CSE191, Spr'11 — Final Exam — May 9, 2011

Open book, open notes, closed neighbors, 170 minutes. Please do ALL TEN problems in the exam booklets provided. Please *show all your work*; this may help for partial credit. The exam totals 240 pts., subdivided as shown.

(1) (5 × 6 = 30 pts. total)

Put each of the following into the form $P \longrightarrow Q$ or $(\forall x)[P(x) \longrightarrow Q(x)]$. In the latter case say what is playing the role of the variable "x" and what its domain could be. For example, if the phrase is "Every dog has its day," then you can say x ranges over animals and write $(\forall x)[\text{Dog}(x) \longrightarrow \text{HasDay}(x)]$. Note that there is some "flex' in your answers, since x could range over pets or all life forms, and you could even say x ranges over dogs and say that P is the trivial `true` proposition (note that `true` $\longrightarrow Q$ is equivalent to just Q). In some cases full credit will be given for having "x" when the simpler intent was to do without it.

Writing the $\longrightarrow$ symbol in your answers is *required*, so as to keep them clear. There are also a few "style points" for phrasing predicates positively and using the negation symbol ($\neg$) rather than phrasing predicates negatively. If you think the answer should really be $P \longleftrightarrow Q$, you must include a justification.

(a) Beggars cannot be choosers.

(b) An honors degree requires a 3.0 average.

(c) If you cannot stand the heat, you cannot be in the kitchen.

(d) Respond only when asked.

(e) To get credit for a full hour, it is enough for a person not to leave early.

(2) (6 + 18 = 24 pts.)

A baseball bullpen divides pitching responsabilities this way:

(a) If George or Harry pitches, then Ira does not pitch.

(b) At least one of Harry and Ira always pitches.

(c) If Harry pitches or George does not pitch, then Jim pitches.

(d) George pitches, or Harry does not pitch.

The goal is to deduce (e) Either Jim pitches or Harry does not pitch.

Directions: First abbreviate "George pitches" to just G, and similarly with H, I, J. Then translate the above statements (a-e) into propositional logic. Using your translations, give a formal proof of (e) from the premises (a-d), showing each proof rule used. *Also answer:* if a premise is not needed in your proof, say so.

(3) (6 + 6 × 3 = 24 pts.)

Let the domain of the variable s be students and c be courses, at some university. Let $T(s,c)$ stand for "student s is taking course c." Consider the following list of first-order logic statements:

(a) $(\exists s)(\forall c);T(s,c)$

(b) $(\exists c)(\forall s)\ T(s,c)$

(c) $(\forall s)(\exists c)\ T(s,c)$

(d) $\neg(\exists s)(\exists c)\ T(s,c)$

(e) $(\exists s)(\forall c)\ \neg T(s,c)$

(f) $(\forall s)(\forall c)\ T(s,c)$

(g) $(\forall c)(\exists s)\ T(s,c)$

(h) $(\exists c)(\forall s)\ \neg T(s,c)$

(i) $\neg(\exists c)(\forall s)\ T(s,c)$

(j) $(\forall s)(\exists c)\ \neg T(s,c)$.

I. Simplify (d) and (i) by bringing the negation inside. (6 pts.)
II. For each of the following, give the letter in (a)–(j) of its correct translation.

(i) Some student is taking every course.

(ii) There is a course that no student is taking.

(iii) Some student is taking no courses.

(iv) No course is being taken by all students.

(v) Every student is taking some course.

(vi) No student is taking all the courses at once.

(4) (9 + 12 + 3 = 24 pts.)

Prove that if $A \subseteq B$, then for any set C, $A \cap C \subseteq B \cap C$. Notice that what you are proving in full is that for any sets A, B, C, the statement R holds, where R is:

$$A \subseteq B \longrightarrow (A \cap C \subseteq B \cap C).$$

(a) Convert R into an equivalent Boolean logic proposition P with variables a, b, c corresponding to the sets A, B, C. (More precisely, a is short for "$x \in A$" which comes up when you translate $A \subseteq B$ as $(\forall x)[x \in A \longrightarrow x \in B]$, and so on.)

(b) Prove that P is a tautology. For full credit, use a 'syntactic proof' using proof rules, rather than truth tables (which will take more time and gain at most 9/12 pts.).

(c) Conclude that R is true.

