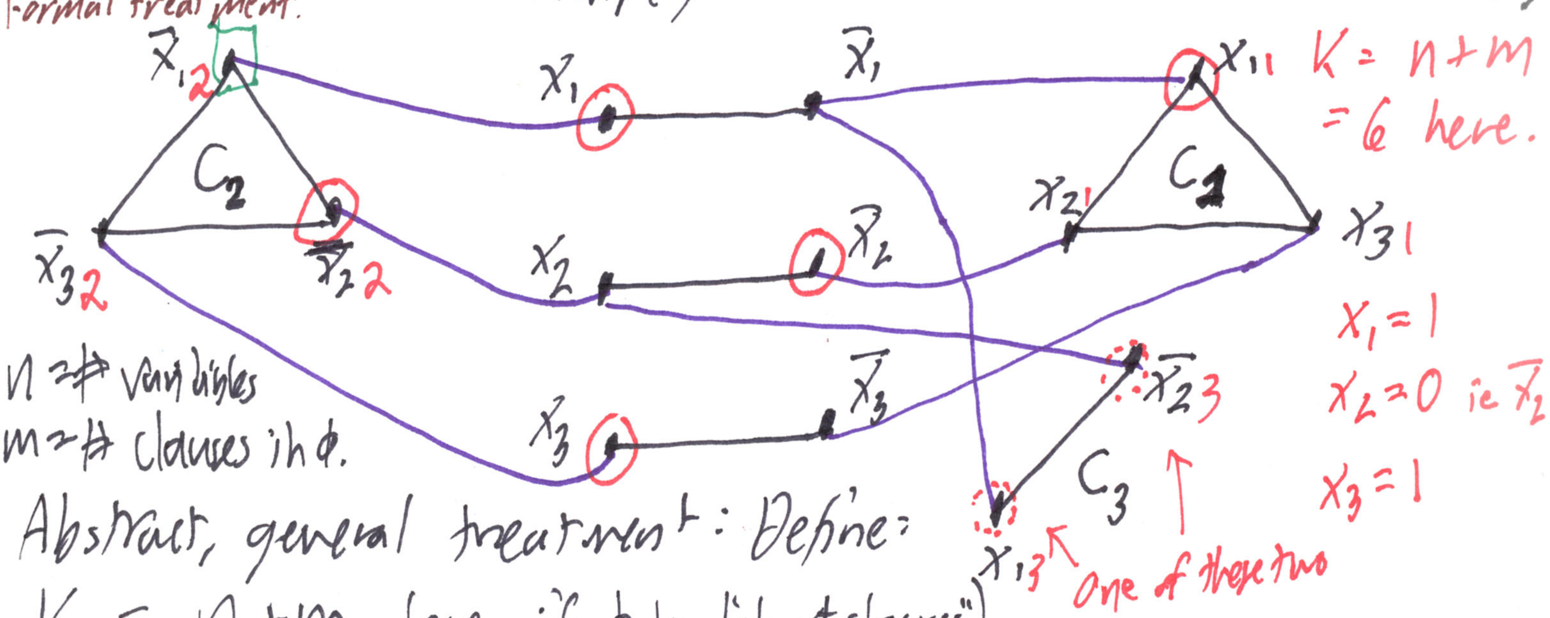
CSE596 Lecture Wed. Oct. 31 Fall 2018

3SAT $\leq_m^p$ IND_SET: Example for $\phi = \overset{C_1}{(x_1 \vee x_2 \vee x_3)} \wedge \overset{C_2}{(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)} \wedge \overset{C_3}{(x_1 \vee \bar{x}_2)}$

Formal treatment.



n = # variables
m = # clauses in ϕ.

Abstract, general treatment: Define:

$K_\phi = n + m$ (even if ϕ has "short clauses")

$G_\phi = (V_\phi, E_\phi)$ where: $V_\phi = \{ x_i, \bar{x}_i : 1 \leq i \leq n \}$ "rung nodes"

$\cup \; \{ x_{ij} : x_i$ appears as a positive literal in clause $j \}$

$\cup \; \{ \bar{x}_{ij} : \bar{x}_i$ appears negated in $C_j \}$.

$\|V_\phi\| = 2n + \sum_{j=1}^{m} \|C_j\|$

$= 2n + 3m$ when all clauses have the same size 3.

$E_\phi = \{ (x_i, \bar{x}_i) : 1 \leq i \leq n \}$ "rung edges"

$\cup \; \{ (\pm x_{ij}, \pm x_{kj}) :$ literals in the same clause $j \}$

$\cup \; \{ (x_i, \bar{x}_{ij}) : \bar{x}_i$ appears in clause $j \}$

$\cup \; \{ (\bar{x}_i, x_{ij}) : x_i$ appears in clause $j \}$.

These edges can be stamped out in $\tilde{O}(n+m)$ time, in one "pass" over ϕ.

"Crossing clear edges" and rigorous

Correctness: ⓐ • If a is a sat assgt to ϕ, then a gives us an ind. set of size K

ⓑ • If S is an ind. set of size K, then we must have a sat assgt. to ϕ.

ⓐ. Let any sat. assgt a to ϕ be given. Define S to consist of:

- The rung nodes made true in a
- One satisfied literal $+x_{ij}$ or $\bar{x}_{ij}$ from each C_j.

Then $\|S\| = K = n+m$ and S is independent because ... (go thru E)

ⓑ Let any ind set S of size n+m be given. Then S must have:

- one node from each clause
- and one node from each rung

because K = n+m is the max which "saturates" G_ϕ.

Because every X_{ij} is connected to $\overline{X_i}$ in a rung, and $\overline{X_{ij}}$ is connected to X_i in a rung,

the chosen node X_{ij} or $\overline{X_{ij}}$ in each clause must have the same sign + or − as the node X_i or $\overline{X_i}$ in the rung. Therefore the truth assignment induced by the choices in the rungs satisfies each C_j.

ⓐ & ⓑ give us $\phi \in SAT \iff G_\phi \in L_{IND SET}$, so $3SAT \le_m^p$ IND_SET. $\boxtimes$

☆ Reductions Are Reusable:

VERTEX COVER (VC)

Also clearly in NP

$n = |V|$

INST: A graph G, an integer $K' \le n$

QUES: Does G have a set S' of size at most K' s.t. every edge involves a node in S'?

easily checkable.

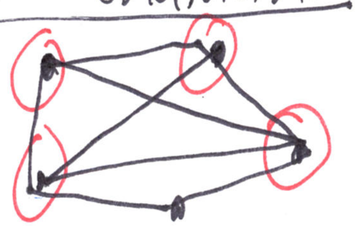
Fact: S' is a vertex cover ⇔ $S = V \setminus S'$ is an independent set.

Thus VC with $K' = n - K$ is the same problem as IND SET with n, K

The same reduction shows that VC is NP-complete

☆ Reductions Are Chainable:

CLIQUE:



K = 4 is max.

INST: G, K

QUES: Does G have a clique of size at least K, i.e. a set S of K nodes, every two connected by an edge?

IND_SET $\le_m^p$ CLIQUE via the map $g(G,K) = \overline{G}, K$ because S is a clique in G ⇔ S is an ind-set in the complementary graph $\overline{G} = (V, \binom{V}{2} \setminus E)$. ∴ $3SAT \le_m^p$ CLIQUE too. $\boxtimes$