]:

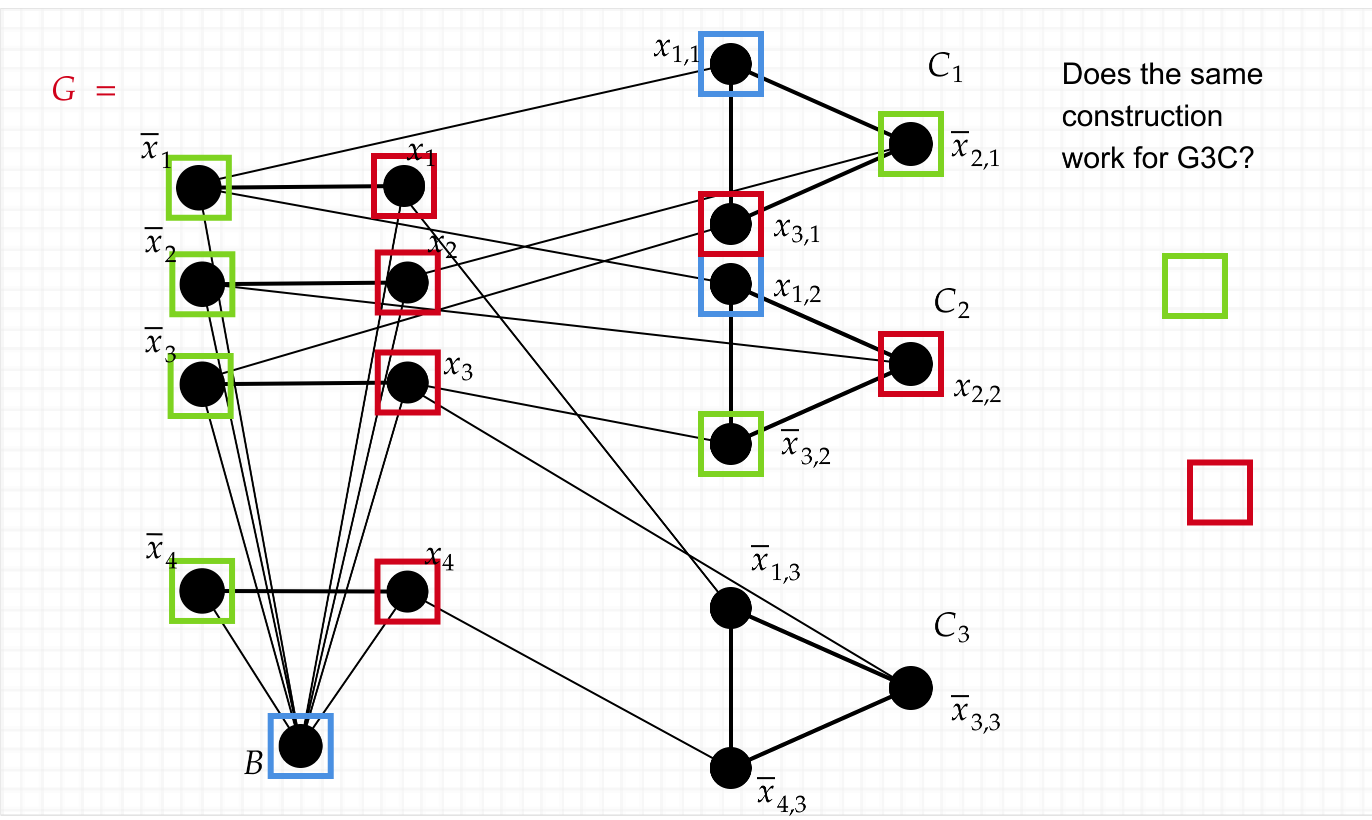


$$\phi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$

This forces each rung to use one R and one G. Now incidentally, $\chi(B) = B$ is not something the reduction is able to define---it is not part of G. But any good coloring remains good under any of the 6 permutations of the colors, so it is "wlog." that we presume $\chi(B) = B$. This leaves R and G for the rung nodes. It is natural to have G stand for the literals that are made true, R for false, but this is where we have to be careful. The permutation that swaps R and G while keeping B fixed stays good, but if flipping an assignment a like 1010 to 0101 satisfies ϕ one way but not the other, there could be a mismatch on correctness requirements.

