

Linear Algebra of Quantum Computation: $n=1$ $N=2$ $n=2$ $N=4$ $n=3$ $N=8$

A pure quantum state ϕ of n qubits is notated as a vector of length $N=2^n$ of complex numbers (a_1, a_2, \dots, a_N) s.t. $|a_1|^2 + |a_2|^2 + \dots + |a_N|^2 = 1$.

When measured, ϕ outputs an index $i \in \{0, \dots, N-1\}$ with probability $|a_i|^2$. We call a_i the complex amplitude of the outcome i .

Let's index strings in $\{0, 1\}^n$ in standard order $0^n, 0^{n-1}1, \dots$ also called lex order.

Generally we have $\phi = \sum_{x \in \{0, 1\}^n} a_x e_x$

If $a_x = 1$, all other $a_y = 0$, the measurement always outputs x . Then ϕ equals the standard basis vector $e_x = (0, \dots, 0, 1, 0, \dots, 0)$ Also called $|x\rangle$ in Dirac notation.

Since $N=2^n$, this is exponentially big notation. How can it be compact? How to get big vectors from smaller components?

Tensor Product: $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 4 & 3 \end{pmatrix}$ General $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$

$A \otimes B = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 & a_1 b_4 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 & a_2 b_4 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 & a_3 b_4 \\ a_4 b_1 & a_4 b_2 & a_4 b_3 & a_4 b_4 \end{pmatrix}$

Examples with $e_0 = |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $e_1 = |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Tensor Products: General Idea: Indices get cross-producted, Values get multiplied.

General Rule: $(A \otimes B)(i, j) = A(i) \cdot B(j)$

For Matrices $C[i, k, j, l] = A[i, j] \cdot B[k, l]$

$A: m \times n$ $A[i, j]$ $B: r \times s$ $B[k, l]$ $C: m \times r \times n \times s$

eg if $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ then $A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & a_{13}B \\ a_{21}B & a_{22}B & a_{23}B \end{pmatrix}$ block matrix.

Examples with basis vectors, $n=1$ so $N=2$

$e_0 = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_0 \otimes e_0 = \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = e_{00} = |00\rangle$

$e_1 = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $e_0 \otimes e_1 = \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = e_{01}$

$e_1 \otimes e_0 = \begin{pmatrix} 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = e_{10} = |10\rangle$

$e_1 \otimes e_1 = \begin{pmatrix} 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = e_{11} = |11\rangle$

Matrix Example $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

Helpful Rule: If $A \cdot B$ and $C \cdot D$ are legal matrix products, Tensor Products then $(A \otimes C)(B \otimes D)$ is a legal matrix product and equals $(AB) \otimes (CD)$.

Unit vector means that $\|\phi\|_2 = \sqrt{\sum_{i=1}^n |a_i|^2} = 1$.

NOT the same as $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & CD \end{pmatrix}$ if A, B, C, D are all $N \times N$ then $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$ is $2N \times 2N$ not $N^2 \times N^2$

AB operation "evolves" separately from the CD operation. Hence tensor products also convey that the Similar, $\phi \otimes \psi$, where ϕ has N outcomes and ψ has R outcomes, convey the product distribution of $N \cdot R$ pairs of independent outcomes.

Fact: Then $\phi \otimes \psi$ is also a unit vector.

Defn: A quantum state ϕ is separable if there are smaller ϕ_1, ϕ_2 such that $\phi = \phi_1 \otimes \phi_2$. Otherwise, ϕ is entangled. However

Examples: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
 $= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

is entangled. $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} \therefore ac=1, ad=0, bc=1, bd=1$
 "The Basis" = $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$
 after John Bell. Aka EPR pair
 mean $a=0$ or $b=0$ so $ac=1, bd=1$ are not both possible

General Rule: eg. $A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$
 $A \otimes B = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \\ a_3 b_1 \\ a_3 b_2 \end{pmatrix}$
 "Shape of A multiplied content of B "

Examples with basis vectors $e_0 = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
 $e_1 = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $e_0 \otimes e_0$
 "ket" $e_1 \otimes (1,0) p_1 = \dots$