Closed book, closed-notes except for 1 sheet, closed neighbors, 48 minutes. Please do ALL THREE problems on the separate exam sheet provided. Please *show all your work*; this may help for partial credit. Reminder: $\mathbb{N}$ includes 0. The exam totals 80 pts., subdivided as shown.

(1) (26 pts. total)

    For each of the following relations, say whether it defines r as a function of integers m and/or n. Real numbers that aren't integers are disallowed as values as well as arguments. Then for every *function*, say whether it is 1–1, and say whether it is onto the set $\mathbb{Z}$ of integers.

  (a) $m + r = n$.

  (b) $r^2 + r = m$.

  (c) $r = (n^2 + n)/2$.

  (d) $r = 4m - 2n$.

  (e) $r = 4m - 3n$.

(2) (9 + 9 + 9 = 27 pts.)

  (a) Is m^2 always congruent to $(m - n)(m + n)$ modulo n? Prove your answer algebraically or give a counterexample.

  (b) Now prove that if $m < n - 1$, then $n^2 - m^2$ cannot be a prime number.

  (c) Give an example where $2 \leq m < n - 1 < 10$ and yet $n^2 - m^2$ is relatively prime both to m and to n.

(3) (12 + 3 + 3 + 9 = 27 pts)

    This problem concerns the logical propositions Q_n and R_n defined for $n \geq 1$ by:

$$
\begin{aligned}
Q_n &= (A \longrightarrow S_1 \wedge S_2 \wedge \cdots \wedge S_n) \\
R_n &= (A \longrightarrow S_1) \wedge (A \longrightarrow S_2) \wedge \cdots \wedge (A \longrightarrow S_n).
\end{aligned}
$$

  (a) Prove by induction on n that Q_n is equivalent to R_n, for all n.

  (b) Is $n = 1$ sufficient as a base case, or did you need the case $n = 2$ separately?

  (c) Let $P_n = (Q_n \longleftrightarrow R_n)$, which is what you are proving is always a tautology in (a). Suppose we define P_n, Q_n, R_n also for $n = 0$. Then what does P_0 say? Is it true?

  (d) Translate Q_n and R_n into assertions Q'_n and R'_n about *sets*, that is re-interpreting $S_1, \ldots, S_n$ and A as *sets*. Give an intuitive argument for $Q'_n \iff R'_n$; drawing a Venn-diagram-style sketch is OK.