CSE610 Week 4: Qubit Representations and Physical States

A **qubit** is a physical system whose **state** ϕ is described by a pair (a, b) of complex numbers such that $|a|^2 + |b|^2 = 1$. The components of the pair *index* the *basic outcomes* **0** and **1**. There are two ways we can gain knowledge about the values a and b:

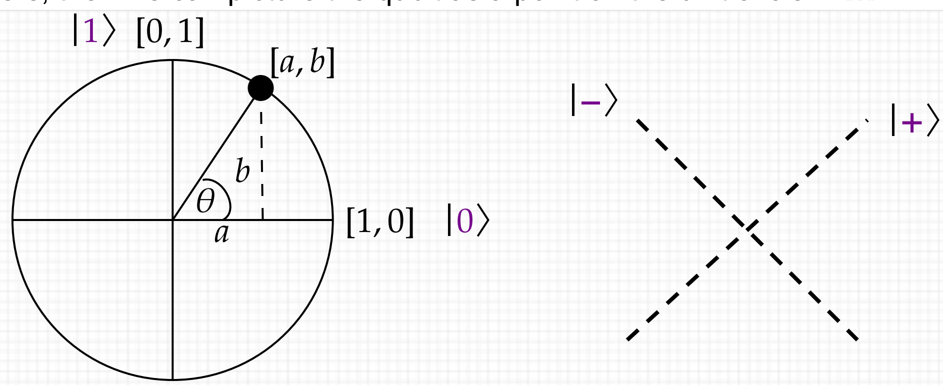
- We can **prepare** the state from the known initial state $e_0 = (1, 0)$ by known quantum operations, which here can be represented by 2×2 matrices.
- We can **measure** the state (with respect to these basic outcomes), in which case:
  - We either **observe** **0**, whereupon the state becomes $\mathbf{e_0}$, or we observe **1**, in which case the state becomes $\mathbf{e_1} = (0, 1)$.
  - The probability of observing **0** is $|a|^2$, of getting **1** is $|b|^2$. This is called the **Born Rule**, after Max Born.

If both a and b are real numbers, then we can picture the qubit as a point on the unit circle in $\mathbb{R}^2$:

$$|a|^2 + |b|^2 = 1$$

$$a = \cos\theta$$

$$b = \sin\theta$$



(transpose notation omitted here)

The qubit state $\phi = [a, b]$ represents $ae_0 + be_1 = a[1,0] + b[0,1] = a|0\rangle + b|1\rangle$

If $\theta = \frac{\pi}{3}$ then $\cos\theta = \frac{1}{2}$, so $|a|^2 = 0.25$. And $\sin\theta = b = \frac{\sqrt{3}}{2}$ so $|b|^2 = 0.75$. Note that $a = \langle\phi|0\rangle$ and $b = \langle\phi|1\rangle$. What the measurement does is **project** onto the standard basis.

