(1) Let an undirected graph $G = (V, E)$ have distinguished nodes s_1, s_2, t_1, t_2. Initially we place a chess queen on s_1 and a chess king on s_2. If the king and queen are ever on nodes connected by an edge, then the king is in check. The question is, can they execute a sequence of moves "asynchronously"—meaning either can move at any time and can make multiple moves in a row, not strict alternation or "lockstep" moves—so that the queen reaches t_1, the king reaches t_2, and the king is never in check?

Your task is to determine whether this decision problem belongs to NL or is NP-complete. Note that if the king and queen move simultaneously in lockstep, then this is the problem on Prelim II which is in NL. But if the queen could be "superposed" anywhere along its path to s_2 at once, so that the king would need to find a path that gives zero risk of being checked from the queen's path, then this becomes the same as the "Edge Disjoint Paths" problem which we saw to be NP-complete. So which problem is this like? You must prove your answer, at least up to the part of the Prelim II key that is relevant to your answer. (24 pts.)

Answer: The problem is in NL, hence in P. The important contrast with the edge-disjoint paths problem is once again that the absence of edges need only be checked at two focal points, here the current locations of the king and queen. So we design an NTM N that has exactly the same memory map as the one in the Prelim II answer. The only difference in operation is that instead of guessing a neighbor for both u and v "in lockstep," N nondeterministically chooses one of them and updates only it. The check is then that there is no edge (u', v) (if u was updated to u'), or alternatively no edge (u, v') (if v was updated to v'). The rest is as before.

The avenue of defining a modified graph G' also works, though is IMHO a trickier to visualize when the original graph G is bushy (and in particular, not a planar graph). You need the new edge set to consist of pairs-of-pairs $((u, v), (u', v'))$ such that either $u' = u$ and G has an edge (v, v'), or $v' = v$ and G has an edge (u, u'). Thus far it is like the definition of the **line graph** of G, but that applies only where (u, v) is an edge in G. Here we always need (u, v) **not** to be an edge. So what we actually need is G' to be the line graph of the *complement* of G. Then BFS in this graph G' works. IMHO, the element of negation involved in its being the line graph of the *complement* makes this more difficult to visualize than the proof via G' on Prelim II. Whereas, the demonstration via NL-machine is as easy as before.

(2) Compute the tensor product of the row vector $u = \frac{1}{\sqrt{2}}(1, i)$ with the column vector $v = \frac{1}{\sqrt{2}}\binom{1}{i}$. You will get a 2×2 matrix A. Does it matter whether you do the tensor product as $A = u \otimes v$ or as $A = v \otimes u$? Also answer: Is A unitary? (15 pts. total)

Answer: The constant factors $\frac{1}{\sqrt{2}}$ multiply to give simply $\frac{1}{2}$, so we can temporarily put that aside and focus on the matrix contents. They give:

$$u \otimes v = \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} & i \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} \end{pmatrix} = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}.$$

The other way is

$$v \otimes u = \begin{pmatrix} 1 \cdot (1,\ i) \\ i \cdot (1,\ i) \end{pmatrix} = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix},$$

which is the same matrix A. Is this an accident or a general rule? Well, for a general $u = (a, b)$ and any column vector $v = \binom{c}{d}$ we get

$$\begin{aligned} u \otimes v &= \left(a \cdot \begin{pmatrix} c \\ d \end{pmatrix} \;\; b \cdot \begin{pmatrix} c \\ d \end{pmatrix} \right) = \begin{bmatrix} ac & bc \\ ad & bd \end{bmatrix}. \\ v \otimes u &= \begin{pmatrix} c \cdot (a,\ b) \\ d \cdot (a,\ b) \end{pmatrix} = \begin{bmatrix} ca & cb \\ da & db \end{bmatrix}. \end{aligned}$$

They are the same, but FYI that is because ordinary scalar multiplication is commutative. As for whether A is unitary, note that its adjoint is

$$A^* = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}.$$

The -1 did not change sign, because it is a real number, but the two occurrences of i (which were symmetric to each other) did change sign. Now we get (still leaving aside the $\frac{1}{2}$ part):

$$AA^* = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2i \\ 2i & 2 \end{bmatrix}.$$

Whoops—the off diagonal entries did **not** cancel. So A is not unitary.

(3) Now change u to be the row vector $\frac{1}{\sqrt{2}}(1, -i)$, keeping v the same. Now does $u \otimes v = v \otimes u$? Is the 2×2 matrix you get, either way, unitary now? (15 pts. total)

Answer: We know from the general demonstration in (2) that $u \otimes v = v \otimes u$. