CSE596 Friday Sept. 13

The relation $x \sim_L y \equiv (\forall z \in \Sigma^*)\, L(xz) = L(yz)$

is an equivalence relation (for any language L):

- Reflexive: $x \sim_L x$ (immediate)
- Symmetric: $x \sim_L y \Leftrightarrow y \sim_L x$ (immediate "by form").
- Transitive: $w \sim_L x \wedge x \sim_L y \Rightarrow w \sim_L y$

$$(\forall z_1)\, L(wz_1) = L(xz_1) \;\wedge\; (\forall z_2)\, L(xz_2) = L(yz_2) \overset{?}{\Rightarrow} (\forall z_3)\, L(wz_3) = L(yz_3)\; ✓.$$

Hence the relation $\sim_L$ partitions Σ^* into equivalence classes.

Defn: A set S is a system of representatives for an equiv. relation $\sim$ if it $\equiv$ PD set. does not contain more than one member from any equiv. class. Complete if it has one from each.

$y \sim_L x \Rightarrow yc \sim_L xc$ because the 'c' could be part of any "z"

Myhill Nerode Theorem: A language L is regular if and only if $\sim_L$ partitions Σ^* into finitely many equiv. classes. I.e., iff all PD sets are finite.

Which direction, "$\Rightarrow$" or "$\Leftarrow$," did we show by proving that having an infinite PD set makes L non-regular?

Proof: We showed $\Rightarrow$ by contrapositive. Now show "$\Leftarrow$".

Design an automaton $M = (Q, \Sigma, \delta, s, F)$ where Q is the set of equivalence classes,

Note: if $x \in L$ and $x \sim_L y$, then $y \in L$ (consider $z = \epsilon$)

F is the set of equiv. classes R_x where $x \in L$.

$s = R_\epsilon$, and for any $x \in \Sigma^*$, $c \in \Sigma$:

$$\delta(R_x, c) = R_{xc}. \quad \text{Then } L(M) = L.$$

This is well-defined because if y is any other representative of R_x, then $R_{yc} = R_{xc}$, so "$\delta(R_y, c)$" gives the same answer.

Given that Q is finite, M is a DFA, so L is regular. ☒

Corollary: If $k = |Q|$, then no other DFA M' s.t. $L(M') = L$ can have fewer than k states, because a complete sys. of representatives is a PD set of size k. Moreover, M is unique among DFAs of k states. That is, $\boxed{\text{statement in box}}$.

$\boxed{\therefore \text{ Every regular language has a unique minimum-size DFA } M \text{ s.t. } L(M) = L.}$

Proofs of Nonregularity by MNT: Fill out a "proof script"

Take $S = \$^* = \{\varepsilon, \$, \$\$, \$\$\$ \ldots\}$ (Justify if needed that S is infinite.)

Let any $x, y \in S$, $x \neq y$, **be given**. Then we can helpfully write $x = \$^m$ and $y = \$^n$ where $m \neq n$,

indeed where $m < n$ without loss of generality since "x" can be the shorter one.

Take $z = D^n$. Then $L(xz) \neq L(yz)$ because $xz = \$^m D^n$ gets killed since $m<n$ but $\$^n D^n$ survives.

Since $x, y \in S$ are arbitrary, S is PD for L, and since S is infinite, L is nonregular by MNT. ⊠

Which direction, $\Rightarrow$ or $\Leftarrow$, did we show by proving that having an infinite PD set makes L non-regular?

Likewise $\{a^n b^n : n \geq 0\}$ is nonregular.

Example: L = "the spears-and dragons language if you could save any number of spears".

$L' = \{ x \in \{(,)\}^* :$ there is a suffix w of)'s that makes xw balanced.

If $(\exists S \subseteq \Sigma^*)(\forall x, y \in S, x \neq y)(\exists z) : L(xz) \neq L(yz)$ then L is not regular

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Example: $L_0 = \{$balanced-paren strings$\}$, called BAL.

Take $S = (^*$, clearly infinite. Let any $x, y \in S$, $x \neq y$, be given. Then $x = (^m$, $y = (^n$, where $m \neq n$. Take $z =)^m$.

Then $xz = \underbrace{(((\cdots(}_{m}\underbrace{)))\cdots)}_{m}$ is balanced, but $yz = (^m)^n$ is not.

So $BAL(xz) \neq BAL(yz)$, so S is an infinite PD set for BAL. ⊠

Note that if $m > n$, $(^m)^n$ and $(^m)^m$ are both in L'.

So we need "wlog $m<n$" trick to make this work for L'.

In fact, L' is the same as spears-and-dragons with $(= \$$, $) = D$.