CSE191, Spr'11 Prelim II Answer Key Apr. 20, 2011

(1) (21 + 3 = 24 pts. total)

For all $n \geq 1$, define $s(n) = \sum_{r=1}^{n} \frac{1}{r^2}$.

(a) Prove by induction that for all $n \geq 1$, $s(n) \leq \frac{2n}{n+1}$. (*Hint:* at the end, carefully cross-multiply by n^2 on one side and $(n+1)$ on the other—if you use "k to $k+1$" style you will have similar but different quantities here.)

(b) Deduce that the infinite sum $s(\infty) = \sum_{r=1}^{\infty} \frac{1}{r^2}$ converges—by giving the least integer t you can such that $s(\infty) \leq t$.

Answer: Prove $(\forall n \geq 1)P(n)$, where $P(n) \equiv s(n) \leq 2n/(n+1)$.

Basis ($n = 1$): $P(1)$ states $s(1) \leq 2/(1+1)$, which is $1 \leq 1$, *check*.

Induction ($n \geq 2$): Assume (IH) $P(n-1)$, which states that $s(n-1) \leq 2(n-1)/(n-1+1)$, i.e. $s(n-1) \leq (2n-2)/n$. Goal: show $P(n)$. To do it, note that

$$\begin{aligned} s(n) &= s(n-1) + 1/n^2 \\ &\leq (2n-2)/n + 1/n^2 \qquad \text{(by IH)} \\ &= (2n^2 - 2n + 1)/n^2. \qquad \text{Now, is this} \leq 2n/(n+1)? \end{aligned}$$

By cross-multiplying, that's true iff $(n+1)(2n^2 - 2n + 1) \leq 2n^3$. The left-hand side multiplies out to $2n^3 + 2n^2 - 2n^2 - 2n + n + 1$, which after cancellation leaves $2n^3 - n + 1 \leq 2n^3$, i.e., $1 - n \leq 0$, which is true since $n \geq 2$. Thus $P(n)$ holds, and $(\forall n)P(n)$ follows by induction.

For (b), note that $2n/(n+1) < 2n/n = 2$, so always $s(n) < 2$. This implies $s(\infty) \leq 2$, and 2 is clearly the least possible such integer. (That $s(n)$ converges follows because it always increases but stays bounded by 2—this is easier than the lecture example with $1 - 1/3 + 1/5 - 1/7 + 1/9 \ldots$. In fact it converges to $\pi^2/6$, which is about 5/3.)

(2) (3+3+6+1+1+3 = 17 points total)

For all integers n, define $f(n)$ to be the nearest multiple of 3 to n. Note that if n is a multiple of 3, then $f(n) = n$.

(a) Give the values $f(n)$ for $n = 0, 1, 2, 3, 4, 5$. *Answer:* $0, 0, 3, 3, 3, 6$.

(b) What, therefore, is $f(\{0, 1, 2, 3, 4, 5\})$? What is its cardinality? *Answer:* As a *set*, it is just $\{0, 3, 6\}$, which has cardinality 3.

(c) What is $f^{-1}(\{3, 4, 5, 6\})$? Find a proper subset $S \subset \{3, 4, 5, 6\}$ such that $f^{-1}(S)$ gives the same answer. *Answer:* The values 4 and 5 are not in the range because they are not multiples of 3, so we can use $S = \{3, 6\}$ which gives the same answer $f^{-1}(S) = \{2, 3, 4, 5, 6, 7\}$.

(d) Is f 1-to-1? *Answer:* No, since e.g. $f(2) = f(3) = 3$.

(e) Is f onto the set of integers? *Answer:* No, since e.g. 4 is not in the range.

(f) Which of the following is a formula for $f(n)$ when $n \geq 0$? (Recall $\lfloor x \rfloor$ means the greatest integer y such that $y \leq x$.)

(i) $3\lfloor \frac{n+1}{3} \rfloor$ *Answer:* This one. The others are wrong for $n = 4$ or for $n = 5$.

(ii) $3\lfloor \frac{n}{3} \rfloor$

(iii) $3\lfloor \frac{n+2}{3} \rfloor$

(iv) $3\lfloor n+1 \rfloor$

(3) (9 + 12 + 3 = 24 points total)

Let A, B, C be subsets of some universe U. Consider the proposition

$$P \equiv A \setminus (\tilde{B} \cup \tilde{C}) \subseteq B \cap C.$$

(Note: the text would write $A - (\bar{B} \cup \bar{C}) \subseteq B \cap C$ instead.)

(a) Write the corresponding logical proposition, using just a for "$x \in A$", and b similarly for B, c similarly for C. Call it ρ (Greek rho).

(b) Prove that ρ is a tautology. Any covered proof method, e.g. truth-tables or some other "semantic proof," proof rules or some other "syntactic proof," is AOK.

(c) Deduce that P itself is always true.

Answer: (a) Translating set-difference as "and not" and $\subseteq$ as implication, we have $\rho = [a \wedge \sim (\neg b \cup \neg c)] \longrightarrow (b \wedge c)$.

(b) Quickest is to give a syntactic proof using DeMorgan's Laws and simplification: Borrow the left-hand side of ρ. By DeMorgan's Laws, $\sim (\neg b \cup \neg c)$ becomes simply $b \wedge c$, so $\rho \equiv a \cap b \cap c \longrightarrow b \cap c$. Remember we borrowed the left-hand side which is now $a \cap b \cap c$, but the right-hand side $b \cap c$ now follows by simplification, so we can redeem ρ itself. Thus ρ is a tautology.

(c) Super-formally, P "expands" into the logical assertion

$$(\forall x \in U)[x \in A \setminus (\tilde{B} \cup \tilde{C}) \longrightarrow x \in B \cap C].$$

which is equivalent to

$$(\forall x \in U)[(x \in A \wedge \neg(x \notin B \cup x \notin C)) \longrightarrow (x \in B \wedge x \in C)].$$

Because ρ is a tautology, the part in $[\ldots]$ is true for all x, so $(\forall x \in U)[\ldots]$ is true, which means P is true. (Any reasonable explanation of the conncetion between set relations and logic was full-credit here.)

(4) (5 × 3 = 15 pts.)

True/False. Please write out the words `true` and `false` in full. Brief justifications are not needed, but might help for partial credit.

(a) The power set of the empty set is the empty set. *Answer:* `false`—it's $\{\emptyset\}$ not $\emptyset$.

(b) The power set of the empty set has cardinality $2^0 = 1$. *Answer:* `true`: $\{\emptyset\}$ has cardinality 1, which allows it to *be* 1 in "set-theory math."

(c) If $P(0)$ is true, $P(1)$ is true, and for all $n \geq 2$, $P(n-2) \longrightarrow P(n)$ is true, then $(\forall n)P(n)$ is true, where n ranges over the domain of natural numbers. *Answer:* `true`—the even and odd cases induct separately.

(d) The complement of the union of two sets is always a subset of one of the sets. *Answer:* `false`—it excludes both sets.

(e) The difference of two sets is always a subset of their intersection. *Answer:* `false`—it's the first set *minus* the intersection.