(5) (6 + 18 = 24 pts.)

(a) Write a predicate $P(n)$ of logical arithmetic over the natural numbers expressing that n is 1 more than a multiple of 3.

(b) Prove that for all natural numbers n, n is a multiple of 3 if and only if $2n$ is a multiple of 3. Use the style of proving $X \longleftrightarrow Y$ by proving $X \longrightarrow Y$ and $Y \longrightarrow X$. You may take for granted that every natural number is congruent to 0, 1, or 2 modulo 3, using cases defined by predicates such as in (a).

(6) (24 pts. total)

Prove by induction on n that $n^3 + 2n$ is always a multiple of 3. Be sure to do this in careful steps: (a) place this in the form "$(\forall n)P(n)$", (b) tell whether $b = 0$ or $b = 1$ (or etc.) should be the base case, (c) prove $P(b)$, (d) state what the induction hypothesis $P(n-1)$ says, (e) prove $P(n-1) \longrightarrow P(n)$, and (f) finish the proof.

(7) (24 pts. total)

Consider the depth-2 linear recurrence equation $r(n) = 4r(n-1) - 3r(n-2)$, with initial conditions $r(0) = 0$, $r(1) = 2$.

(a) Compute the values $r(n)$ up through $n = 6$.

(b) Guess and prove a solution. Do you need to use induction?

[Note: In Spring 2011 I covered what in the new edition of Rosen appears as section 8.2, and the actual exam problem was to *solve* $r(n) = 2r(n-1) + 3r(n-2)$ with $r(0) = 0$, $r(1) = 1$. The above case is more amenable to the "guess-and-verify" kind of solving, which is in-bounds for you.]

(8) (3 + 3 + 3 + 6 + 6 = 21 pts. total)

Define $f(n) = n(n-1)/2$, and consider the set $A = \{3, 4, 5, 6\}$.

(a) Find the set $B = f(A)$.

(b) Find the least natural number that is not in the range of f.

(c) Is f 1-to-1? Justify your answer.

(d) Find the set $C = f^{-1}(A)$. Is there a proper subset $A' \subset A$ such that $C = f^{-1}(A')$?

(e) Now define $g(0) = 0$, and for $n \geq 1$, $g(n) = f(n) - f(n-1)$. Is g onto the set of natural numbers? What is the function $g(n)$?

(9) (3 + 3 + 6 + 6 + 3 = 21 pts. total)
Define $V = \{0, 1, 2, 3, 4, 5\}$. Define a graph $G = (V, E)$ by letting the edges be:

$$E = \{ (a, b) : |a - b^2| \leq 1 \vee |b - a^2| \leq 1 \}.$$

(a) Is the relation defined by E symmetric, so that G can be viewed as an undirected graph?

(b) Does G as-defined have any self-loops? If so, let G' be the graph obtained by deleting the self-loops. If not, let $G' = G$.

(c) Draw the graph G'.

(d) Is G' transitive? Prove it or give a place where transitivity fails.

(e) Is G connected?

(10) (8 × 3 = 24 pts.) *True/False:* No justifications are asked for, though they could help for partial credit. Please write `true` or `false` in full.

(a) For every finite set A, the power set $P(A)$ of A always has cardinality 2^k for some natural number $k \geq 0$.

(b) The $\subseteq$ relation on sets is transitive and reflexive.

(c) The $\subset$ relation on sets is transitive and reflexive.

(d) The relation $A \subseteq B \wedge B \subseteq A$ is an equivalence relation on sets.

(e) If f is 1-1 and g is 1-1 and defined on the range of f, then the composition $g \circ f$ is a 1-1 function.

(f) If $(\exists x)P(x)$ is true and $(\exists x)Q(x)$ is true, then $(\exists x)[P(x) \wedge Q(x)]$ is true.

(g) If $P(n) \longrightarrow P(2n) \wedge P(2n + 1)$ for all n, and $P(0)$ holds, then $(\forall n)P(n)$ holds by the principle of strong induction.

(h) Given any subsets $A_1, A_2 \subseteq B$ such that $A_1 \cup A_2 = B$, the relation

$$R(x, y) \equiv (x \in A_1 \wedge y \in A_1) \vee (x \in A_2 \wedge y \in A_2)$$

is always an equivalence relation on the elements of the sets.

END OF EXAM