(0) Not graded—a challenge for class discussion: The first two pages of my "Week 9" notes add a proof of the $\Theta(n^2)$ expected time for a uniform random walk to travel from node 0 to node n in the finite path graph P_n of n edges and $n+1$ nodes. They then redo the recursion for the case where the graph is infinite to the left of 0, in the direction away from node n. What can you say about the expected time then? How about the expected time to reach either of node $-n$ or node $+n$? What if the graph has a $0.5+\epsilon$ bias toward moving right? Or if the bias at node $\pm m$ is away from the origin node 0 when $m \leq n/2$ but toward 0 when $m > n/2$? What if the graph has other edges—do they give similar effect to the latter bias?

(1) This problem continues (5) and (6) from the previous set. All parts are about a general n-qubit graph-state circuit C_G where G is an undirected graph on n nodes, possibly allowing self-loops but not multiple edges. They also refer to the "maze diagram" visual-aid used in lectures. For better visual clarity we abbreviate $\langle 0^n|C_G|0^n\rangle$ as $\langle 0^n|G|0^n\rangle$ and so on.

(a) Consider inputs x to C_G other than 0^n. Let the rows be indexed by binary strings $u \in \{0,1\}^n$ that of course may be different from x. Give the rule for the middle section of row u to begin with a -1 phase. You may find the text's Lemma 5.1 about the Hadamard transform helpful.

(b) Use the rule in (a) to show that $\langle 0^n|G|x\rangle$ equals $\langle 0^n|G_x|0^n\rangle$, where G_x is the graph obtained from G by adding self-loops to the nodes i such that $x_i = 1$. (You may suppose that the original G has no self-loops in your argument. It will then extend fairly readily to say that if G already has a self-loop at node i, then "adding a self-loop to node i" means removing it. Put another way, only the even-odd parity of edges and loops matters. Note also that the "input" goes on the right in $\langle 0^n|G|x\rangle$. Actually, because all the gates are self-adjoint, this case is perfectly left-right symmetric, a fact that might help you in the next part.)

(c) Now show that if $u \oplus v = x \oplus y$, then $\langle y|G|x\rangle = \langle v|G|u\rangle$. You may be able to take (and justify) one of various shortcuts.

(d) Conclude further from this that all possible cases of $\langle z|G|y\rangle$ are "covered" by cases of $\langle 0^n|G_x|0^n\rangle$ for appropriately-defined graphs G_x.

(e) *Added:* One can interpret the initial and final $\mathbf{H}^{\otimes n}$ transforms as transforming from the standard basis (a.k.a the $\mathbf{Z}$-basis) to the Hadamard basis (a.k.a. the $\mathbf{X}$-basis). We can instead work within the $\mathbf{Z}$-basis. In order to make an operator A defined in one basis produce output within the new basis as well as take input from it, one must sandwich A by the transform and its inverse. Since $\mathbf{H}^{\otimes n}$ is self-inverse, this just means replacing $\mathbf{CZ}$ by

$$\mathbf{E} = (\mathbf{H}^{\otimes 2})(\mathbf{CZ})(\mathbf{H}^{\otimes 2}) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

(This also equals sandwiching a CNOT gate between two Hadamards on its control line only. My $\mathbf{E}$ just means "edge"; I can't find a standard name for it.) The transform of $\mathbf{Z}$ is just $\mathbf{HZH} = \mathbf{X}$. Say in a few sentences what happens if you redo your answers, in particular (b) and (c), in the other basis. Does it help in confirming them? $(6+12+12+6+9 = 45$ pts.)

(2) Lipton-Regan text, exercise 14.7 on page 165. (Show the spectral method expressly. 18 pts.)

(3) Lipton-Regan text, exercise 14.10 on page 165. (Justify your answer briefly, for 9 pts.)

(4) Lipton-Regan text, exercise 8.4 on page 96 (left as-is; 12 pts., for 84 on the set).