Let us go ahead. The next question is, can we re-use the clauses-as-triangles idea? With the same crossing edges? Let's try it for the same example formula:

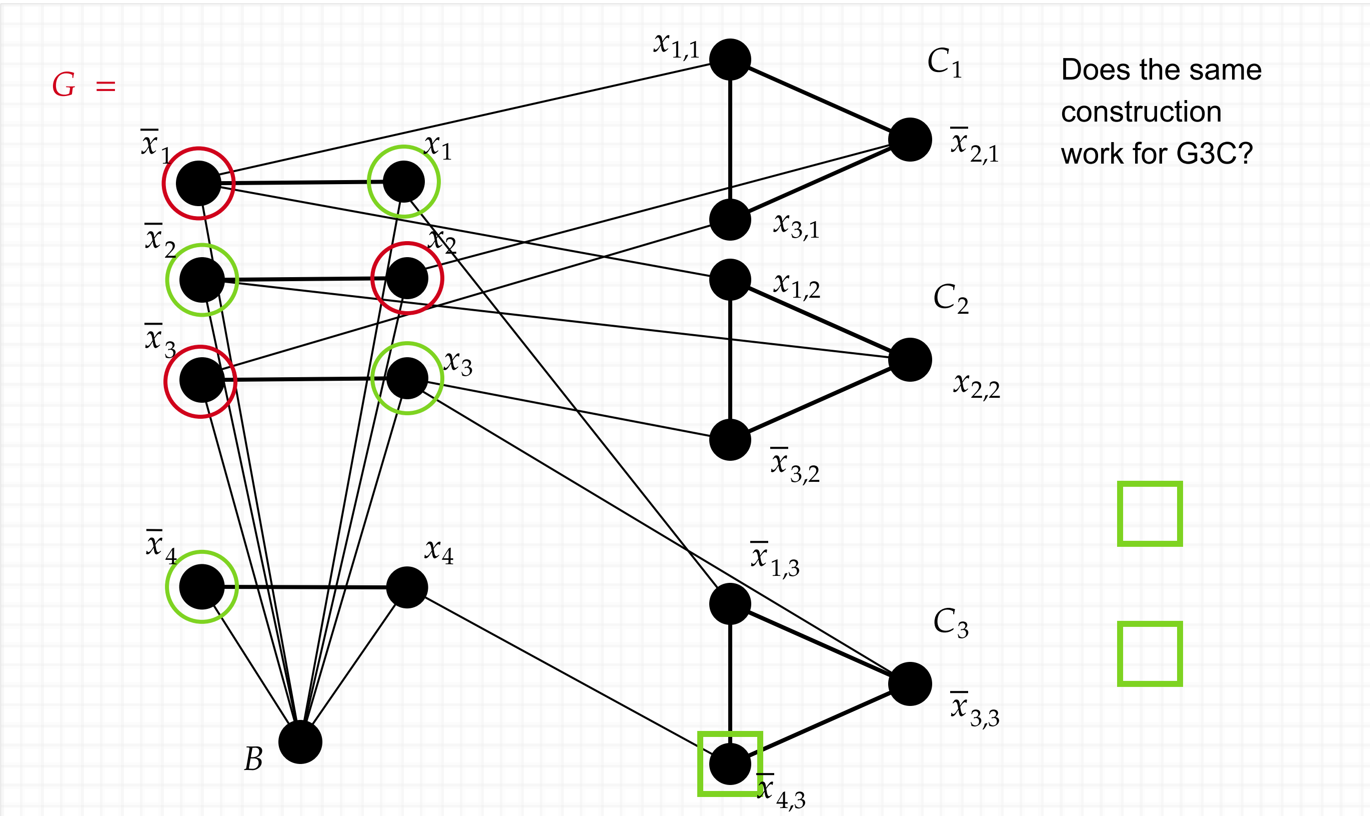
$$\phi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$



Here's the deal: If we have a 3-coloring χ, it has to use G once in each clause triangle and once in each rung. If x_{ij} is green in clause C_j then its crossing edge goes to $\bar{x}_i$ in rung i. This had to be red, so x_i in the rung is green. This means x_i was set true, so C_j is satisfied. The reasoning for a negative literal $\bar{x}_{ij}$ being green in C_j is symmetrical: the crossing edge goes to x_i in the rung, which must be red, so x_i is set false, so $\bar{x}_i$ satisfies C_j. Therefore we get the ($\Longleftarrow$) direction that G being 3-colorable implies ϕ is satisfiable.

