CSE491/596 Lecture Wed. Dec. 2: Quantum Computing I

How Does Nature Compute?

$\Downarrow$

How Does Nature Concatenate?

Strings: Trivial operation: $a \bullet b = ab$, $x \bullet y = xy$

Nature uses (linear) operators that we represent as matrices (over a standard basis).

$A \bullet B = \{x \cdot y : x \in A \wedge y \in B\}$

When you think of matrices and vectors, the first idea that pops into mind is the ordinary matrix product AB of an $\ell \times m$ and an $m \times n$ matrix. But this is "lossy," whereas concatenation must be lossless (except possibly for memory of the place where the strings got concatenated). Instead, Nature uses **tensor product**, which applies also to vectors and doesn't need the "shapes" of the operands to agree.

Main Point: Matrix multiplication is

Instead, it is Tensor Product:

$$A \otimes B = \begin{vmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & & \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{vmatrix}$$

How to visualize it: Say $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$:

As a matrix, if A is $n \times n$ and B is $r \times r$, then $A \otimes B$ is $(nr) \times (nr)$. If we form $\underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ items}}$, then the matrix $A^{\otimes k}$ has size n^k. If $k \approx n$, this is exponential size

If A is $\ell \times m$ and B is $n \times r$ then $A \otimes B$ is $\ell n \times mr$, so the dimensions can be anything. In particular, A and B can both be column vectors with $m = r = 1$, whereupon $A \otimes B$ is a column vector of length ℓn.

Example Hadamard Matrix (without normalizing)

$H = H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (normalized: $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$).

$$H \otimes H = \begin{bmatrix} 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & -1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \quad 4 \times 4$$

$H \otimes H \otimes H$ is $2^3 \times 2^3$ ie 8×8

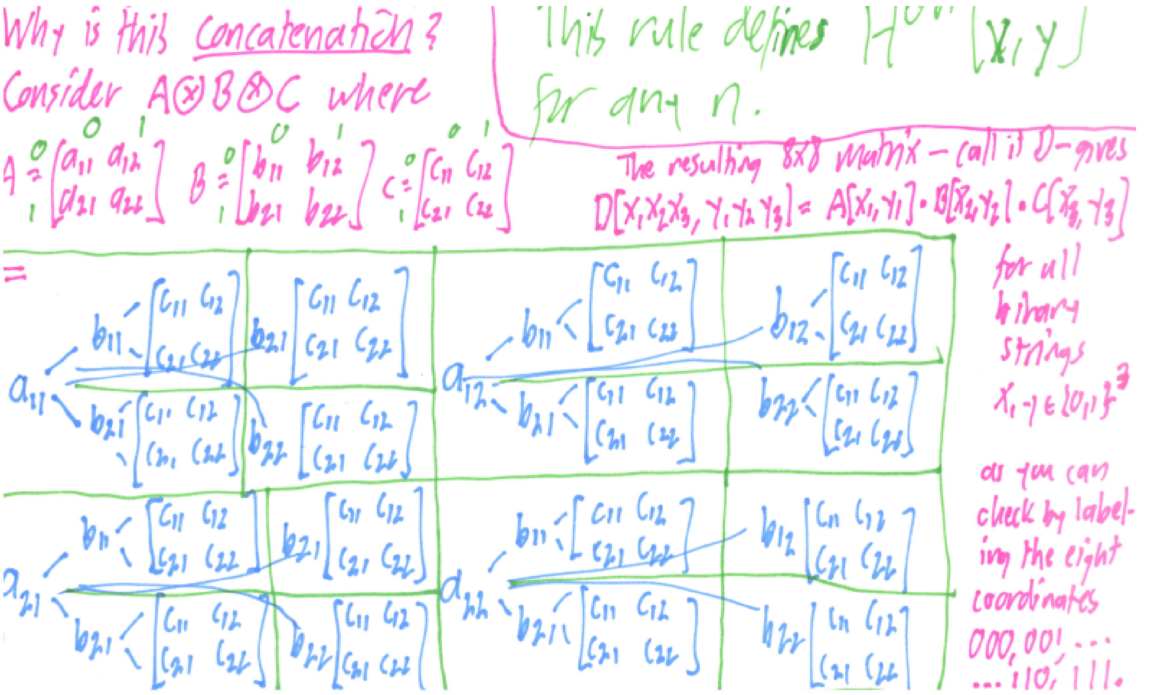
$H^{\otimes n}$ is $N \times N$ where $N = 2^n$

Rule: $+1$ if $\langle row \bullet col \rangle = 0$
Multiply $row_i \cdot col_i$ for all i then add up mod 2.

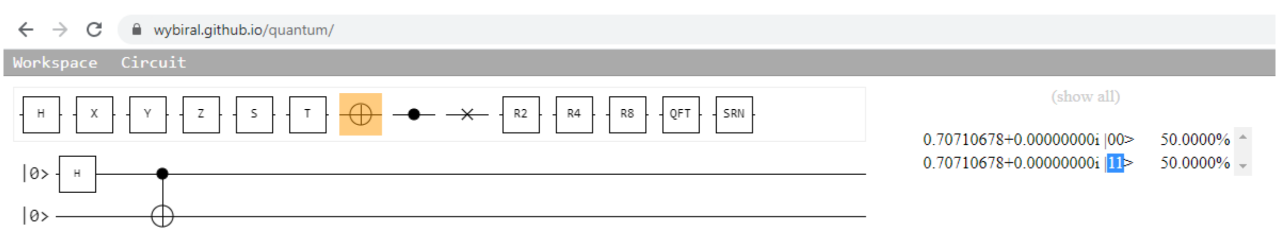
The entries of $H^{\otimes n}$ are indexed by binary strings x, y of length n. Take the Boolean inner product mod 2 of x and y. If it is 0, then $H^{\otimes n}[x, y] = 1$, but if it is 1, then $H^{\otimes n}[x, y] = -1$.

E.g. $\langle 00, y \rangle = 0$ for any y, so the row for 00 is all 1s. But $\langle 01, 01 \rangle = 1$, so the entry $H^{\otimes 2}[01, 01] = -1$. And $\langle 11, 11 \rangle = 2 = 0\ (mod\ 2)$, so $H^{\otimes 2}[11, 11] = +1$ back again.

This rule defines $H^{\otimes n}$ for any n as an $N \times N$ matrix. On paper that is exponential size, but in a **quantum circuit diagram** on n qubits, it is $O(n)$ gates. Is it linear effort for Nature to compute? Because the computation is unitary, hence **reversible** and ideally accompanied by zero entropy, it might be zero effort. Or, because it represents splitting beams of particles, possibly serially, it might be exponential effort after all.



At the end I showed the applet for quantum circuits by Davy Wybiral:



There is also IBM's recently-upgraded qwidget: