CSE610, Fall 2021 Problem Set 4 Due Mon. Nov. 29

(1) For any integer $k \geq 1$, consider the following process applied to the qubit $|0\rangle$: For $j = 1$ to k, measure the qubit in the basis where direction $\frac{j\pi}{2k}$ means "yes." If you obtain a "no" answer, break and reject; else continue measuring the resulting qubit. If the loop ends at $j = k$ with "yes," then the resulting qubit is $|1\rangle$. Define $p(k)$ to be the probability of acceptance.

If $k = 1$, then the first measurement will be in the basis with $\frac{\pi}{2} = 90°$ for "yes," and the answer will certainly be "no," so $p(1) = 0$. For $k = 2$, the first measurement will have $|+\rangle$ meaning "yes," and will succeed with probability $\cos^2(\frac{\pi}{4}) = 0.5$. The next measurement of $|+\rangle$ will have a 50-50 chance of succeeding with output $|1\rangle$, so $p(2) = 0.25$.

The *problem* is: what happens to $p(k)$ as $k \to \infty$? Does it decrease back to 0, increase to 1, or level off at a maximum value strictly between 0 and 1? Derive the maximum value in any case. You may find it possible to use trigonometric approximations to justify your answer. (18 pts.)

(2) Suppose Alice and Bob play the CHSH game with qubit pairs in the slightly-depolarized state that is a $(1-\epsilon)$ mixture of a Bell pair $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ with ϵ of the completely mixed state of two qubits. The density matrix for such a qubit pair is

$$\boldsymbol{\rho}_\epsilon \quad = \quad (1-\epsilon)\,|\Phi^+\rangle\,\langle\Phi^+| + \frac{\epsilon}{4}\mathbf{I} \quad = \quad (1-\epsilon)\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{bmatrix}.$$

Find the value of ϵ that knocks the quantum CHSH strategy down to the same 75% expectation as the best classical strategy. For a shortcut and hint, it is enough to consider the play where Bob (holding the little-endian qubit) measures first in the basis $\{[\cos(\theta), \sin(\theta)]^T, [-\sin(\theta), \cos(\theta)]^T\}$ where $\theta = \frac{\pi}{8}$ and the former means "yes" (note that Bob's other choice of basis has $\theta = \frac{3\pi}{8}$) and Alice measures in the standard basis with $|0\rangle$ meaning "yes." This is one of the three cases where Alice and Bob win if their yes/no answers agree. Convert Bob's basis into a projective measurement $\mathbf{M}$, apply it to $\boldsymbol{\rho}$, and for either of the two equally-likely outcomes, compute the rule to update the state for Alice. Then calculate Alice's chance of getting the same yes/no outcome, and solve for ϵ vis-à-vis 75%.

Finally answer: Is the resulting probability the same as if, on any play of the CHSH game, Alice and Bob have a $(1-\epsilon)$ chance of being given a perfect pair, and an ϵ chance of being given a completely mixed state—whereupon the quantum strategy will give them only 50% not 75% chance of success? (This could feed a discussion of whether nature really goes through trigonometric identities with fourth-powers like you've done above. $24 + 6 = 30$ pts. total)

(3) Consider the 3-node "lollipop graph" G with a loop at node 1 and edges $(1,2)$ and $(2,3)$. Construct the corresponding graph state $|\Phi_G\rangle$ without the second bank of Hadamard gates.

(a) Construct the density matrix $\boldsymbol{\rho}_G$. Then show the result of tracing out node 3. Is the result a completely mixed state of two qubits? Then trace out node 2 as well and say what the first qubit is unto itself. (Yes, please do write out the 8×8 matrix of $+1$ and -1; you'll use it in part (d) too.)

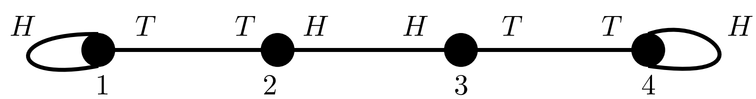
(b) Now add a loop at node 3. Does this change the answers to part (a)? (Here and in (c) you may lean on *Quirk*, mindful of the little-endian display.)

(c) Call node 3 “Charlie,” the others “Alice” and “Bob.” Now let Charlie apply a gate that does not commute with **Z** and **CZ**—try the square-root-of-**Y** gate. (If you’re curious to do this by hand as well as via *Quirk*, use the simplified form at the bottom of page 149 in section 14.6 of the text.) Does it matter now whether you place the $\mathbf{Y^{1/2}}$ before or after the **CZ** gate that involves Charlie?

(d) Now throw a CNOT gate from control on node 1 to target on node 3 after the **CZ** gate. On a density matrix, the action of a permutation is carried out both swapping rows and swapping columns—here swapping 5 and 6, then 7 and 8. Then trace out Bob and Charlie in one go, on paper. Is Alice at node 1 left with a mixed or pure state? $(12 + 3 + 6 + 9 = 30$ pts.)

(4) Now let G be the “Q-tip graph” with an extra node in the middle:

$G =$



(a) Let P be the 8×8 permutation matrix of the walk on the space $\{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T\}$. Design a deterministic quantum circuit that implements P on three qubit lines. (Hint: every permutation is a composition of transpositions.)

(b) Then compose P with the Hadamard coin matrix. Adapt the Python code linked in my “Quantum Trick or Treat” post to program the walk. Is the behavior chaotic, or does it have a period?

(c) Check your Python output against a quantum circuit simulator for the first few iterations. (Is there one that can do any number of iterations?)

(d) Change the coin to the matrix called **J** in the post. Compare the result to what you get with the Hadamard coin. Also try $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & -1 & 1\end{bmatrix}$, which is the rotation matrix $\mathbf{R}_\theta$ for $\theta = \frac{\pi}{4}$, i.e., a counterclockwise rotation by 45°.

(e) Now use the rotation matrix for a counterclockwise rotation by 30°. What do you observe now? $(18 + 12 + 6 + 6 + 6 = 48$ pts., for 126 total on the set)