Open notes but not text or Internet access, closed neighbors, 170 minutes. Please do ALL TEN problems in the exam booklets provided. Please *show all your work*; this may help for partial credit. The exam totals 240 pts., subdivided as shown.

The difference of sets A and B can be written either $A - B$ as in the text, or $A \setminus B$ as in lectures. The symmetric difference is $A \oplus B$ as in the text; although in other sources you may have seen the notation $A \Delta B$ for it.

(1) ($5 \times 6 = 30$ pts. total)

Note that $P \longleftrightarrow Q$ and $P \oplus Q$ each have 2 true and 2 false cases, whereas $P \wedge Q$, $P \vee Q$, $P \longrightarrow Q$, and P NAND Q (among some others) are split 3–1: either one true and 3 false cases, or one false and 3 true cases. And the trivial functions giving always-true and always-false are split 4–0. Call propositions of the $\longleftrightarrow$ and $\oplus$ and trivial kind "even," and call the others "odd."

Each of the following assertions has two propositional variables and/or predicates (or one of each) expressed in prose, possibly negated. For each one, write it more symbolically, *eliminating negations if you can*, and classify it as "even" or "odd." For example, "Anyone who can't stand the heat must get out of the kitchen" can be rewritten using the contrapositive as

$(\forall x) : \texttt{InKitchen}(x) \longrightarrow \texttt{CanStandHeat}(x)$,

and is odd. If you add the clause, "and anyone who can stand the heat must stay," then it becomes an equivalence, which is even. The assertions are:

(a) Buy yourself an ice cream cone, or at least don't drown your sorrows in whiskey.

(b) $P \longrightarrow Q$ or $Q \longrightarrow P$, but not both.

(c) A function $f : X \longrightarrow Y$ fails to be onto exactly when $Y \setminus Ran(f)$ is not empty.

(d) If the werewolves won't attack, the zombies won't attack.

(e) The quarterback threw a good pass, but the receiver did not catch it.

Note: You may perceive multiple ways to interpret one of these, but before asking a question, please see if the two ways give the same even/odd answer. In any event, *please show your reasoning in the exam booklet(s) provided.*

(2) (15 + 24 = 39 pts.)
Consider the following five premises:

1. A writer who is devious does not really understand human nature.
2. All writers who allow others to use an early draft are devious.
3. All true poets understand human nature.
4. Only a true poet could have written *Hamlet*.
5. Shakespeare wrote *Hamlet*, after being allowed to use another writer's early draft.

In addition, here are two premises which the original author of a similar exercise took for granted. They are written in a formal style which suggests how you can formalize the above:

- $(\forall X, H) : \texttt{wrote}(X, Y) \longrightarrow \texttt{writer}(X)$.
- $(\forall X, Y, H) : \texttt{allowedToUseOtherWriter'sDraft}(X, Y, H) \longrightarrow Y \neq X$.

Well, you could add "$\texttt{writer}(X) \wedge \texttt{writer}(Y)$" to the right-hand side as well. But you also don't have to—you may assume instead that the whole domain of X and Y is *writers*, so `shakespeare` is automatically a *writer*. You may not assume every writer is a "true poet," however. (This is where early course discussion about domains is coming in.)

First write the above five numbered premises symbolically. Then for each of the following propositions, say whether it follows from the above premises. If so, give a proof using symbolic logic rules. If not, give a reason why not.

(a) Shakespeare was devious.

(b) Some other writer was devious.

(c) Shakespeare was not devious.

(3) (24 pts. total)
For each of the following assertions about sets A, B, C, say whether it is always true. If so, give a truth table for the corresponding logical formula; if not, give a Venn diagram where it is false. Recall that $\oplus$ stands for symmetric difference.

(a) $A \cap B \subseteq A \oplus B$.

(b) $A \subseteq C \wedge B \subseteq C \longrightarrow A \oplus B \subseteq C$

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(4) ($4 \times 3 \times 2$ = 24 pts.)