The $\Longrightarrow$ direction hits a possible snag, however: Suppose ϕ is satisfiable, but only by assignments that make all three literals in some clause true. It's not just that we can't color all three nodes in the clause green, it's that their crossing edges go to red nodes in the rungs. Suppose this happens for clause C_1 in our example:

$$\phi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$



Now the clause C_1 is "redlocked": we can't color any of its nodes red, so we cannot color it. Note, however, that when an assignment fails to satisfy a clause, the resulting "greenlock" is exactly what we want for correctness in the $\Longrightarrow$ direction. This is what happens to C_3 if we set x_1, x_3, and x_4 all true. So we cannot fix the "redlock" issue without damaging the "greenlock" feature.

Unless, that is, we can invoke an extra condition that "redlock" never happens: that no assignment can satisfy all three literals in a clause. This is a condition that the Cook-Levin reduction, together with the idea of inserting an always-false variable z, allow us to invoke. Then the $\Longrightarrow$ direction goes through: In every C_j, take one node that is satisfied and the other not satisfied. The crossing edges make it good to color the former green and the latter red. The blue color B can then be used for the third node in the clause. We can rigorize this by stating a **variant** of 3SAT:

NAE-3SAT

Instance: A Boolean formula $\phi(x_1, \ldots, x_n) = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ in 3CNF.

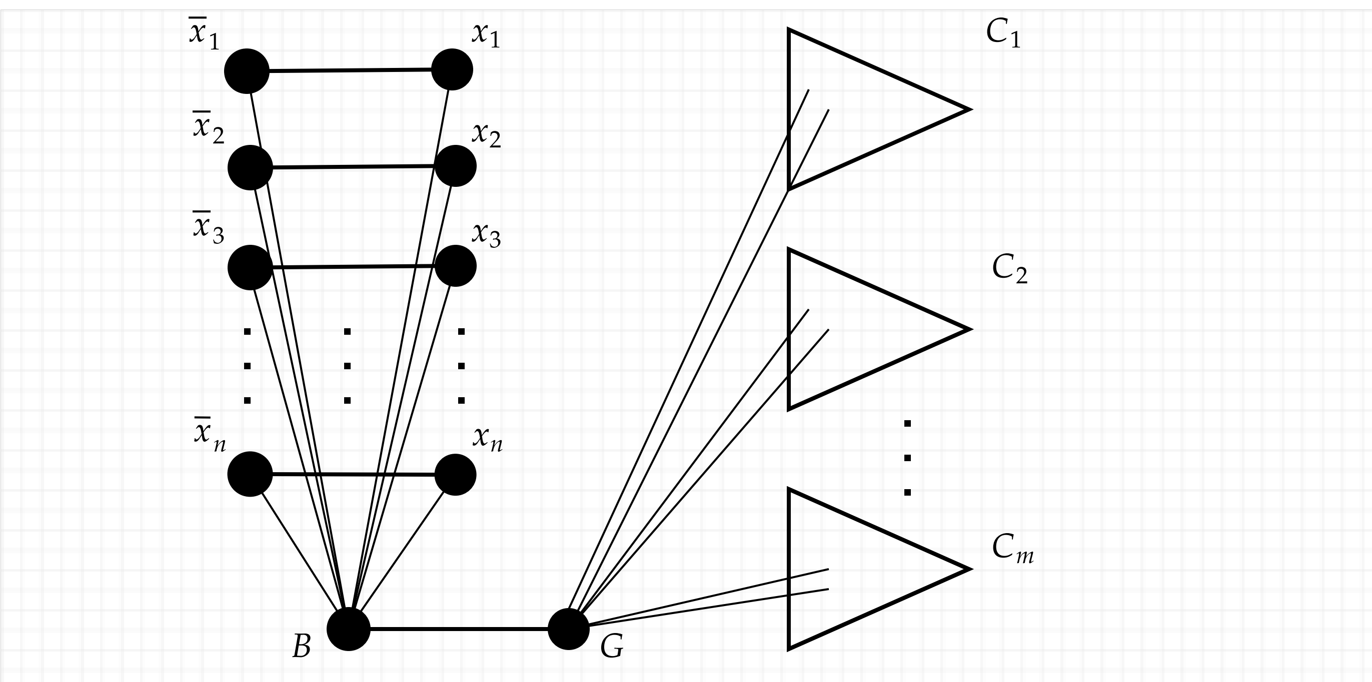
Question: Is there an assignment $\vec{a} = a_1 a_2 \cdots a_n \in \{0,1\}^n$ such that in every clause, one or two of its three literals are made true? (I.e., the values of its literals are not all equal.)

