CSE610 Week 8: The CHSH Game and Bell's Theorem

First, a summary of recent lectures. A key property of the traceout operation is that the sum of each slot where an identity-matrix "stencil" would be placed to "trace out Bob" is preserved by any unitary operation $\mathbf{U}$ that Bob applies to his qubits---i.e., to his part of the joint system. This is an extension of the fact that the operation $\rho \to \mathbf{U}\rho\mathbf{U}^*$ preserves the trace of ρ when $\mathbf{U}$ has the same dimension as ρ. The conclusion that none of Bob's unitary actions change her density matrix is called the **No-Communication Theorem** in Aaronson's notes.

- If Alice has no entanglement with Bob, this is obvious: if there is never any communication from Bob to Alice at all, there is just no effect.
- The point is that this is true even if they share however many entangled qubits (or qutrits etc.): absent any communication, Alice cannot detect any unitary actions by Bob.

If Bob *measures*, however, then there is an instantaneous effect on Alice's density matrix. For an entangled Bell pair, Alice's density matrix $\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ becomes one of the following, depending *both* on Bob's actions *and* on Bob's results:

1. If Bob uses the $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ measurement and gets $|0\rangle$, Alice has $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
2. If Bob uses the $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ measurement and gets $|1\rangle$, Alice has $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
3. If Bob uses the $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ measurement and gets $|+\rangle$, Alice has $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$.
4. If Bob uses the $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ measurement and gets $|-\rangle$, Alice has $\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$.
5. If Bob uses $\{|\kappa\rangle\langle \kappa|, |\lambda\rangle\langle \lambda|\}$ where $|\kappa\rangle = [0.6, 0.8]^T$ and $\lambda = [0.8, -0.6]^T$, then Alice has either $\begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}$ or $\begin{bmatrix} 0.64 & -0.48 \\ -0.48 & 0.36 \end{bmatrix}$, again depending on Bob's outcome.

If Bob measures in the standard basis and gets an unusually large number of $|1\rangle$ results on many qubits shared with Alice---say 70 out of 100---and if Alice measures in the same basis, then Alice will learn that Bob got an unusual statistical outcome. If Alice measures in $\{|+\rangle\langle +|, |-\rangle\langle -|\}$, however, then she will remain ignorant of Bob's stroke of luck. Yet even if Alice does know in advance that Bob measures all the qubits first (for those cognizant of relativity, this means "first" as judged in Alice's reference frame) in the standard basis, she does not get any willful message from Bob---she merely learns the chance results he got.

The question that concerned Einstein is whether Bob can send a willful message to Alice through their entanglement by choices of measurement bases. My use of "willful" here is willful: *pace* quantum-based arguments against free will, it is IMHO the clearest way to frame the technical argument. All

agree that Alice gains *information* of Bob's random outcomes, though that information was "pre-paid" by
the interactions that set up n entangled qubits to begin with. The point of superdense coding is that Bob could distinguish among $\mathbf{4}$ willful actions by Alice *after* the initial exchange of one entnagled qubit, when it was *followed by* her sending $\mathbf{1}$ other qubit. Can something like this be done *without any further interaction*---and over time intervals shorter than the time for light to travel between Alice and Bob?

Most in particular, can Alice gain any willful information---other than unstructured randomness---from how Bob orients his measurements? The answer is no. If they share $|00\rangle + |11\rangle$ (over $\sqrt{2}$) you might think Bob could guarantee a '1' by measuring in the $|+\rangle, |-\rangle$ basis, but no: that was the first decoherence example with Alice. Any basis Bob uses is the same as a unitary $\mathbf{U}$ to convert to the standard basis followed by a measurement there, and $\mathbf{U}$ has no effect on what Alice will see.

This makes it all the more amazing that there are situations where willful choice of measurement basis does make a difference when there is followup classical communication---a difference that has been quantified in actual experiments.

The CHSH Protocol

Alice and Bob share n Bell pairs and can have as much prior classical communication to agree on strategies as they please. Between the start and end of a *trial*---one play of the game---they may not communicate with each other, but they may observe common sources. The common source can not only be random---such as from patterns of solar flares both Alice and Bob can see---it can be controlled by an oracle "Ozzie" who is trying to help Alice and Bob. Each trial operates via classical communication with a third party, "Ralph" (to sound like ref, referee) and goes like this:

1. Ralph sends a random bit a to Alice and a bit b to Bob. Neither can see the other's bit.
2. Alice sends a response bit u to Ralph and Bob simultaneously sends his response bit v to Ralph.
3. Ralph declares that Alice and Bob win the trial if $u \oplus v \;=\; a \wedge b$.

We may suppose that Alice and Bob receive a and b in sealed boxes, and give their respective u and v within a nanosecond of opening their boxes. Without loss of generality, we may suppose that any other influence from observations or "Ozzie" has been registered by that instant. At that point, Alice's u is a one-bit Boolean function of a alone. We use $0, 1$ for the inputs to this function but give the outputs as **Y** for "yes" or **N** for "no" in order to keep inputs and outputs visually separate. There are just four functions that she can use:

- The always-true function: yes to 0 and yes to 1, which we call **YY**.
- The always-false function, which we similarly call **NN**.
- The identity function, giving Ralph the same bit back, which is **NY**.
- Flipping Ralph the bit, which is **YN**.

Bob has the same four options, so there are in total **16** different strategies they can use for any trial. Meanwhile, Ralph has his own four possible actions. Here is the entire matrix of possibilities. The matrix entries are numeric rather than Boolean: 1 if Alice and Bob win, 0 if they lose. The rows are the four options by Ralph, in order a, b so that for instance, if Alice and Bob adopt the strategy in the third column and find that Ralph chose $1, 0$, then Alice says $\mathbf{N}$ while Bob says $\mathbf{Y}$---and they lose because their answers disagreed while $1 \wedge 0$ is false.

Alice	NN	NN	NN	NN	NY	NY	NY	NY	YN	YN	YN	YN	YY	YY	YY	YY
Bob	NN	NY	YN	YY	NN	NY	YN	YY	NN	NY	YN	YY	NN	NY	YN	YY
0,0	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
0,1	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
1,0	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
1,1	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0

Note that when Ralph plays randomly, Alice and Bob can assure 75% winning if they choose any of the eight columns with three 1s as their joint strategy. **They cannot do better**, because every column has a case where Ralph could send them something that makes their joint strategy lose---and the randomized Ralph does so with 25% probability.

The amazing fact is that sharing one entangled Bell pair enables Alice and Bob to do much better: to win **over 85%** of the time.

A side note if you are familiar with matrix game theory: could Ralph do better with non-random play if he
knew Alice and Bob's strategy? Certainly if Alice and Bob always play one fixed column, such as both always saying $\mathbf{N}$, then Ralph could always deny them by giving the one losing combination a, b. If you know the **Minimax Theorem** of zero-sum matrix game theory, then you already know that because 25% is Ralph's optimum when he has to move first, Alice and Bob *must* have a *classically randomized* strategy that assures them 75% even if Ralph is told about it in advance. We can find it easily by first removing the eight "obviously stupid" joint strategies---those with only one 1 in their column---leaving:

Alice	NN	NN	NY	NY	YN	YN	YY	YY
Bob	NN	NY	NN	YN	NY	YY	YN	YY
0,0	1	1	1	0	0	1	1	1
0,1	1	0	1	1	1	1	0	1
1,0	1	1	0	1	1	0	1	1
1,1	0	1	1	1	1	1	1	0

Now if Alice and Bob use their shared classical randomness to choose one of the leftover strategies at random with probability $1/8$, there is no way Ralph can avoid their winning 75% of the time even if Ralph knows that is their policy. If Ralph could steal their random bits by looking at solar flares *and* knowing *how* and *when* Alice and Bob will decode them, then Ralph could still always send the bad combo. But the order is: Ralph commits to the a, b combo first, then Alice and Bob have a moment to read the shared random source that determines their policies before they open their boxes.

The scientific significance does not require this detail---we just stipulate that Ralph plays randomly.

