CSE191, Fall 2013 — Final Exam — Dec. 11, 2013

Open notes but not text or Internet access, closed neighbors, 170 minutes. Please do ALL TEN problems in the exam booklets provided. Please *show all your work*; this may help for partial credit. The exam totals 240 pts., subdivided as shown.

The difference of sets A and B can be written either $A - B$ as in the text, or $A \setminus B$ as in lectures. The symmetric difference is $A \oplus B$ as in the text; although in other sources you may have seen the notation $A \Delta B$ for it.

(1) ($5 \times 6 = 30$ pts. total)

Note that $P \longleftrightarrow Q$ and $P \oplus Q$ each have 2 true and 2 false cases, whereas $P \wedge Q$, $P \vee Q$, $P \longrightarrow Q$, and P NAND Q (among some others) are split 3–1: either one true and 3 false cases, or one false and 3 true cases. And the trivial functions giving always-true and always-false are split 4–0. Call propositions of the $\longleftrightarrow$ and $\oplus$ and trivial kind "even," and call the others "odd."

Each of the following assertions has two propositional variables and/or predicates (or one of each) expressed in prose, possibly negated. For each one, write it more symbolically, *eliminating negations if you can*, and classify it as "even" or "odd." For example, "Anyone who can't stand the heat must get out of the kitchen" can be rewritten using the contrapositive as

$(\forall x)$: InKitchen$(x) \longrightarrow$ CanStandHeat(x),

and is odd. If you add the clause, "and anyone who can stand the heat must stay," then it becomes an equivalence, which is even. The assertions are:

(a) Buy yourself an ice cream cone, or at least don't drown your sorrows in whiskey.

(b) $P \longrightarrow Q$ or $Q \longrightarrow P$, but not both.

(c) A function $f : X \longrightarrow Y$ fails to be onto exactly when $Y \setminus Ran(f)$ is not empty.

(d) If the werewolves won't attack, the zombies won't attack.

(e) The quarterback threw a good pass, but the receiver did not catch it.

Note: You may perceive multiple ways to interpret one of these, but before asking a question, please see if the two ways give the same even/odd answer. In any event, *please show your reasoning in the exam booklet(s) provided.*

Answer:

(a) `DrownInWhisky` $\longrightarrow$ `BuyIceCream`. Odd but true, that's what it means when both options are stated positively. Either way, it's *odd*.

(b) Instead of giving $P \longleftrightarrow Q$ like "and" would, this gives the exclusive-or $P \oplus Q$, which is even. The other way you could interpret this is as if "$P \longrightarrow Q$" was just a generic proposition A, and "$Q \longrightarrow P$" was B, so this reads "A or B, but not both." In that case it's still exclusive or, and still even, even though the underlying *logical fact* is different.

(c) Here there are really two predicates $O(f)$ meaning "f is onto" and $E(f)$ meaning "empty $Y \setminus Ran(f)$." So this is $(\neg O(f)) \longleftrightarrow (\neg E(f))$, which you should flip around positively to read $O(f) \longleftrightarrow E(f)$. The latter reads more nicely as, "A function $f : X \longrightarrow Y$ is onto exactly when $Y \setminus Ran(f)$ is empty." If you regarded "nonempty" as a positive property (which is fine) you would get an exclusive-or, but any way you slice it the result is *even*.

Or, being really geek-minded, you could note that unlike the other examples, this is a mathematical statement with a definite truth value. It is TRUE. But, the always-true function is still *even*, so this doesn't change the ultimate answer.

(d) Contrapose: `ZombiesAttack` $\longrightarrow$ `WerewolvesAttack`. Note that putting it this way is psychologically frightening, but it's logically equivalent—and anyway, *odd*.

(e) Here the "but" is just a logical AND, so this is `QBThrewGoodPass` $\wedge \neg$`ReceiverCaughtIt`. Or, if you picked up "a" as an existential word, you could elaborate this to

$(\exists p)$[`QBThrew`$(p) \wedge$ `WasGoodPass`$(p) \wedge \neg$ `Caught`(p).

Even if you considered this a 3-way operation, it has a 1–7 truth split and is still *odd*. Also either way, you cannot eliminate the negation with an implication—that works only with logical OR.

(2) (15 + 24 = 39 pts.)
Consider the following five premises:

1. A writer who is devious does not really understand human nature.
2. All writers who allow others to use their early drafts are devious.
3. All true poets understand human nature.
4. Only a true poet could have written *Hamlet*.
5. Shakespeare wrote *Hamlet*, after being allowed to use another writer's early draft.