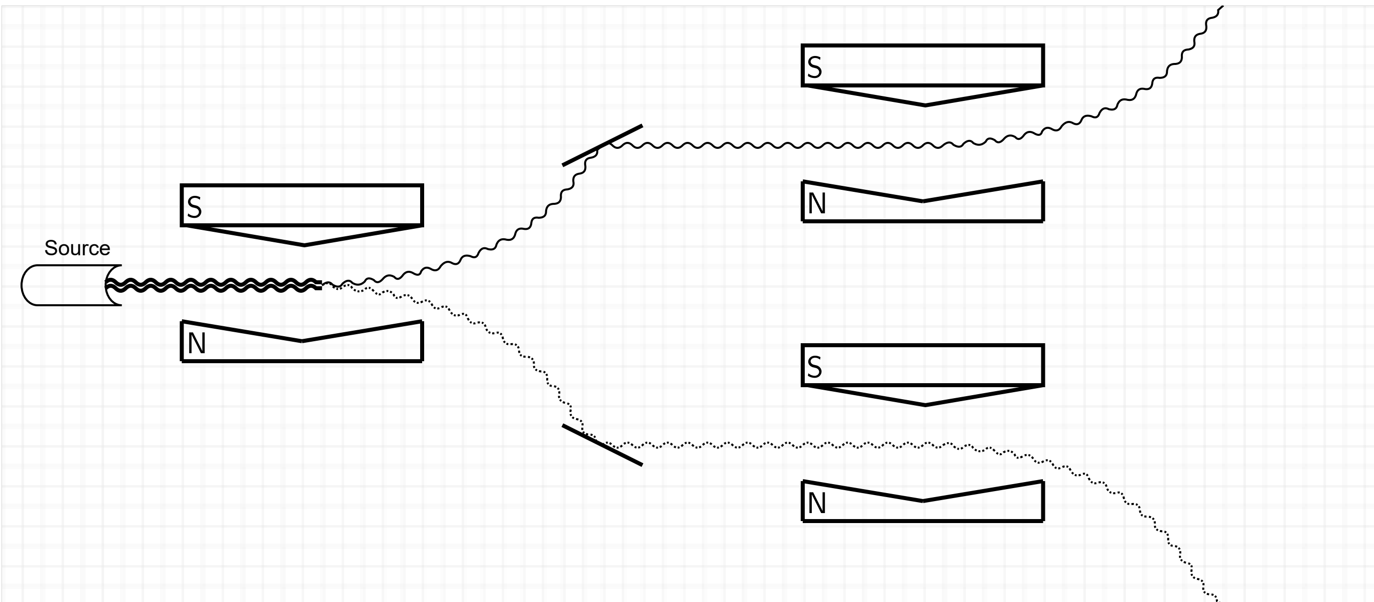
We can get different probabilities by projecting onto a different basis. Note that

$$\langle\phi|+\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\cdot 1 + \frac{\sqrt{3}}{2}\cdot 1\right) = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{2.732...}{2.828...} = 0.9659...$$

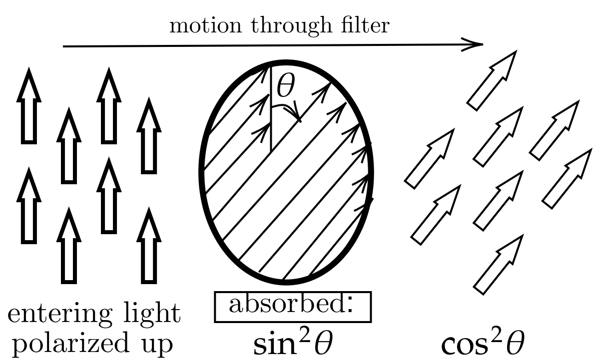
and squaring that gives just over 0.933. Thus, this particular quantum state ϕ gives a higher probability of one result when measured in the $|+\rangle, |-\rangle$ basis---and a near-zero probability of the other result.

What *happens* to ϕ after a measurement? The full picture is much debated, but the local happening is clear: ϕ becomes the basis state corresponding to the result obtained. The fact that we---humans---can *elect* to **measure in** a particular choice of basis will be a major component of quantum communication protocols and the **CHSH Game** on-tap later in Chapter 14. The "election" part is as easy as twirling a polaroid filter (if that is *free will*, mind you).

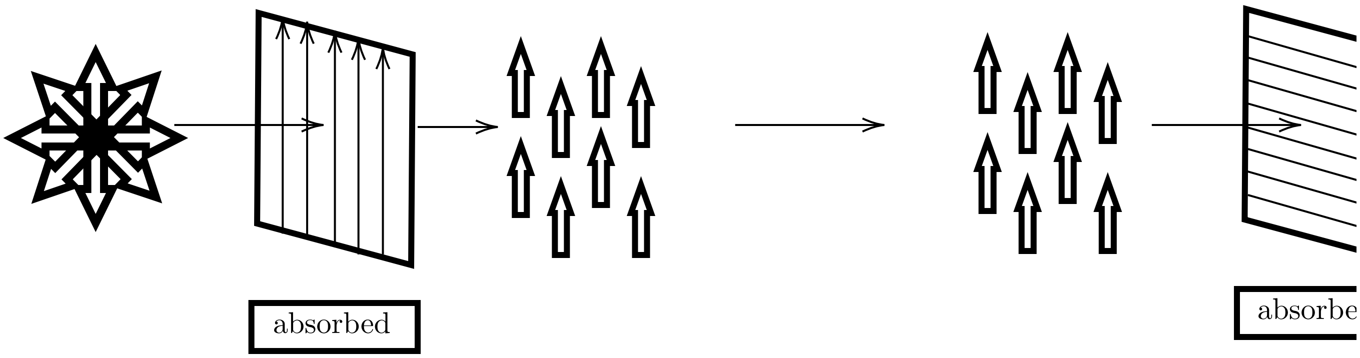
(1) That the particles' states become basis states in the particular measurement frame is shown by the **Stern-Gerlach experiment**. In the setup, the measurable physical state "spin up" is denoted by $|\uparrow\rangle$ and can be treated like $|0\rangle$. There is a distinct physical state called "spin down" and denoted by $|\downarrow\rangle$, which plays the role of $|1\rangle$. These are the only two distinguishable outcomes that manifest when a magnetic field acts on the particle (relative to the orientation of the field; incidentally, "spin" is not-rotation per-se). Once a particle "chooses" between $|\uparrow\rangle$ or $|\downarrow\rangle$, that is its state upon going through a second Stern-Gerlach device with the same orientation.