The Quantum Case

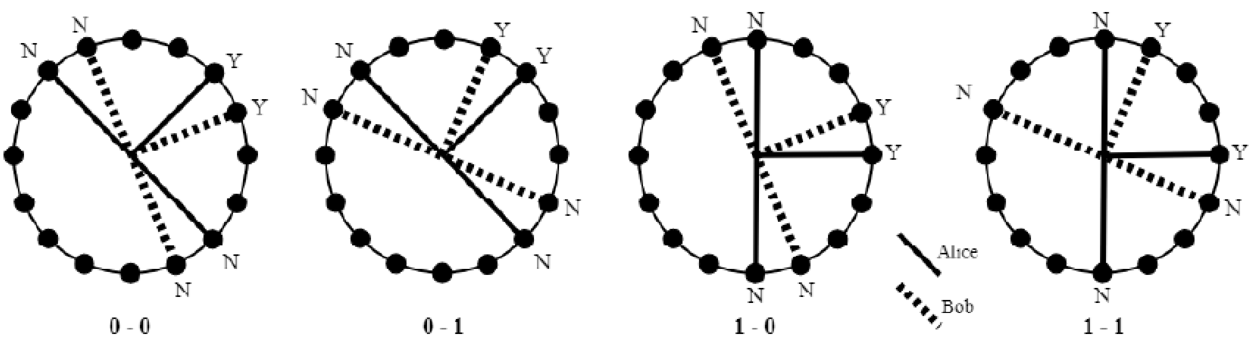
Alice and Bob get an extra option using one shared Bell pair per trial: Each can measure ina basis that depends on the bit received from Ralph. The timing of this option is synchronized as viewed by Ralph. The text describes Alice as measuring first, but we'll make Bob go first for consistency with recent lectures. By symmetry, it does not matter who goes first. What does matter, technically, is that the time lapse from opening the boxes to the second measurement---as viewed by Ralph---must be less than the time it would take light to travel from Bob to Alice. This is in order to avoid one of several possible "loopholes" that could enable a classical explanation.

Rather than the Bloch sphere, this is a case where the Cartesian diagram of state vectors is best for visualization: $|0\rangle$ at east (**E**), $|1\rangle$ at north (**N**), $|+\rangle$ between them facing northeast (**NE**), and $|-\rangle$ to the southeast (**SE**). Alice will use either the $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ or $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ measurement. We let the former outcomes stand for "yes", so we can abbreviate her options as **E** or **NE**. Bob has a funkier set of bases to choose from. He can use the basis that orients his "yes" answer at 22.5°, which we call **ENE** for east-northeast, and puts "no" at 112.5° (or equivalently, at -67.5°, i.e., 292.5°). Or Bob can use the basis that puts "yes" at $+67.5^\circ$, which is **NNE** for north-northeast. Here is the protocol:

1. Alice and Bob open their boxes simultaneously.
2. If $b = 0$, Bob measures his entangled qubit in the basis oriented **ENE**; if $b = 1$, Bob chooses **NNE**.
3. If $a = 0$, Alice instantly chooses **NE**, else she chooses **E**. No more than a nanosecond later than Bob's actions, Alice measures her qubit in her chosen basis.
4. Each sends Ralph "yes" if **ey** received the measurement outcome designated "yes", else "no".

(In the great pronoun debate, I've supported the "Alternative" version {*ey, em, eir, emself*} of the pronouns in Michael Spivak's 1979 handwritten first volume of his *Differential Geometry* text, which I had in a course that year. Those had actually been proposed by Christine Elverson of Chicago in 1975. But I chickened out of using them in my textbook with Lipton, and Wikipedia doesn't say they appeared in Spivak's 1983 published version, only that he used them in a later handbook on LaTeX.)

The upshot can be appreciated ahead of any thinking about the underlying physical reality, just by looking at the diagram of the choices made by Alice and Bob in the four cases Ralph can send them:



Alice's chosen orientations depend only on her bit from Ralph: northeast on 0, due east on 1. Bob likewise reacts independently of Alice. Yet the options combine to make their "yes" orientations come within 22.5° of each other in all of the 00, 01, and 10 cases from Ralph, yet 67.5° apart on 11.

If being one-fourth of a right angle apart meant a one-fourth chance of losing, then the resulting chances would be no different from the classical case: 75% frequency of winning. But in ways we can actually see for ourselves by orienting polarizing filters at these angles and telling how much light gets through, in the first three cases, the chance of her qubit instantly transformed(?) by Bob's outcome giving the same yes/no answer from her 22.5°-apart measurement is greater: $\cos^2\left(\frac{\pi}{8}\right) = 85.3553...\,\%$. And in the fourth case, the frequency of Alice and Bob giving different answers and winning is the same.

Well, saying "transformed" is exactly the kind of *spukhafte Fernwirkung* that Einstein objected to. But this is the straightest path to expressing the explanation for what we observe---which has been verified in actual experiments achieving over 80%. The gap between 80% and 85+% is ascribable in substantial part to the kind of slight-degrading errors we saw in the "depolarization and de-phasing" section (plus to other slight flaws in the apparatus and its nanosecond timing).