In addition, here are two premises which the original author of a similar exercise took for granted. They are written in a formal style which suggests how you can formalize the above; you may abbreviate predicates if it's clear:

- $(\forall X, H)$: `wrote`$(X, H) \longrightarrow$ `writer`(X).
- $(\forall X, Y, H)$: `allowedToUseOtherWriter'sDraft`$(X, Y, H) \longrightarrow Y \neq X$.

First write the above five numbered premises symbolically. Then for each of the following propositions, say whether it follows from the above premises. If so, give a proof using symbolic logic rules. If not, give a reason why not.

(a) Shakespeare was devious.

(b) Some other writer was devious.

(c) Shakespeare was not devious.

Answer: This assumes the domain is all writers, and so does not need the extra premises. $\texttt{UHN}(X)$ abbreviates "X understands human nature," and $\texttt{ATUOWD}(X, Y, H)$ abbreviates the predicate in the second "extra" premise.

1. $(\forall X)[\texttt{devious}(X) \longrightarrow \neg\texttt{UHN}(\texttt{X})]$.
2. $(\forall X, Y, H)[\texttt{ATUOWD}(X, Y, H) \longrightarrow \texttt{devious}(Y)]$.
3. $(\forall X)[\texttt{TruePoet}(X) \longrightarrow \texttt{UHN}(X)]$.
4. $(\forall X)[\texttt{wrote}(X, \texttt{hamlet}) \longrightarrow \texttt{TruePoet}(X)]$.
5. $\texttt{wrote}(\texttt{shakespeare}, \texttt{hamlet}) \wedge (\exists Y)\texttt{ATUOWD}(\texttt{shakespeare}, Y, \texttt{hamlet})$.

Now here are some deductions we can make (rules in the same bunch were fine to lump together):

(a) $\texttt{wrote}(\texttt{shakespeare}, \texttt{hamlet})$; $\texttt{wrote}(\texttt{shalespeare}, \texttt{hamlet}) \longrightarrow \texttt{TruePoet}(\texttt{shakespeare})]$; $\texttt{TruePoet}(\texttt{shakespeare})$

(b) $\texttt{TruePoet}(\texttt{shakespeare}) \longrightarrow \texttt{UHN}(\texttt{shakespeare})$; $\texttt{UHN}(\texttt{shakespeare})$

(c) $\texttt{devious}(\texttt{shakespeare}) \longrightarrow \neg\texttt{UHN}(\texttt{shakespeare})$; $\neg\texttt{devious}(\texttt{shakespeare})$.

Thus we have proved assertion (c). This also means (a) is *false*, which goes as a proof of its not being derivable. As for (b), we can continue right along:

(d) $(\exists Y)\texttt{ATUOWD}(\texttt{shakespeare}, Y, \texttt{hamlet})$; $\texttt{ATUOWD}(\texttt{shakespeare}, c, \texttt{hamlet})$ *where c is labeled existential.*

(e) $\texttt{ATUOWD}(\texttt{shakespeare}, c, \texttt{hamlet}) \longrightarrow \texttt{devious}(c)$ *by universal gen. stepping Y down to existential c.*

(f) $\texttt{devious}(c)$ *where c is labeled existential.*

(g) $(\exists Y)\texttt{devious}(Y)$.

Well actually the conclusion should be $(\exists Y)[\texttt{devious}(Y) \wedge Y \neq \texttt{shakespeare}]$, to catch the meaning of "other." We could get this directly from (c) if we invoke an unnamed(?) logical rule that two arguments on which a predicate differs must be unequal. Or you could modify the above steps by carrying the $y \neq X$ implication of the "extra" premise all the way through. Neither was needed for full credit, but recognizing this could be part of "compensatory credit" for errors elsewhere. This is essentially how logical reasoning is actually programmed.

This was based on one of Lewis Carroll's examples, #18 at http://www.math.hawaii.edu/~hile/math100/logice.htm I do in fact believe that *Hamlet* and certain other plays (chiefly *Romeo and Juliet* and both parts of *Henry IV*, plus the *Sonnets*) arose this way, as the best explanation of why they contain biographical details of the Earl of Oxford, dating even to the 1570's and early 1580's. Oxford was both devious and destitute, partly from profligacy and partly because his plays for Elizabeth's court in those years couldn't make money. It is IMHO not necessary to take any of the extreme positions in the Shakespeare authorship question: not that Shakespeare was and did nothing, nor that Oxford *was* a son of QE1 as the movie "Anonymous" portrays—that he believed it is enough (hey, my dad's father associated with Lindbergh and disappeared in 1930, so my dad used to wonder if he could be the Lindbergh baby). My orthodox Stratfordian high-school teacher proclaimed "Shakespeare stole," to which I just add the words, "from Oxford among others, with approval of all parties, including the Queen who paid Oxford what in our money was about $3,000,000 per year." That Elizabeth's chief advisor spent 3+ years trying to convince the Earl of Southampton—the evident addressee of the *Sonnets*—to marry his granddaughter—who was Oxford's daughter—and used the Latin motto *Cor unum* which was made into the name *Corambis* ("two-hearted") of Polonius in the early drafts of *Hamlet*, completes the picture for me.

