CSE 491/596 Lecture Fri Dec. 2 Fall 2022

A <u>quantum</u> <u>state</u> "is" any vector $(a_1, a_2, \ldots, a_N)$ of complex entries such that $\|a_1\|^2 + |a_2|^2 + \cdots + |a_N|^2 = 1$. <u>Unit vector</u>

When $N = 2^n$, we say n is the number of <u>qubits</u> (implicitly if not explicitly)

The allowed operations A on quantum states "are" $N \times N$ matrices that map any unit vector to a unit vector. Mathematically, this is equivalent to

$$A \cdot A^* = I = A^* \cdot A$$ <u>Unitary</u> {matrix / operator}.

Notation (YMMV): • $\bar{a}$ or a^*: complex conjugate of complex scalar a

E.g. if $\omega = \frac{1+i}{\sqrt{2}}$ then $\bar{\omega} = \frac{1-i}{\sqrt{2}}$. or write it as ω^*

• If $x = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$ then $x^* = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_N)$ as a row vector. $\bar{x}$ would only mean $\begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_N \end{pmatrix}$

• If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, <u>transpose</u> $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ <u>Conjugate</u> <u>transpose</u>, also called <u>adjoint</u> $A^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} \\ \bar{a}_{12} & \bar{a}_{22} \end{bmatrix}$

<u>YMMV</u>: Many (newer) sources write $A^\dagger$ for conjugate transpose, also $x^\dagger$. dagger

<u>OMWV</u>: <u>Dirac</u> <u>Notation</u>: $|x\rangle$ denotes a column vector in which "x" is an attribute we can treat as basic.
"<u>Ket</u> x"

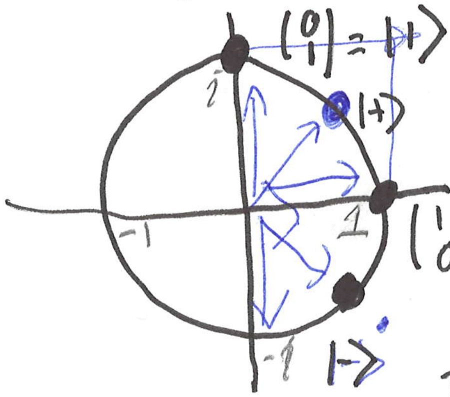
Our mileage will vary. Its conj transpose is $|x\rangle^* = \underline{\langle x|}$ as a row vector "<u>bra-x</u>" $\langle x|y\rangle$ is a <u>bra</u>c<u>ket</u>

One Qubit: $n = 1$, $N = 2$. Standard Basis: $e_0 = \begin{pmatrix}1\\0\end{pmatrix}$ $e_1 = \begin{pmatrix}0\\1\end{pmatrix}$

$\qquad\quad = |0\rangle \qquad\quad = |1\rangle$

Two Ways to Visualize: "Cartesian" and "Bloch"

Cartesian Unit "circle" for a Qubit



$\begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$

Good intuition: Vectors at 90° angle are orthogonal.

The blue point is $\dfrac{\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}}{\sqrt{2}} = \dfrac{\begin{pmatrix}1\\1\end{pmatrix}}{\sqrt{2}}$

Its Dirac name is $|+\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

Also $\dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$ is called $|-\rangle$

Defn: The real inner product for two vectors $X = (x_1 \dots x_K)$
$Y = (y_1 \dots y_K)$
is $X \bullet Y = \sum_{i=1}^{K} x_i y_i$. The complex inner product, however, conjugates X:
$\langle X, Y\rangle = \sum_{i=1}^{K} \overline{x_i}\, y_i$. Dirac: $\langle x|y\rangle = \langle x| \cdot |y\rangle$!

Norm: $\|X\| = \langle X|X\rangle = \sum_{i=1}^{K} \overline{x_i}\, x_i$ Unit vectors have $\|X\| = 1$.

Defn: X and Y are orthogonal if $\langle X, Y\rangle = 0$ Examples:

$\langle 0|1\rangle = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle = \overline{\begin{pmatrix}1\\0\end{pmatrix}}, \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}^{*} \bullet \begin{pmatrix}0\\1\end{pmatrix} = (1\ 0)\begin{pmatrix}0\\1\end{pmatrix} = 0$

$= 1 \cdot 0 + 0 \cdot 1 = 0$ as a scalar

$\langle +|-\rangle = \left\langle \dfrac{\begin{pmatrix}1\\1\end{pmatrix}}{\sqrt{2}}, \dfrac{\begin{pmatrix}1\\-1\end{pmatrix}}{\sqrt{2}} \right\rangle = \dfrac{1}{2}(1\ \ 1)\begin{pmatrix}1\\-1\end{pmatrix} = \dfrac{1}{2}(1 \cdot 1 + 1 \cdot (-1)) = 0.$

Some unitary matrices

Hadamard $H = \dfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$ $H^* = H$ H is self-adjoint, aka Hermitian.

$HH^* = HH = \dfrac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = I$

$X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ note: $X\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ $X\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$ aka NOT, $X^2 = I$ too. $Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ $Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$ $Y^* = Y$