

"Main Landscape Theorems": For any time function $t(n) \geq nH$ and space $f(n) \geq \log_2(n)$:

① $DSPACE[s(n)] \subseteq NSPACE[s(n)] \stackrel{\text{today}}{\subseteq} DTIME[2^{O(s(n))}]$ (IA) $\stackrel{\text{Monday}}{\subseteq} DTIME[2^{O(t(n))}]$

② $NSPACE[s(n)] \subseteq DSPACE[O(s(n))^2]$ (Savitch's Theorem \rightarrow later) $DTIME[t(n)] \subseteq NTIME[t(n)] \subseteq DSPACE[O(t(n))]$ (IB)

③ Presuming $t_1(n)$ and $s_1(n)$ are "clockable" [i.e. time fully constructible and space fully constructible], if $t_1(n) \log t_1(n) = o(t_2(n))$ and $s_1(n) = o(s_2(n))$

Wed \rightarrow then $DTIME[t_1(n)] \not\subseteq DTIME[t_2(n)]$ and $DSPACE[s_1(n)] \not\subseteq DSPACE[s_2(n)]$.

Not Covered:

- Similar "Hierarchy Theorems" for $NTIME$ and $NSPACE$
- "Linear Speed-Up" — IMO it's a lame justification for last notation not a real phenomenon
- Alternation
- "Padding and Translation" — maybe.
- Theorem: $NLIN = NTIME[n+1]$ (Book-Greibach Thm)

note: $\log n \equiv (\log n)^2$

eg leaving out O , OK at base level wrap in \log symbols

$E = DTIME[2^{O(n)}]$

$EXP = DTIME[2^{n^{O(1)}}]$

Examples: $s(n) = \log_2 n$ then $2^{O(s(n))} = 2^{O(\log n)} = \bigcup_{c \geq 0} 2^{c \log n} = \bigcup_{c \geq 0} n^c = n^{O(1)}$

$\therefore NL \subseteq P$ (called NLIN, DLBA, NLBA)

• $t(n) = O(n)$: $DLIN \stackrel{\text{def}}{=} DTIME[O(n)] \subseteq NTIME[O(n)] \subseteq DSPACE[O(n)] \subseteq NSPACE[O(n)]$

These classes are not closed downward under \leq_m^P (nor \leq_m^{\log}). Their closures are the next sequence.

• $t(n) = n^{O(1)}$: $P = DTIME[n^{O(1)}] \subseteq NP \subseteq DSPACE[n^{O(1)}] \subseteq NSPACE[n^{O(1)}]$

Concat-sensitive Grammars: $\bigcup_{k \geq 1} DTIME[n^k]$ By \textcircled{a} , $NSPACE[n^{O(1)}] \subseteq DSPACE[n^{O(1)}]$

$\bigcup_{k \geq 1} DTIME[n^k + k] = DSPACE[n^{O(1)}] = DSPACE[n^{O(1)}]$

note: $\log n \equiv (\log n)^2$

eg leaving out O , OK at base level wrap in \log symbols

Book-Greibach Thm

whether NLBA = DLBA is older than whether NP = P. $\therefore NSPACE = DSPACE = PSPACE$

whether NL = L is part-way contingent on it. BUT: NLBA \subseteq DSPACE[$O(n^2)$] and NL \subseteq DSPACE[$O(\log n)^2$] are still best-known inclusions.

• $L \not\subseteq DSPACE[O(\log n)^2]$ But $DSPACE[O(\log n)^2] \subseteq DTIME[2^{O(\log n)^2}] = DTIME[n^{O(\log n)}]$

• $P \not\subseteq DTIME[n^{O(\log n)}]$ because for any k , $n^k \log n^k = o(n^{O(\log n)})$ so $\not\subseteq P$.

From \textcircled{a} , $NL \subseteq DSPACE[O(\log n)^2]$ and we don't know if this containment is proper. \textcircled{b} not known

• $DTIME[n^k] \not\subseteq DTIME[n^{k+1}]$ for all $k \geq 1$ and $P \not\subseteq DTIME[2^{O(n)}]$ but $PSPACE \subseteq DTIME[2^{O(n)}]$

Main Landscape Theorems: For any time function $t(n) \geq nH$ and space $f(n) \geq \log_2(n)$:

① $DSpace[s(n)] \subseteq NSpace[s(n)] \stackrel{\text{today}}{\subseteq} DTime[2^{O(s(n))}]$ (IA) $DTime[t(n)] \subseteq NTime[t(n)] \subseteq DSpace[O(t(n))]$ (IB)

Examples

- $t(n) = O(n)$
- These class
- $t(n) = P^n$

Proof of (IA): Let $A \in NSpace[s(n)]$, which means we can take an $s(n)$ space-bounded NTM N with read-only input tape such that $L(N) = A$, and st. N does "good housekeeping."

IDs I of N have the form $I = \langle q, x, h_1, \dots, h_k \rangle$
 $I_0(x) = \langle s, x, \epsilon, 1, \dots, 1 \rangle$
 where q is curr state, x is fixed so we can ignore, h_i is position of input tape head, w is contents of work-tapes, $h_2 \dots h_k$ are head positions on the work-tapes.

How big is the string encoding of I ? $O(n) + 0 + (\log_2(n+2)) + s(n) + k \log_2(s(n)) = O(s(n))$ since $s(n) \geq \log_2 n$.

Hence we can build a graph $G_x = \langle IDs, E \rangle$ where $(I, J) \in E \Leftrightarrow I \vdash J$, with source node $s = I_0(x)$ and (by GH) a unique target node $t = \langle q_{acc}, x, 1, \dots, 1 \rangle$.

Since each node has a string label of size $O(s(n))$ there are $2^{O(s(n))} = R(n)$ nodes total. BFS solving GAP for "Is there a path from s to t ?" works in time $O(R(n)^2) = 2^{O(s(n))}$ time again. The answer is yes iff $x \in L(N) = A$, so $A \in DTime[2^{O(s(n))}]$. \square

$\log n \equiv (\log n)^2$ note

Context-sensitive Grammars:
 Whether $ML \in P$?
 Whether $ML = L$?

• $L \subseteq DSpace[O(n)]$
 $E = DTime[2^{O(n)}]$
 $P \subseteq DTime[n^{1/2}]$
 $EXP = DTime[2^{O(n)}]$ From ② $NL \subseteq DSpace$
 • $DTime[n^2] \not\subseteq DTime$