Thus if ϕ is "Not All Equal"-satisfiable then G is 3-colorable. And the original ($\Longleftarrow$) direction also works this way: the red node in the clause cannot be satisfied. Thus we actually get **NAE-3SAT** $\leq_m^p$ **G3C**. This is good enough to show that **G3C** is NP-complete. And to top it off, if a is an "NAE" satisfying assignment, then so it its flip a'. So the symmetry in the coloring is a feature, not a bug.

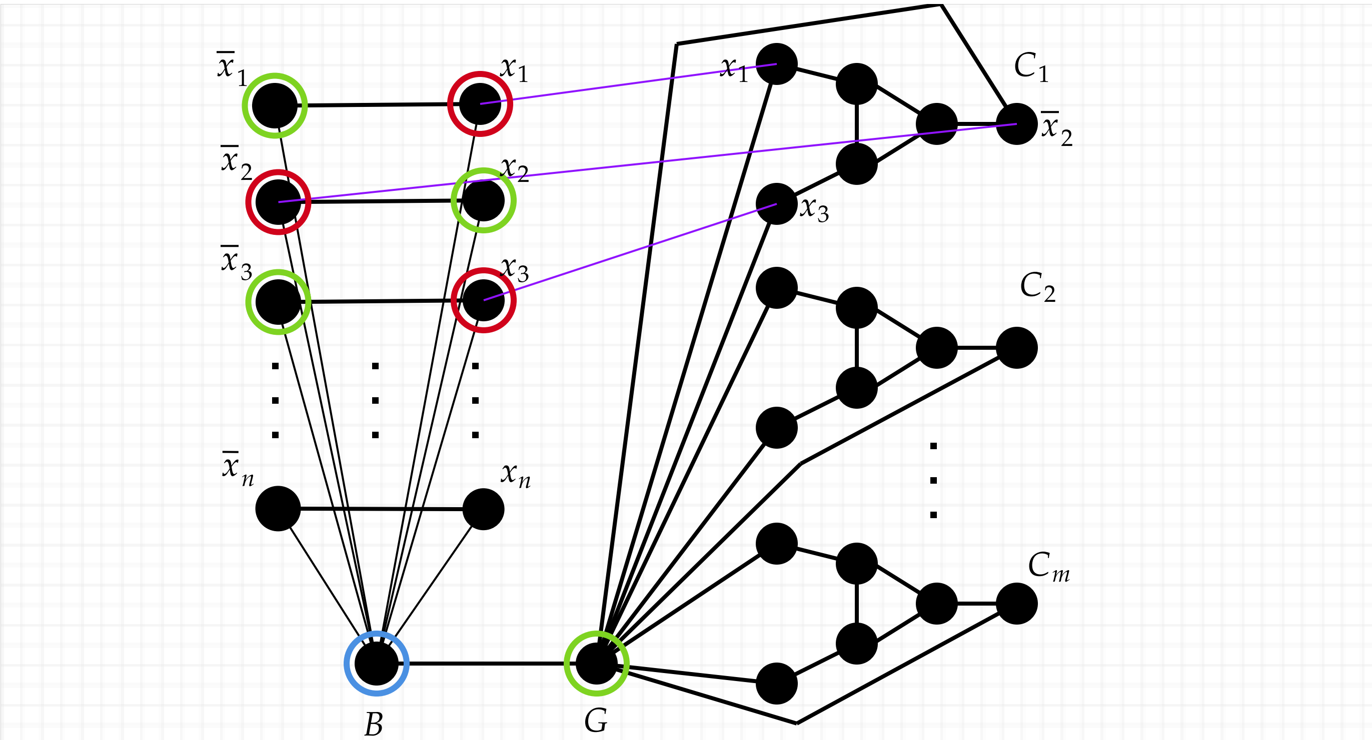
[2023 Note: The fire alarm happened just before I was set to define **NAE-3SAT**. I wound up speaking the definition in the courtyard. The delay caused me to skip the part that does the reduction to **G3C** without using **NAE-3SAT**, so that I could cover **Dominating Set** in full. Thus what follows until then---which is the way my proof in ALR chapter 28 does it---is FYI for you.]

