CSE610, Fall 2021 **Problem Set 2** **Due Thu. Oct. 21**

The problems employ a simple plain-text notation for describing quantum circuits, placing gates in left-to-right order on some fixed number n of qubits. The qubits themselves are numbered here from 1, whereas several popular simulators number them from 0. The circuits themselves do not care whether the qubits are ordered in "big-endian" fashion as in the text and Wybiral's simulator (https://wybiral.github.io/quantum/) or "little endian" as in *Quirk* (https://algassert.com/quirk). In big-endian notation numbering from 1, the basis state $|011\rangle$ means that qubit 1 has value $|0\rangle$ while qubits 2 and 3 have value $|1\rangle$. In little-endian numbering from zero, and speaking of the qubits as "quantum registers" $q_0, q_1, q_2, \ldots$, we would say $q_0 = |1\rangle$, $q_1 = |1\rangle$, and $q_2 = |0\rangle$. Output from *Quirk* can be viewed either way by grabbing the "Reverse" widget from lower left in the "Order" section, but after trying it I've concluded that this can cause more confusion than it is worth—mainly because the amplitude display at right stays little-endian.

(1) Text, exercises 6.9 and 6.10 on page 70. Include the computation of **HSHS*** from problem 3.13 as part of what you show. (24 pts. total)

(2) Design quantum circuits that given the all-zero basis state as input create the states

(a) $\frac{1}{2}(|000\rangle + |001\rangle + |010\rangle - |111\rangle)$, and

(b) $\frac{1}{2}(|000\rangle + |001\rangle - |010\rangle - |111\rangle)$.

You may check your work with a quantum circuit simulator—recall that multiplying everything by any unit scalar, in particular -1, gives the same quantum state. For a warmup, note that the 2-qubit circuit `H 1 H 2 CZ 1 2` applied to $|00\rangle$ produces the state $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$. This is plain-text notation for the circuit with Hadamard gates on qubits 1 and 2 followed by a **CZ** gate between them. $(9 + 12 = 21$ pts.)

(3) Design a 4×4 unitary matrix U such that $U|00\rangle$ equals the state $|\phi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle$. For full credit, make U Hermitian as well as unitary. $(9 + 3 = 12$ pts. Note the difference from the state given as warmup in problem (1).)

(4) A progression of ideas, taking the state $|\phi\rangle$ in problem (2) as the springboard.

(a) Spend some time trying to design a quantum circuit C of the basic gates seen so far in the text and notes that computes your transformation U in problem (2), or at least such that $C|00\rangle = |\phi\rangle$. (If you succeed, open-ended extra credit. Else, for a paragraph explaining what you tried and why you think it might be impossible, 6 pts.; 12 points extra if you find a proof)

(b) Show that you stand a 75% chance of creating that state on the first two qubits if you run the circuit `H 1 H 2 Tof 1 2 3` on input $|000\rangle$ and measure qubit 3. Do so formally by noting that the two projection operators corresponding to the third-qubit measurement (in big-endian notation) are:

$$P_0 = |000\rangle\langle 000| + |010\rangle\langle 010| + |100\rangle\langle 100| + |110\rangle\langle 110|$$
$$P_1 = |001\rangle\langle 001| + |011\rangle\langle 011| + |101\rangle\langle 101| + |111\rangle\langle 111|$$

Use Definition 14.3 on page 146 to verify both that the probability of the outcome for P_0 is $p_0 = 0.75$ and that the resulting state is $|\phi\rangle \otimes |0\rangle$. (9 pts.)

(c) **Postselection** is a "super-natural" operator that allows assuming the result of a measurement. The *Quirk* simulator provides it as an operator on a single qubit at upper left in the "Toolbox"—with the suggestive labels $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. Show the circuit in *Quirk*, first using P_1 to post-select the outcome $|1\rangle$, and then using P_0 to post-select on $|0\rangle$. In the latter case, note that the amplitude display at right is little-endian, so in particular, what it shows as $|001\rangle$ is $|100\rangle$ in the notation above. Show your verification from the amplitudes that post-selecting on the $|0\rangle$ outcome really yields the state $|\phi\rangle \otimes |0\rangle$ in the simulator. (6 pts., for 21 total. This is really like a work-through tutorial of an example.)