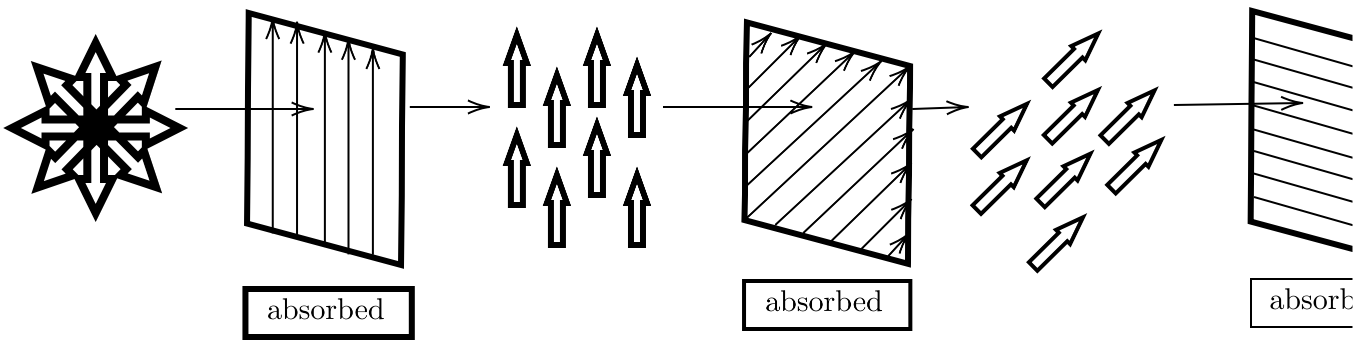


(2) But if the second device changes the orientation, then the particles once again behave nondeterministically with respect to the changed orientation. This is shown more cheaply using polarizing filters, except for not being able to identify the particles (of light) individually.



The individual photons do **not** "lose mojo" after their orientation "*collapses*" onto the basis state. It appears that way because of the physical fact that those photons giving the opposite outcome are absorbed by the filter.





In the second situation, the first filter produces light that is polarized up. The second filter absorbs $\cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}$ of that light and the other $\frac{1}{2}$ is passed through with diagonal polarization (analogous to the $|+\rangle$ basis state). The third filter absorbs $\frac{1}{2}$ again of that light. Positioning the middle filter at any angle θ between 0 and $\frac{\pi}{2}$ allows $\cos^2(\theta) \cdot \sin^2(\theta)$ of the light from the first filter to go through. This goes to zero as θ approaches either 0 or 90° and is maxed for $\theta = 45^\circ$. The Born Rule in action!

For most work with quantum circuits, we may suppose that a single measurement is taken at the end, and the output is read from the basis state $|y\rangle$ that is returned. Or we may run a circuit multiple times, thus **sampling** y from the output distribution. The **principle of deferred measurement**, which we will see soon in Chapter 6, makes this be "without loss of generality" in most computing situations. Quantum communication protocols, however, require a fuller formulation of measurement via linear algebra. This will come hand in hand with **mixed states**, which "are" classical probability distributions over unit vectors which are quantum **pure states**. Doing this is facilitated by the **Bloch Sphere** representation of qubits.