If, on the other hand, we want to do the reduction strictly from 3SAT without special Cook-Levin appeal, then we need to modify G---as the ALR chapter does. This builds on the "governing blue node" idea to enforce an asymmetry between red and green as well. Of course, by just happening to choose **3SAT** as the "language $A \in$ NP" in the Cook-Levin proof, we get **3SAT** $\leq_m^p$ **NAE-3SAT,** so **3SAT** $\leq_m^p$ **G3C** follows by transitivity. But it is useful to illustrate **3SAT** $\leq_m^p$ **G3C** directly.

The first thing we need is to add to the B node a second node G so that the colors used for those nodes wlog. count as "blue" and "green". Connections from the G node to the clause gadgets can fix the problem of symmetry betweed "red" and "green", which we need to do for reduction from **3SAT** though not from **NAE-3SAT**.

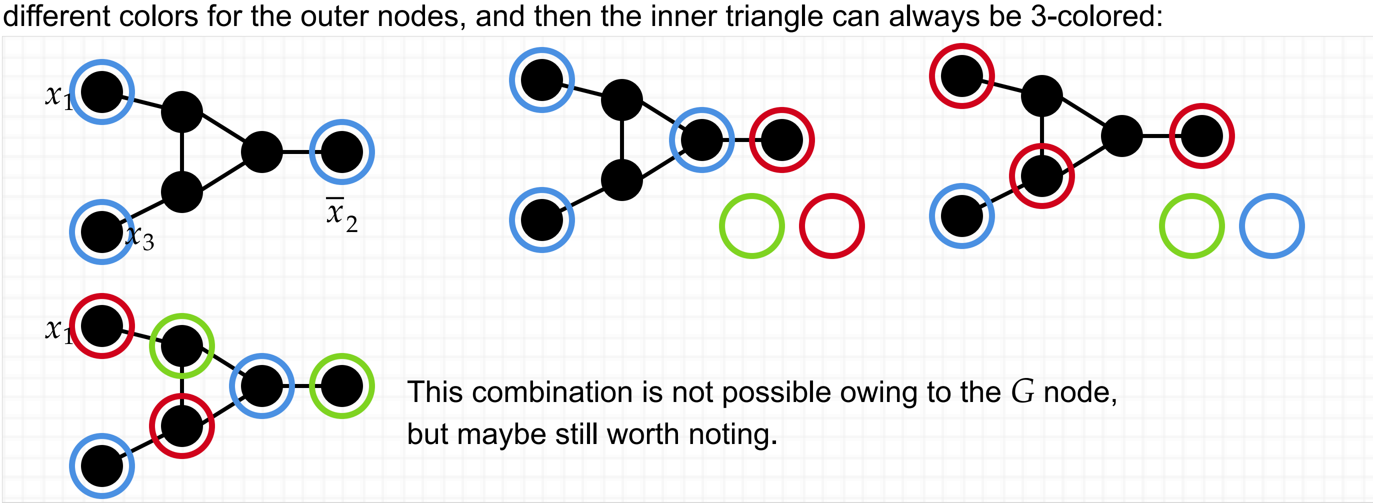


The second change is to include an outer layer of 3 nodes in the clause gadgets. The nodes will get an automatic "greenlock" from the G node. If they get a "redlock" from the rungs---which we want to mean all three literals being made false---then the 3 nodes are forced to be blue. This is without connecting the outer 3 nodes to each other. The resulting "bluelock", however, will prevent an inner triangle of each clause from being 3-colored. If, however, all three literals in the clause are made true, then the outer layer will see "greenlock" twice, and that is no problem. Here is the idea abstractly, showing only crossing edges between the rungs and the first clause $(x_1 \vee \overline{x}_2 \vee x_3)$:

