CSE491/596 Lecture Wed. 11/18: PSPACE and Oracle Turing Machines

First, to recap the PSPACE-completeness of TQBF from a bottom-up rather than top-down view, we begin with the relation

$$\Phi_0(I,J) \equiv (I = J) \vee I \vdash_M J$$

Recall that the latter notation means that ID I can go to ID J by one step of the machine M (which the proof doesn't care whether it is deterministic or nondeterministic). Thus $\Phi_0(I,J)$ means that I can go to J by *at most one step*. The predicate $I = J$ actually stands for the bitwise equivalence of variables $i_1, i_2, \dots, i_s$ standing for the binary encoding of an ID I and $j_1, j_2, \dots, j_s$ standing for J:

$$(i_1 \vee \bar{j}_1) \wedge (\bar{i}_1 \vee j_1) \wedge (i_2 \vee \bar{j}_2) \wedge (\bar{i}_2 \vee j_2) \wedge \cdots \wedge (i_s \vee \bar{j}_s) \wedge (\bar{i}_s \vee j_s) \ .$$

Here s is basically $s(n)$ times $\log_2|\Gamma|$. The predicate $I \vdash_M J$ is coded in much the same way as the conversion from Turing machines to Boolean circuits employed in the Cook-Levin proof; again the size of the formula needed is $O(s) = O(s(n))$.

Now to express that an ID I can go to an ID K in *at most* **2** steps, it would be natural to write

$$\Psi_1(I,K) \equiv (\exists J): \Phi_0(I,J) \wedge \Phi_0(J,K)\,.$$

(The colon $:$, by the way, means that the quantifier grabs everything to the end of the formula.) Then to express "at most **4** steps" we would recurse:

$$\Psi_2(I,K) = (\exists J): \Psi_1(I,J) \wedge \Psi_1(J,K)\,.$$

But when we expand this out, it starts getting very bushy:

$$\Psi_2(I,K) = (\exists I',J,K'): \Phi_0(I,I') \wedge \Phi_0(I',J) \wedge \Phi_0(J,K') \wedge \Phi_0(K',K) \ .$$

For $\mathbf{2^r}$ steps, we would have 2^r terms---exponentially many. By using an alternation rather than keep everything existential, we cut down the expansion *syntactically*---but apparently not *semantically* (unless $\mathsf{P} = \mathsf{PSPACE}$, that is), even though it initially looks bushier:

$$\Phi_1(I,K) \equiv (\exists J)(\forall I',J'): [(I' = I \wedge J' = J) \vee (I' = J \wedge J' = K)] \longrightarrow \Phi_0(I',J') \ .$$

[Interesting thought: while composing this lecture I realized that for the unrolling part it is better to change J' to read K' here:

$$\Phi_1(I,K) \equiv (\exists J)(\forall I',K'): [(I' = I \wedge K' = J) \vee (I' = J \wedge K' = K)] \longrightarrow \Phi_0(I',K') \ .$$

]

When we expand the implication ($a \longrightarrow b \equiv \neg a \vee b$) and use De Morgan's Laws, the Boolean part does become more complicated than CNF or DNF:

$$\Phi_1(I,K) \equiv (\exists J)(\forall I',K') : [(I' \neq I \vee K' \neq J) \wedge (I' \neq J \vee K' \neq K)] \vee \Phi_0(I',K') .$$

The point is what happens when we step up to 4 (and then 8,16,...):

$$\Phi_2(I,K) \equiv (\exists J)(\forall I',K') : [(I' \neq I \vee K' \neq J) \wedge (I' \neq J \vee K' \neq K)] \vee \Phi_1(I',K')$$

$$\equiv \begin{array}{l} (\exists J)(\forall I',K')(\exists J')(\forall I'',K''): \\ \qquad [(I' \neq I \vee K' \neq J) \wedge (I' \neq J \vee K' \neq K)] \vee \\ \qquad [(I'' \neq I' \vee K'' \neq J') \wedge (I'' \neq J' \vee K'' \neq K')] \vee \Phi_0(I'',K''). \end{array}$$

It is weird that the only predicates in the growing body are inequalities. But the chess analogy in the previous lecture can help us understand this. The one thing to realize is that the Turing machine is not doing a checkmate where the opponent plays, but rather what is called a **series-mate.** For example, from https://www.ozproblems.com/problem-world/seriesmover