The Bloch Sphere

Last Thursday's lecture showed the limitations of the Cartesian picture for viewing even the simple computation $[a,b]^T = \mathbf{HTH}|0\rangle$. So we will study one that gives a different picture of physical reality.

The first point is that the complex numbers $a = x + iy$ and $b = u + iv$ involve 4 real numbers, but the requirement $|a|^2 + |b|^2 = 1$ imposes one constraint, thus essentially cutting the "real degrees of freedom" down to 3. A second factor cuts it down to 2. The following definition will be useful for quantum states of multiple qubits as well:

Definition: Two quantum states ϕ, ϕ' are **equivalent** if there is a unit complex number c such that $\phi' = c\phi$.

For example, $\frac{1}{\sqrt{2}}(-1,1)$ is equivalent to $\frac{1}{\sqrt{2}}(1,-1)$, but neither is equivalent to $\frac{1}{\sqrt{2}}(1,1)$, nor any of these to our basic states $(1,0)$ and $(0,1)$. In the line for the matrix $\mathbf{Y}$, $i\mathbf{e}_1$ is simply equivalent to just $\mathbf{e}_1$, $-i\mathbf{e}_0$ to $\mathbf{e}_0$, $-i\mu$ to μ, and $i\pi$. We could also regard $\mathbf{Y}$ as equivalent to

$$i\mathbf{Y} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which makes clearer that it is a combination of $\mathbf{X}$ and $\mathbf{Z}$ (indeed, $i\mathbf{Y} = \mathbf{ZX} = -\mathbf{XZ}$). Finally, to finish the line for $\mathbf{Z}$, $\mathbf{Z}\mathbf{e}_1 = -\mathbf{e}_1 \equiv \mathbf{e}_1$.

Regarding our saying *equivalence*, note that if $c = a + bi$, then

$$\frac{1}{c} = \frac{1}{a+bi} = \frac{a - bi}{(a+bi)(a-bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a - bi}{1} = a - bi = \bar{c}\,,$$

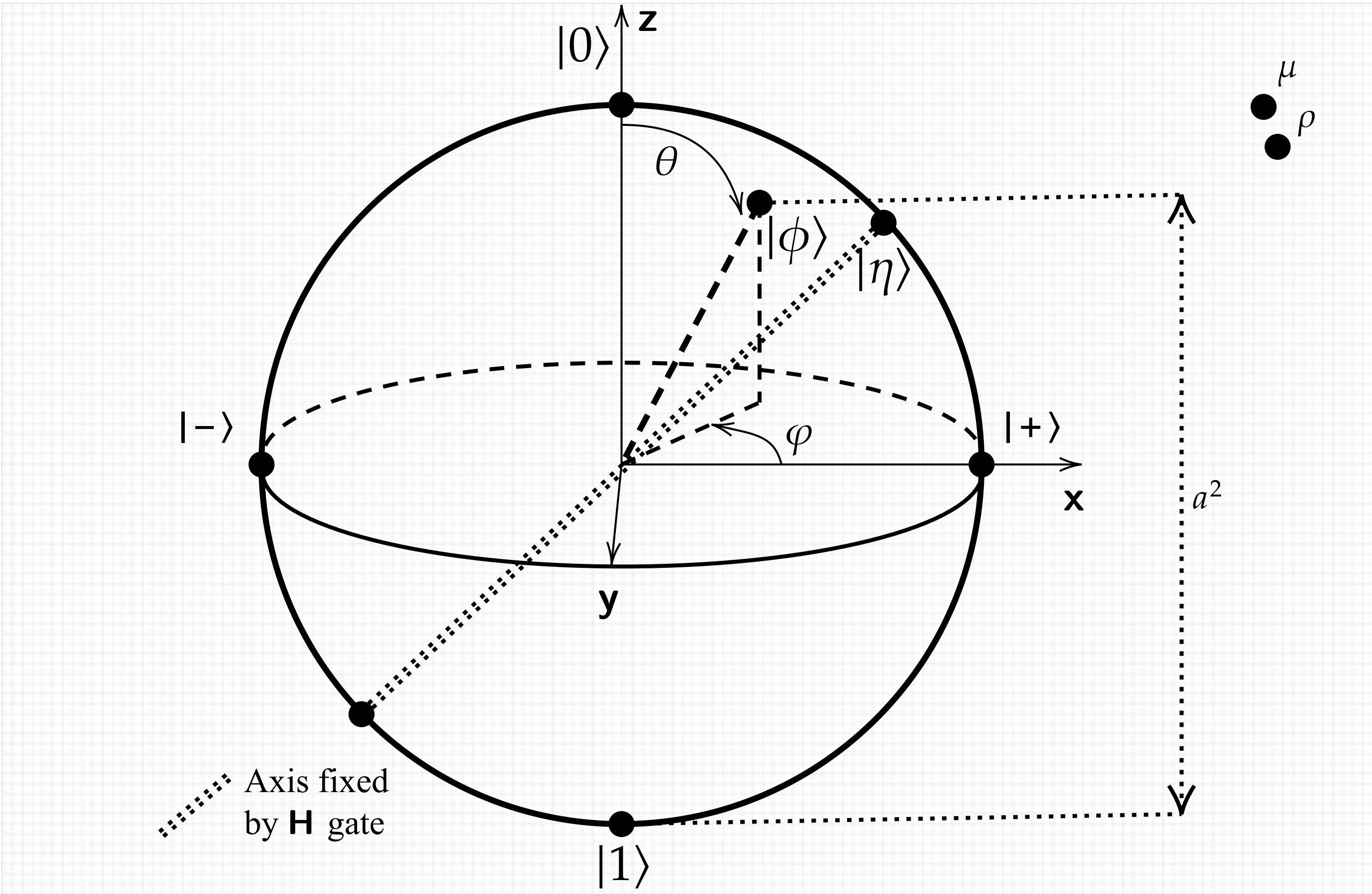
which is the **complex conjugate** of c and is likewise a unit complex number. Since $\phi = \bar{c}\phi'$ the relation is symmetric. That the product of two unit complex numbers is a unit complex number makes it transitive, and being reflexive is immediate with $c = 1$, so this is an equivalence relation.