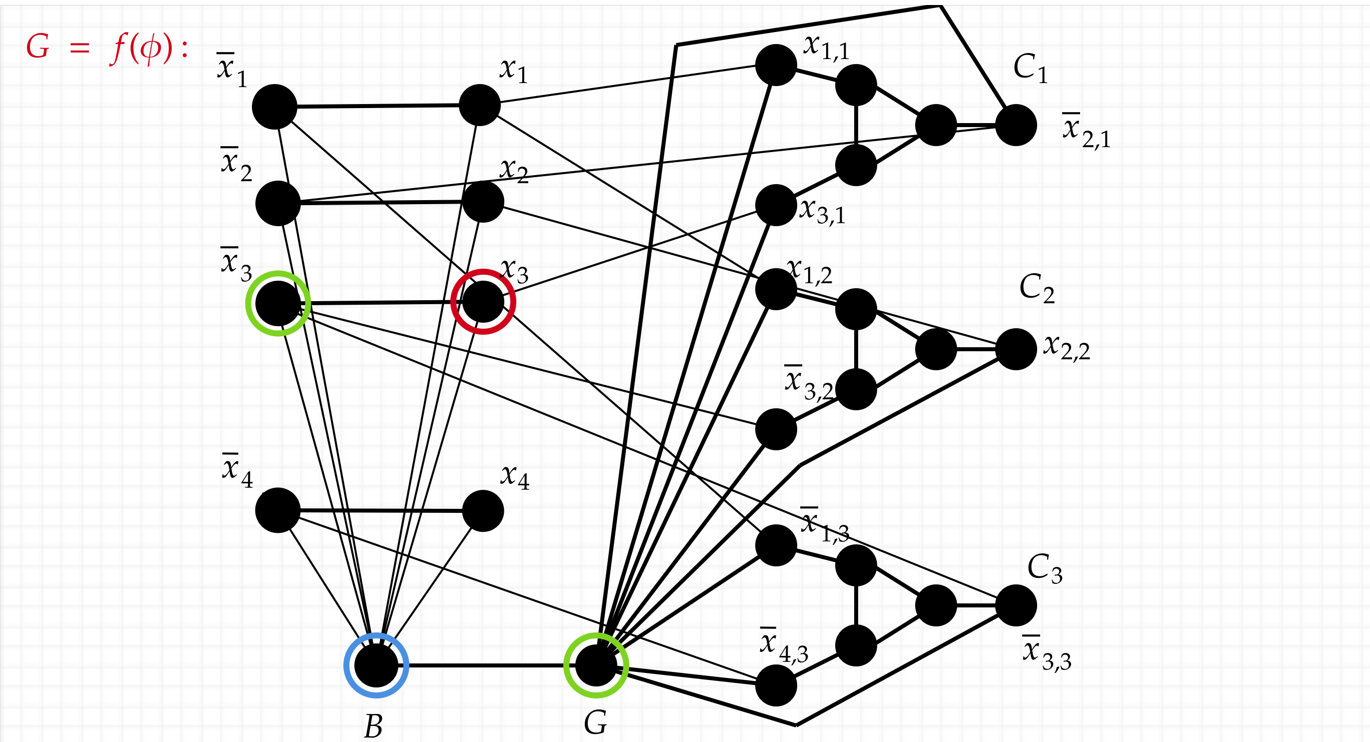


(Note: These are the opposite connections from the ALR notes, where I made the opposite choice of connecting x_1 in a clause to $\overline{x}_1$ in the rung to stay consistent with the reduction to **IND. SET**.)

If x_1 and x_3 are made false and x_2 true, so that clause C_1 fails, then each of the outer nodes of the C_1 gadget "sees red" as well as green from node G. This forces each outer node to be blue, but then the inner triangle of the C_1 gadget cannot be 3-colored. Any other assignment, however, allows using two different colors for the outer nodes, and then the inner triangle can always be 3-colored:



Here is the whole reduction carried out for our example formula
$\phi = (x_1 \vee \overline{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_3 \vee \overline{x}_4)$



