

**Decision Problems of "Instance Type":** "A graph  $G$  and a number  $k = \langle G, k \rangle$   $G=(V, E)$

**Associated Optimization Problems:** Find  $S$  to IND SET:  $Q: (\exists S \subseteq V, |S| \geq k) \wedge (u, v) \notin E$   
 Maximize  $k$  such that CLIQUE:  $Q: (\exists S \subseteq V, |S| \geq k) \wedge (u, v) \in E$   
 A more precise name would be "Vertex To Edge Cover".  
 Nevertheless,  $IND SET \leq_m^P CLIQUE$  via the map  $\langle G, k \rangle \mapsto \langle \bar{G}, k \rangle$  where  $\bar{G}$  is the complementary graph.  
 Forming  $\bar{G}$  is not the same as negating the problem.  
 Logical:  $(u, v) \in E(\bar{G}) \iff (u, v) \notin E(G)$  for  $u, v \in V$

**VERTEX COVER (VC):**  $Q: (\exists S \subseteq V, |S| \leq k) \wedge (u \in S \vee v \in S)$   
 Fact: The complement  $S' = V \setminus S$  of a vertex cover  $S$  is always an independent set, and vice versa. Thus  $G$  has an independent set  $S$  of size  $\geq k$  if and only if  $G$  has a vertex cover  $S'$  of size  $\leq k' = n - k$ . Note that  $S$  is part of the instance.  
 Nevertheless,  $IND SET \leq_m^P VC$  via the map  $f(G, k) = (G, n - k)$ . Thus VC is also NP-complete.

**GRAPH 1-COLORING:** QUESTION: Can we assign colors  $c_1, \dots, c_k$  to vertices  $u, v, \dots \in V(G)$  such that for each  $(u, v) \in E(G)$ ,  $color(u) \neq color(v)$ ?  
 Logical:  $(\exists color: V \rightarrow \{c_1, \dots, c_k\}) \wedge (u, v) \in E \implies color(u) \neq color(v)$   
 Quantification over the  $k^{|V|} = k^n$  possible finite functions. Still exponential even if we fix  $k=2$  or  $k=3$ .

**GRAPH 2-COLORING (G2C):** Instance "Just a graph  $G$ "  
 QUESTION: Can  $G$  be 2-colored? I.e., is  $G$  bipartite?  
 In P: Choose black or white for  $V_1$ , then the colors of all vertices connected to  $V_1$  are forced, and either you succeed or fail.  
 G3C: Inst  $\langle G \rangle$ , Ques: Can  $G$  be 3-colored? Theorem:  $3SAT \leq_m^P G3C$

**GRAPH COLORING:** Inst: also  $\langle G, k \rangle$   
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Proof: We will map  $\phi \in C \mapsto G_\phi$  where  $G_\phi = (V_\phi, E_\phi)$  and  $V_\phi$  starts with some special features:

$V_\phi = \{ \underbrace{V_G, V_B}_{\text{wlog. colored green, blue}} \} \cup \dots$

$\{ x_i, \bar{x}_i : i = 1 \text{ to } n \}$  ( $i=3$  for illustration)

$V$  vertices for "gadgets" for the  $m$  clauses  $C_1, \dots, C_m$  that make up  $\phi$ .

With the rung edges and edges to  $V_B$ , we have enforced a 1-1 correspondence between truth assignments and legal R-G (-B) colorings of the rung nodes.

$\phi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3)$

Design Trick: We want to make each gadget  $C_i$  3-colorable unless the truth assignment makes clause  $C_i$  unsatisfied. Unsatisfied  $\equiv$  that its three outer nodes are forced to be all blue by the rung coloring and  $V_G$ . Then being able to color a node red in the outer ring means ability to color  $C_i$ .

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This construction is poly-time since we translate each clause  $C_i$  to a six-node gadget with clear internal edge and crossing-edge connections. It is correct because  $\phi$  is satisfiable  $\Leftrightarrow$  no gadget is forced to have all-blue colors in its outer 3 nodes...

... what about if a truth assignment to  $\phi$  makes all 3 nodes in connecting edges green? OK too — not forced to have all red.

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I stopped short at the end of observing that in fact the last "what about if..." question can be assumed never to occur. In the Cook-Levin reduction to 3SAT which I showed, one never has a clause satisfied with all 3 literals being made true. This can be taken as a general feature if-and-when needed. Here it is not needed, but it could be.