(3) (24 pts. total)

For each of the following assertions about sets A, B, C, say whether it is always true. If so, give a truth table for the corresponding logical formula; if not, give a Venn diagram where it is false. Recall that $\oplus$ stands for symmetric difference.

(a) $A \cap B \subseteq A \oplus B$.

(b) $A \subseteq C \land B \subseteq C \longrightarrow A \oplus B \subseteq C$

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

Answer: (a) is false—false in general Venn diagram. (6D)

(b) (b) is true, since $A \oplus B \subseteq A \cup B$. The logical formula is

$$(A \longrightarrow C) \land (B \longrightarrow C) \longrightarrow ((A \oplus B) \longrightarrow C).$$

(c) is false: any Venn diagram with C not a subset of A works.

(4) $(4 \times 3 \times 2 =$ **24 pts.)**

For each of the following functions of real numbers, and for each of the properties of (i) being defined on all of $\mathbb{R}$, (ii) being 1–1 on its domain, and (iii) being onto $\mathbb{R}$, say whether the property is

- True,
- Almost True—that is, true except for 1 or 2 real numbers, or
- Really False—that is, false and not "almost true."

You should have a grid of answers with 5 rows and 3 columns, and entries "True," "Almost True," or "Really False." Here $|x|$ means the absolute value of the number x.

(a) $f(x) = x^3$.

(b) $f(x) = x^2/|x|$.

(c) $f(x) = 1/x$.

(d) $f(x) = \frac{2x-3}{x-3}$.

Answer:

(a) True, True, True.

(b) Almost True (not defined at 0, though it could be judged 0 by continuity), False (symmetric about y-axis), False (never negative).

(c) Almost True—just ignore 0, True, Almost True—again ignore 0.

(d) Almost True—ignore 3, True: $x = (3 - 3y)/(2 - y)$, Almost True: 2 is not in the range.

(5) (18 pts.)
Use the Euclidean algorithm to find integers s and t such that $24s + 35t = 1$.

Answer:

$$\begin{aligned}
35 &= 24 * 1 + 11 \\
24 &= 11 * 2 + 2 \\
11 &= 5 * 2 + 1 \quad \text{so indeed rel. prime, but more needed} \\
1 &= 11 - 5 * 2 \\
2 &= 24 - 11 * 2 \\
1 &= 11 - 5 * (24 - 11 * 2) \\
11 &= 35 - 24 \\
1 &= 35 - 24 - 5 * 24 + 5 * 2 * (35 - 24) = 11 * 35 - 16 * 24.
\end{aligned}$$

(6) (24 pts. total)
Suppose a big roulette wheel has n red slots and n black slots, plus one green slot. Suppose *two* balls go around when the wheel is spun, with the fact that they must fall into different slots.

(a) In how many ways can the balls fall into two slots of the same color, or have one ball fall into black and one into green? (Hint: think what if the green slot were colored black too.)

(b) Prove that this number is always a perfect square. Did you need to use induction?

(Footnote: Ignore the fact that most roulette wheels in casinos have *two* green slots.)

Answer: $C(n, 2) + C(n + 1, 2) = n(n - 1)/2 + n(n + 1)/2 = n(n - 1 + n + 1)/2 = n * 2n/2 = n^2$.

(7) (24 pts.)

Consider the recurrence relation $r(n) = 5r(n-1) - 6r(n-2)$ with initial conditions $r(0) = 0$, $r(1) = 1$. Prove by induction that for all n, $r(n) = 3^n - 2^n$.

Answer:
Prove $(\forall n)P(n)$, where $P(n) \equiv r(n) = 3^n - 2^n$. Since the recurrence has depth 2, this needs "semi-strong" induction using $P(n-1)$ and $P(n-2)$ as induction hypotheses.

Basis ($n = 0, 1$): $P(0)$ states $r(0) = 3^0 - 2^0$. This holds since $r(0) = 0$ and $3^0 - 2^0 = 1 - 1 = 0$. And for ($n = 1$), one can observe $r(1) = 1 = 3^1 - 2^1 = 3 - 2$.