(5) Diagram the graph-state circuit C_G corresponding to a three-node undirected graph G with edges $(1,1),(1,2),(2,3)$. (It looks like a lollipop with a stick and a self-loop at the top.) You can draw it by hand or take a screenshot from any of the quantum circuit simulators we've discussed, mindful of any indexing differences. Then do the following:

(a) Show that the graph G is "net-zero," meaning that $\langle 000 \,|\, C_G \,|\, 000\rangle = 0$. Do so without any 8×8 matrix-vector multiplications by using the "maze" visualization from lectures. You do not need to diagram the parts of the maze for the initial and final $\mathbf{H}^{\otimes 3}$ transforms, only the three middle sections for the three edges. Show where the phase flips (-1 entries) happen in each of the eight rows and three columns. Staring with eight "mice" all having $+1$ phase at left, show how many mice get flipped to -1 phase after the $\mathbf{Z}$ gate and the two $\mathbf{CZ}$ gates—and this will tell you how far they cancel (when they would all reunite at $z_0 = |000\rangle$ at upper right if we drew the right-hand $\mathbf{H}^{\otimes 3}$ maze gadget).

(b) Now let us reckon it a different way. Consider the eight possible ways to color each vertex B or W. Call a coloring "even" if it makes an even number of B-B edges (counting zero as an even number) and "odd" otherwise. Note that in G, the coloring BWW counts as odd because the loop $(1,1)$ then counts as a B-B edge. Show how the odd-even colorings correspond to the $+1$ and -1 phases in the rows corresponding to the eight basis states.

(c) Argue that in general for any n-node graph G, not just this one, that the amplitude of $\langle 0^n \,|\, C_G \,|\, 0^n\rangle$ equals the number of even colorings minus the number of odd colorings, divided by 2^n. Do so by identifying each coloring with a basis state and saying why each B-B edge gives a multiplier of -1 from the corresponding $\mathbf{CZ}$ gate (or $\mathbf{Z}$ gate in case of a loop edge).

(d) Now move the loop from vertex 1 to vertex 2 in the middle. Is the resulting graph G'' still net-zero? Give the amplitude value $\langle 000 \,|\, C_{G''} \,|\, 000\rangle = 0$. $(6 + 6 + 12 + 6 = 30$ pts.)

(6) Let us take any n-node undirected graph $G = (V, E)$ and vertex u of G and add a "stick" at u using two new vertices v, w and two new edges (u, v) and (v, w). The resulting graph $G' = (V', E')$ has $V' = V \cup \{v, w\}$ and $E' = E \cup \{(u, v), (v, w)\}$. An example is that the "lollipop" graph in problem (4) was obtained by adding the "stick" of vertices 2 and 3 to the simple loop graph on one vertex—which we saw in notes is net-zero.

Show the following identity for the amplitude: $\left\langle 0^{n+2} \,\middle|\, C_{G'} \,\middle|\, 0^{n+2}\right\rangle = \frac{1}{2}\langle 0^n \,|\, C_G \,|\, 0^n\rangle$. Conclude that G' is net-zero if and only if G is. (Hint: Consider separately the colorings χ of the original graph G that make $\chi(u) = B$ and those that make $\chi(u) = W$. Show in each case how many colorings of the extra nodes v, w flip the parity of B-B edges. 24 pts., making 132 total on the set. For up to 18 pts. extra credit, assuming nodes v, w are numbered $n+1$ and $n+2$, prove the answer to whether we get $\langle x00 \,|\, C_G \,|\, x00\rangle = \frac{1}{2}\langle x \,|\, C_{G'} \,|\, x\rangle$ for all $x \in \{0,1\}^n$.)