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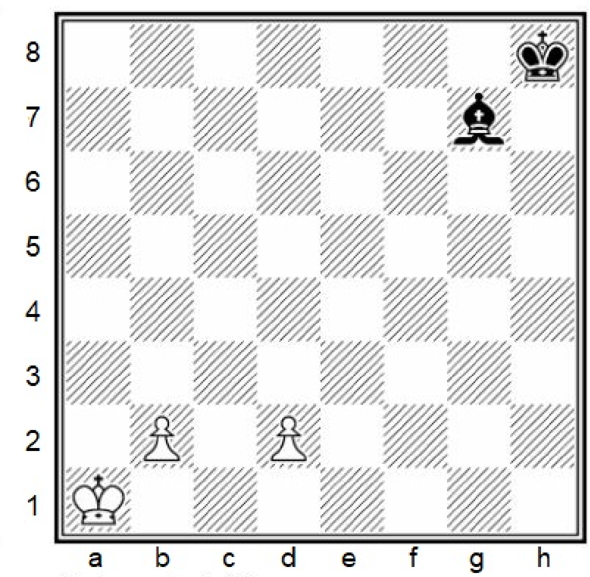
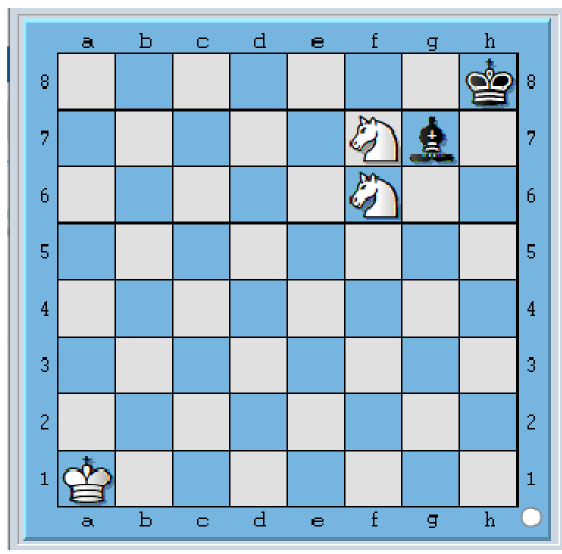


Series-mate in 13

End position:



In normal chess this game would be a draw, but in series chess, White can give checkmate by making 13 moves in a row: First the d-pawn moves up from d2 to d4 to block Black's bishop and allow the b-pawn to move. That pawn advances to b8 and promotes to a knight---not a queen since that would give Black check and make further moves illegal. That Knight then makes two hops to d7 and f6 to block Black's bishop so the other pawn can move. It, too, promotes to a knight, which finally hops from d8 to f7 to give checkmate:

So what the formula $\Phi_2(I,K)$ intuitively means---and note that it becomes a sentence when you specify I as the starting configuration and K as the goal configuration---is this:

[Either you didn't set up the pieces to make I'' and K'' represent the before-and-after positions of one of your 4 moves in the sequence] $\vee$ [you did set up I'' and K'' correctly and the move is legal].

Plus, in the chess case, a single predicate saying *BlackIsMated*. In the Turing machine case, there is no need for an extra predicate because there is only one final ID I_f and we initialize K to I_f and I to $I_0(x)$. Well, 4 moves is not 13 moves, but maybe we can get a feel from the unrolling for 8 moves:

$$\begin{aligned}\Phi_3(I,K) \;\equiv\; &(\exists J)(\forall I',K')(\exists J')(\forall I'',K'')(\exists J'')(\forall I''',K''') :\\ &[(I' \neq I \vee K' \neq J) \wedge (I' \neq J \vee K' \neq K)] \;\vee\\ &[(I'' \neq I' \vee K'' \neq J') \wedge (I'' \neq J' \vee K'' \neq K')] \vee\\ &[(I''' \neq I'' \vee K''' \neq J'') \wedge (I''' \neq J'' \vee K''' \neq K'')] \vee \Phi_0(I''',K''').\end{aligned}$$

Where this gets dizzying is how the logic for (not) setting up pairs of IDs correctly all becomes hierarchical. But once you get the pattern, the final formula Φ_r is easy to stream out. Here is 16:

$$\begin{aligned}\Phi_4(I,K) \;\equiv\; &(\exists J)(\forall I',K')(\exists J')(\forall I'',K'')(\exists J'')(\forall I''',K''')(\exists J''')(\forall I'''',K'''') :\\ &[(I' \neq I \vee K' \neq J) \wedge (I' \neq J \vee K' \neq K)] \;\vee\\ &[(I'' \neq I' \vee K'' \neq J') \wedge (I'' \neq J' \vee K'' \neq K')] \vee\\ &[(I''' \neq I'' \vee K''' \neq J'') \wedge (I''' \neq J'' \vee K''' \neq K'')] \vee\\ &[(I'''' \neq I''' \vee K'''' \neq J''') \wedge (I'''' \neq J''' \vee K'''' \neq K''')] \vee \Phi_0(I'''',K'''').\end{aligned}$$

Since the base predicate allows equality, this says "I can go in *at most* 16 steps to K" and so covers the case of 13 steps. The ultimate point is that we can get up to 2^r steps with just r rows. Since each row has 8 occurrences of IDs and hence requires $8s$ variables to code, the whole formula size is about $3rs$ for the quantifiers (which also tells that $3rs$ is the number of different variables, not counting those in I and K which get initialized to the binary encoding of $I_0(x)$ and I_f), plus $8rs$ for the rows, plus $2s$ for the final invocation of Φ_0. Since $s = O(s(n))$ and $r = O(s(n))$, we get size $O\left(s(n)^2\right)$.