Induction ($n \geq 2$): Assume (IH) $(\forall m < n)P(m)$. we will actually use $P(n-1)$, which states that $r(n-1) = 3^{n-1} - 2^{n-1}$, and $P(n-2)$ which is similar. Goal: show $P(n)$. Do it:

$$\begin{aligned}
r(n) &= 5r(n-1) - 6r(n-2) \qquad \text{(by definition of the recurrence)} \\
&= 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2}) \qquad \text{(by IH-es)} \\
&= 5 * 3^{n-1} - 5 * 2^{n-1} - 2 * 3 * 3^{n-2} + 3 * 2 * 2^{n-2} \\
&= 5 * 3^{n-1} - 5 * 2^{n-1} - 2 * 3^{n-1} + 3 * 2^{n-1} \\
&= (5-2) * 3^{n-1} + (3-5) * 2^{n-1} = 3^n - 2^n.
\end{aligned}$$

This shows $P(n)$, and so $(\forall n)P(n)$ follows by induction.

(8) (24 pts. total)

Let V be the power set of $\{1, 2, 3\}$. Define a directed graph $G = (V, E)$ by drawing an edge from a subset A of $\{1, 2, 3\}$ to a subset B of A if $|A - B| = 1$, that is if there is an element $i \notin B$ such that $A = B \cup i$.

(a) Draw the graph. For labels on the vertices, even better than giving the subsets, give the corresponding binary bit-string representations.

(b) What does the out-degree of each node equal?

(c) Now consider the undirected graph $G' = (V, E')$ made by ignoring the arrows on the edges. Describe it, including saying what the degrees of the nodes are now.

(d) Does the directed graph G have any cycles? How about the undirected graph G'? Is G' connected?

(e) In G', show that for every node A there is a node A' such that the neighborhoods of A and A' do not overlap. Put another way, every path between A and A' has length at least 3. Can you say what A' "is" in terms of A?

Answer: (a) The graph is a directed cube, with say an edge from 101 to 100 because we took away element 3. (b) The out-degree equals the cardinality of the set. (c) The undirected graph—still a cube—has all nodes of degree 3. (d) The directed graph was acyclic, but the undirected graph has a 4-cycle for each face of the cube, and is connected. (e) The node A' corresponds to the complement of the set A.

(9) (15 pts. total)

This time define $V = \{2, 3, 4, 5, 6\}$. Define a graph $G = (V, E)$ by putting (i, j) into E if and only if i and j are relatively prime.

(a) Give an adjacency matrix for G. Is G directed or undirected? You are welcome to draw G.

(b) Is there any node whose neighborhood is all of G?

(c) Show that G is not bipartite, by finding a 3-cycle in G (that is, a triangle).

(d) If you delete node 5 and all the edges into it, is the resulting graph connected?

Answer: Everything is connected except 2-4-6-2 and 3-6. (a) Since the relation is symmetric, the graph is undirected.

$$\begin{matrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}$$

(b) Yes, node 5. (c) 2-3-5, all prime numbers. Can use 4 in place of 2. (d) No, deleting 5 isolates node 6, because it shares a divisor with 2,3,4.

(10) (6 $\times$ 3 = 18 pts.) *True/False:* No justifications are asked for, though they could help for partial credit. Please write `true` or `false` in full.

(a) For all sets A and B, the symmetric difference $A \oplus B$ is empty if and only if $A = B$.

(b) There is no bijection between $\mathbb{N}$ and $\mathbb{Z}$.

(c) Cartesian product of sets is associative and commutative.

(d) If $f : \mathbb{N} \longrightarrow \mathbb{N}$ and $g : \mathbb{N} \longrightarrow \mathbb{N}$ are 1–1 functions, then the sum function $h(x) = f(x) + g(x)$ is 1–1.

(e) If $f : \mathbb{N} \longrightarrow \mathbb{N}$ and $g : \mathbb{N} \longrightarrow \mathbb{N}$ are onto $\mathbb{N}$, then the sum function $h(x) = f(x) + g(x)$ is onto $\mathbb{N}$.

(f) With all quantifiers over the same domain, if $(\forall x)P(x)$ is true and $(\exists x)Q(x)$ is true, then $(\exists x)[P(x) \wedge Q(x)]$ is true.

Answer:

(a) True, since it is $A \setminus B \cup B \setminus A$, and both parts are empty when and only when the sets are equal.

(b) False, they are both countably infinite; enumerate $\mathbb{Z}$ in order $0, +1, -1, +2, -2, +3, \ldots$

(c) False—certainly not commutative since $A \times B = B \times A$ only when $A = B$ (homework), and technically not associative if one distinguishes $((a, b), c)$ from $(a, (b, c))$ (also per homework).

(d) False—take f to be the identity, and g the same except for $g(0) = 1$, $g(1) = 0$. Then the sum has two 1 values.

(e) False—the same example works, or just take f and g both to be the identity, because of all the even numbers in the range.

(f) True—technically since $(\exists x)Q(x)$ being true guarantees that the domain is nonempty, so universal instantiation is OK.

END OF EXAM