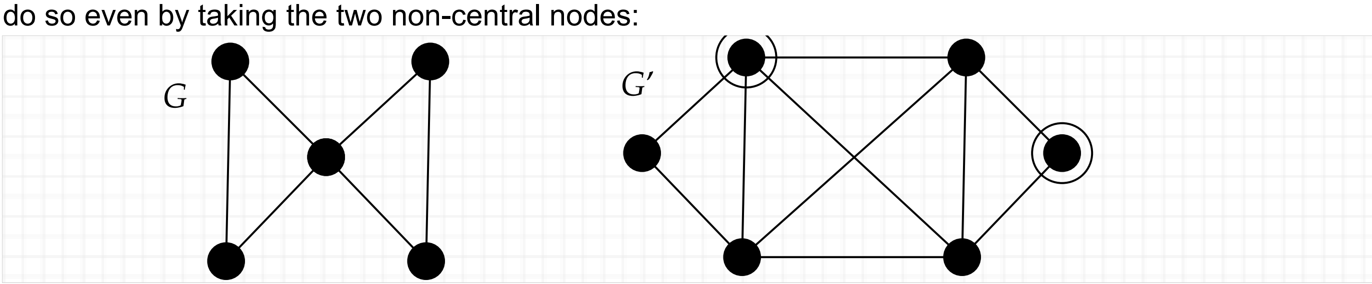
Again the reduction is linear-time computable in one sweep through ϕ. Correctness still needs to be re-checked in the other direction: If G has a good 3-coloring, then for every clause C_j, at least one of its 3 outer nodes x_{ij} or $\overline{x}_{ij}$ must be colored $\boldsymbol{R}$. Since we are now sending crossing edges to the rung node with the *same* sign, this means the same-sign rung node must be colored G. In turn, this means the literal satisfies C_j. Thus any good coloring uniquely yields an assignment that satisfies all clauses. ☒

[---------------End of segment that is FYI for 2023----------------]

Now we will consider a different reduction where both the "rungs" and the "clause gadgets" get different treatment. The target problem is:

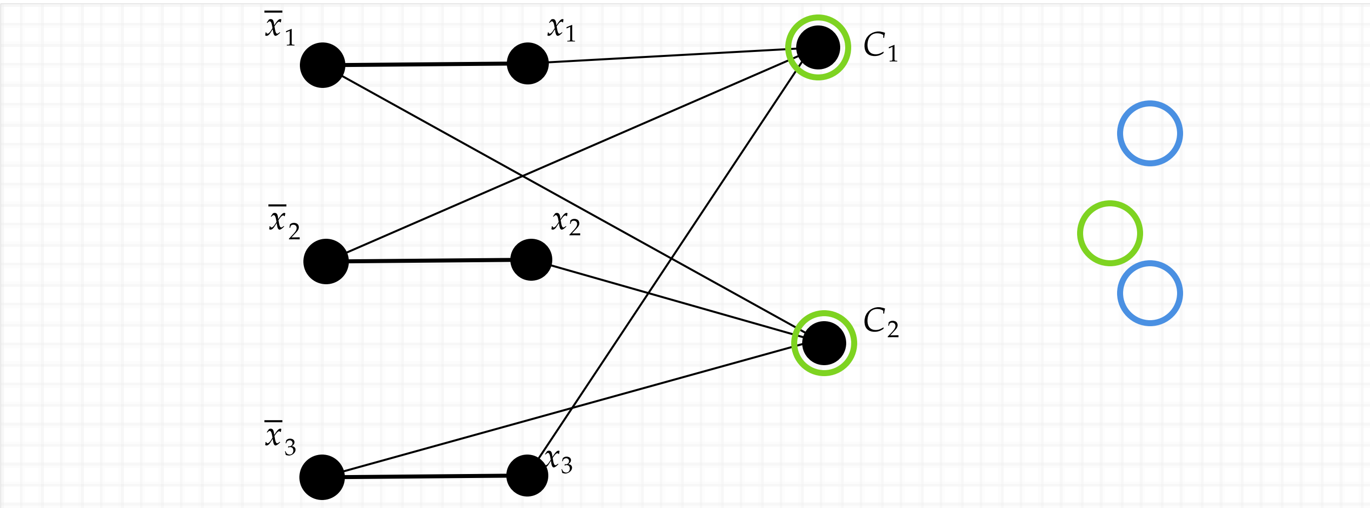
Dominating Set (**Dom Set**)
Instance: An undirected graph $G = (V, E)$ and an integer $k \geq 1$.
Question: **Is there** $S \subseteq V$, $|S| \leq k$, such that every node **not** in S is adjacent to a node in S?