- In the reduction to **TQBF**, $s(n)$ is $O\left(n^k\right)$ for some k, being the space used for M accepting the given language $A \in$ PSPACE, so $s(n)^2 = O\left(n^{2k}\right)$, which is still polynomial. There are $O\left(n^{2k}\right)$ binary variables, but each has a numerical index of size $\log_2\left(n^{2k}\right) = 2k \log n$ which is $O(\log n)$, so a log-space reduction can manage the indices while streaming. Thus TQBF is PSPACE-complete under $\leq_m^{\log}$.
- For Savitch's Theorem, what happens is that whether the resulting QBF instance Φ_r is true can be decided **deterministically** in space linear in its size, so space $O\left(s(n)^2\right)$. Since A can come from an arbitrary **N**TM running in $s(n)$ space, this proves that (provided $s(n) \geq \log n$) NSPACE$[s(n)] \subseteq$ DSPACE$\left[s(n)^2\right]$. [Dropping the O-notation in DSPACE$\left[s(n)^2\right]$ is a permissible fib.]

Some words to the wise:

1. The savings of 2^r-to-r only works because a Turing machine (DTM or NTM) plays *solitaire*.
2. If we have an opponent making moves too, then the space savings does not work over 2^r steps.
3. But if our alternating-move games are limited to r moves, say where r is polynomial in n, then

we can get an $O(r)$-size formula directly just by the direct formulation shown at the outset in the previous lecture.

Theorem. Chess extended to $n \times n$ boards (with more of the same kinds of pieces, still just one king) is PSPACE-hard under $\leq_m^{\log}$. It is PSPACE-complete under $\leq_m^{\log}$ in the presence of a "generalized 50-move rule" limiting games to length $n^{O(1)}$. (But with no such limit it is---amazingly, IMHO---complete for EXP, i.e., exponential time $2^{n^{O(1)}}$. This is thought to imply that $n \times n$ chess without a polynomial length limit on games requires exponential space too.)

The further development is that many two- or multi-player games of strategy, not just chess, are PSPACE-complete to solve (presuming a reasonable limit on the length of games). Whereas NP-completeness characterizes myriad solitary (or co-operative) optimization problems, PSPACE-completeness characterizes optimization in the face of competition.

Another facet of PSPACE as a "catchall" for polynomial-based complexity comes from **oracle Turing machines**. An **OTM** M is allowed instantaneous access to an external database in the form of an **oracle function** $f : \Sigma^* \to \Sigma^*$. At any point in its computation, M can write a **query string** y on a special oracle tape and enter a special **query state** $q_?$. In that state, the query tape is magically erased and replaced by the value $f(y)$ with the head reading the first bit of it. This idea was in Turing's original paper, and anything M does is said to **Turing-reduce** to f. In particular, we write $L\left(M^f\right)$ for the language of M *with oracle f* or *relative to f*.

Here is my favorite small-scale example using a function. Whether the product xy of two n-bit integers can be computed in $O(n)$ time is a major open problem. The algorithm learned in school takes time order-n^2. *Karatsuba's method* runs in time $O\left(n^{1+\epsilon}\right)$ for any prescribed $\epsilon > 0$ but the lower ϵ, the higher the constant in the O, and this tradeoff issue affects the known $\widetilde{O}(n)$ time methods even more. But if there were magically a linear-time algorithm for *squaring* a number, then we would get such an algorithm for multiplication in general.

Theorem: Except over fields of characteristic 2, integer multiplication Turing-reduces to the squaring function in linear time via the formula $xy = \frac{1}{2}\left[(x+y)^2 - x^2 - y^2\right]$.

If f is the 0-1 valued characteristic function of a language A, then we just write M^A. After any query y is submitted, M is left reading either 0 for "no, $y \notin A$" or 1 for "yes" on its tape. Many sources have M go to one of two special answer states q_y or q_n instead---this difference is immaterial. If C is a class of *machines*, then we write C^A to be the class of languages $L\left(M^A\right)$ over all machines M:

- $\mathsf{P}^A = \left\{L\left(M^A\right) : \textit{The DTM } M \textit{ runs in polynomial time } (\textit{with oracle } A)\right\}$
- $\mathsf{NP}^A = \left\{L\left(N^A\right) : \textit{The NTM } N \textit{ runs in polynomial time } (\textit{with oracle } A)\right\}$
- $\mathsf{PSPACE}^A = \{L(M^A) : \textit{The DTM } M \textit{ runs in polynomial space } (\textit{with oracle } A)\}.$