A unit complex number can be written in polar coordinates as $c = e^{i\gamma}$ for some angle γ, which represents a "global phase." Thus, dividing out by this equivalence relation emphasizes the **relative phase** φ of the two components. So let us write our original quantum state ϕ in polar coordinates as $(ae^{i\alpha}, be^{i\beta})$ where now a, b are real numbers between 0 and 1. Choose $\gamma = -\alpha$, then $c\phi = (a, be^{i\varphi})$ with $\varphi = \beta - \alpha$. Since $a^2 + b^2 = 1$, the value of b is forced once we specify a. So a and φ are enough to specify the state. These are the 2 true degrees of freedom.

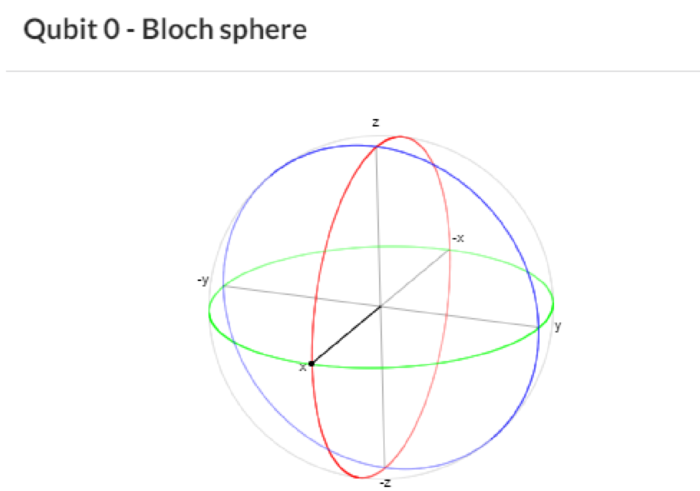
We can uniquely map points (a, φ) to the sphere by treating φ as a longitude and a^2 (rather than a) as a latitude where the north pole is 1, the equator is 0.5, and the south pole is 0. Then the latitude gives the probability of getting the outcome 0. All states that give equal probability of 0 and 1 fan out along the equator. The north pole is $|0\rangle$ and the south pole is $|1\rangle$. And again:

- $\frac{1}{\sqrt{2}}(1,1) \;=\; \frac{1}{\sqrt{2}}\big(|0\rangle \;+\; |1\rangle\big)$ is called $|+\rangle$, the "plus" state.
- $\frac{1}{\sqrt{2}}(1,-1) \;=\; \frac{1}{\sqrt{2}}\big(|0\rangle \;-\; |1\rangle\big)$ is called $|-\rangle$, the "minus" state.