For each of the following functions of real numbers, and for each of the properties of (i) being defined on all of $\mathbb{R}$, (ii) being 1–1 on its domain, and (iii) being onto $\mathbb{R}$, say whether the property is

- True,
- Almost True—that is, true except for 1 or 2 real numbers, or
- Really False—that is, false and not "almost true."

You should have a grid of answers with 5 rows and 3 columns, and entries "True," "Almost True," or "Really False." Here $|x|$ means the absolute value of the number x.

(a) $f(x) = x^3$.

(b) $f(x) = x^2/|x|$.

(c) $f(x) = 1/x$.

(d) $f(x) = \frac{2x-3}{x-3}$.

(5) (18 pts.)

Use the Euclidean algorithm to find integers s and t such that $24s + 35t = 1$.

(6) (24 pts. total)

Suppose a big roulette wheel has n red slots and n black slots, plus one green slot. Suppose *two* balls go around when the wheel is spun, with the fact that they must fall into different slots.

(a) In how many ways can the balls fall into two slots of the same color, or have one ball fall into black and one into green? (Hint: think what if the green slot were colored black too.)

(b) Prove that this number is always a perfect square. Did you need to use induction?

(Footnote: Ignore the fact that most roulette wheels in casinos have *two* green slots.)

(7) (24 pts.)

Consider the recurrence relation

$$r(n) = 5r(n-1) - 6r(n-2)$$

with initial conditions $r(0) = 0$, $r(1) = 1$. Prove by induction that for all n, $r(n) = 3^n - 2^n$.

(8) (24 pts. total)

Let V be the power set of $\{1, 2, 3\}$. Define a directed graph $G = (V, E)$ by drawing an edge from a subset A of $\{1, 2, 3\}$ to a subset B if $|A - B| = 1$, that is if there is an element $i \notin B$ such that $A = B \cup \{i\}$.

(a) Draw the graph. For labels on the vertices, even better than giving the subsets, give the corresponding binary bit-string representations.

(b) What does the out-degree of each node equal?

(c) Now consider the undirected graph $G' = (V, E')$ made by ignoring the arrows on the edges. Describe it, including saying what the degrees of the nodes are now.

(d) Does the directed graph G have any cycles? How about the undirected graph G'? Is G' connected?

(e) In G', show that for every node A there is a node A' such that the neighborhoods of A and A' do not overlap. Put another way, every path between A and A' has length at least 3. Can you say what A' "is" in terms of A?

(9) (15 pts. total)

This time define $V = \{2, 3, 4, 5, 6\}$. Define a graph $G = (V, E)$ by putting (i, j) into E if and only if i and j are relatively prime.

(a) Give an adjacency matrix for G. Is G directed or undirected? You are welcome to draw G.

(b) Is there any node whose neighborhood is all of G?

(c) Show that G is not bipartite, by finding a 3-cycle in G (that is, a triangle).

(d) If you delete node 5 and all the edges into it, is the resulting graph connected?

(10) ($6 \times 3 = 18$ pts.) *True/False:* No justifications are asked for, though they could help for partial credit. Please write `true` or `false` in full.

(a) For all sets A and B, the symmetric difference $A \oplus B$ is empty if and only if $A = B$.

(b) There is no bijection between $\mathbb{N}$ and $\mathbb{Z}$.

(c) Cartesian product of sets is associative and commutative.

(d) If $f : \mathbb{N} \longrightarrow \mathbb{N}$ and $g : \mathbb{N} \longrightarrow \mathbb{N}$ are 1–1 functions, then the sum function $h(x) = f(x) + g(x)$ is 1–1.

(e) If $f : \mathbb{N} \longrightarrow \mathbb{N}$ and $g : \mathbb{N} \longrightarrow \mathbb{N}$ are onto $\mathbb{N}$, then the sum function $h(x) = f(x) + g(x)$ is onto $\mathbb{N}$.

(f) With all quantifiers over the same domain, if $(\forall x)P(x)$ is true and $(\exists x)Q(x)$ is true, then $(\exists x)[P(x) \wedge Q(x)]$ is true.

END OF EXAM