The difference between a dominating set and a vertex cover is that the nodes don't have to cover every edge. The bowtie graph has a dominating set of size just 1, while its line graph needs $k = 2$ but can do so even by taking the two non-central nodes:



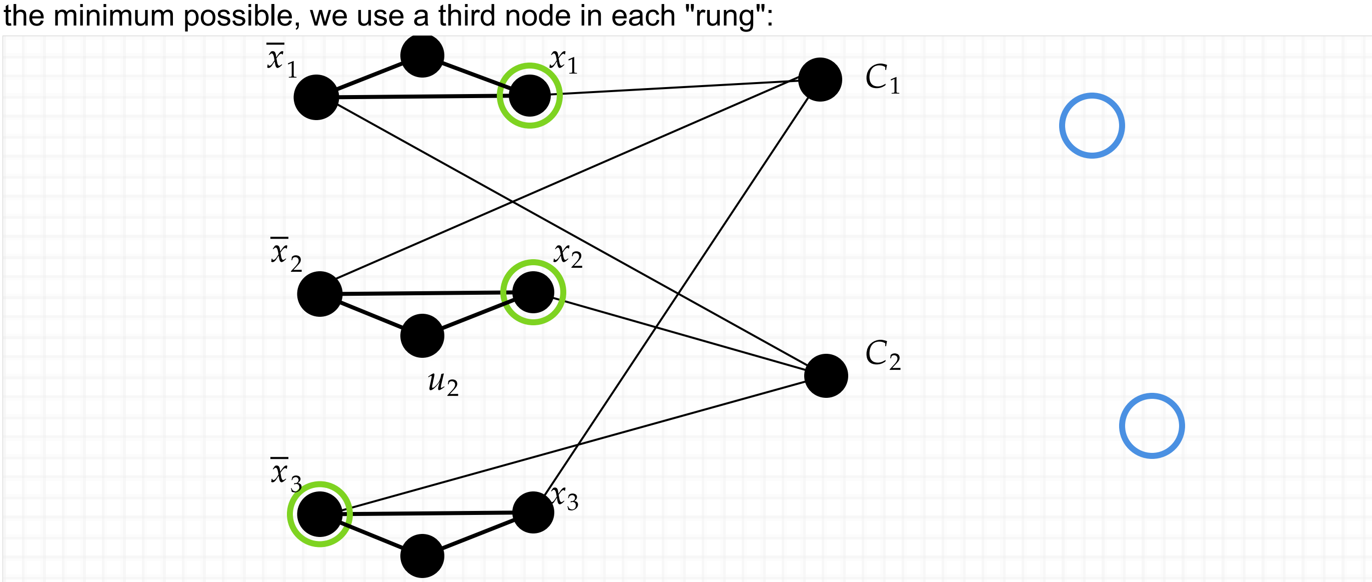
(Does the line-graph function give a reduction from **Edge Cover** to **Dom Set** or vice-versa? Hmmm... But we still want to reduce **3SAT** to **Dom Set** directly.)

The first key idea is the same: the rung nodes chosen in S correspond to those literals set true. The second key idea is simple: *make that true literal dominate every clause it satisfies.* This needs only one node per clause, and suggests taking $k = n$, irrespective of the number m of clauses. Here is how that looks for a simpler formula, $\psi = (x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3)$:



Setting all three variables false dominates the two clause nodes. So would setting them all true, whereas moving just x_2 to be true would fail to dominate (or satisfy) C_1. Is this all we need?

The flaw is that we have not enforced that in each rung, either x_i or $\overline{x}_i$ **must** be in S. This case allows a "surprise" domination by two nodes outside the rungs: use C_1 and C_2. To enforce the correspondence between possibly-good choices of S and truth assignments, and make sure $k = n$ is the minimum possible, we use a third node in each "rung":



Those extra nodes can only be dominated from the rung, and they do not help dominate each other, so n separate nodes are needed to dominate them. This fixes the problem. Defining the reduction formally in general and proving it correct is a self-study exercise.