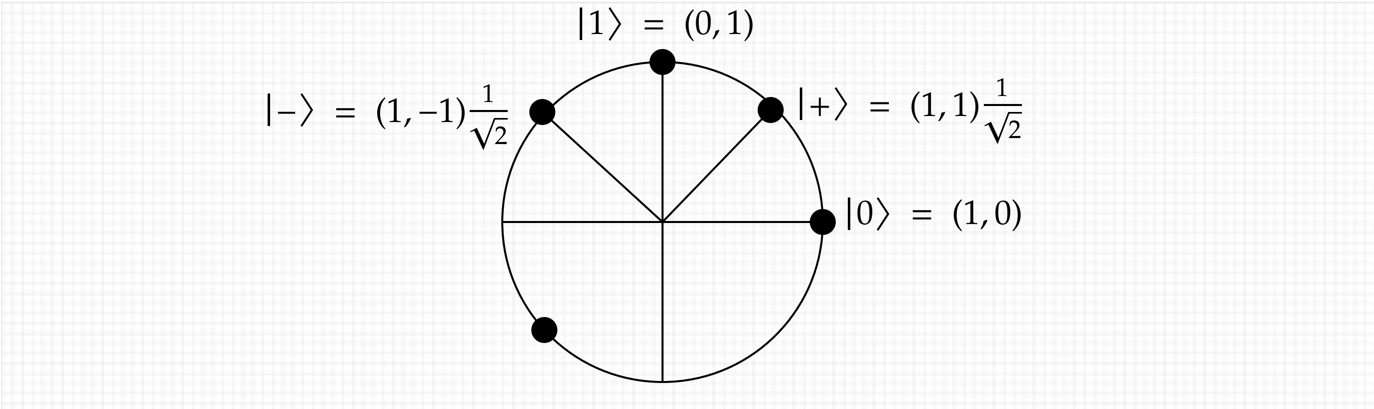
Here they all are, graphed on the **Bloch Sphere**:



Among web applets displaying Bloch spheres for qubits is https://quantum-circuit.com/home (free registration required). Here is its graph for the $|+\rangle$ state. It is more usual to show the **x** axis out toward the reader and **y** at right, but that is less convenient IMHO for picturing $|+\rangle$ and $|-\rangle$.



Some algorithms, however, are IMHO easier to picture using the original planar diagram:



For one thing, this makes it easier to tell that $|0\rangle$ and $|1\rangle$ are orthogonal vectors, that $|+\rangle$ and $|-\rangle$ are likewise orthogonal vectors, and that the orthonormal basis $\{|+\rangle, |-\rangle\}$ is obtained by a linear transformation (indeed, a simple rotation) of the standard basis $\{|0\rangle, |1\rangle\}$.

A downside, however, is that this diagram gives extra points for equivalent space, whereas the Bloch sphere is completely non-redundant. The Bloch sphere is also "more real" than the way we usually graph complex numbers via Cartesian coordinates. In fact, *every unitary 2×2 matrix U induces a rotation of the Bloch sphere and hence fixes an axis, so the axes of the sphere are in 1-to-1 correspondence with lossless quantum operations on a single qubit.* Whereas, the planar diagram gives a cut-down picture of how $\mathbf{H}$ acts as a rotation without fully showing you its axis.

The axis of the $\mathbf{H}$ gate goes through the origin and the point corresponding to the pure state $|\eta\rangle = \left[cos\frac{\pi}{8}, \sin\frac{\pi}{8}\right]$. With this vector, the latitude is $\cos^2\left(\frac{\pi}{8}\right) = 0.85355339...$ That's the number we got from the $\mathbf{HTH}$ computation. Note: the latitude looks like it should be "3/4" but it's not. The equator is 0.5 and the diagonal point is $\frac{1}{\sqrt{2}}$ of the way up from equator to the pole, so the latitude is $0.5 + 0.5\frac{1}{\sqrt{2}} = 0.85355339...$ as required.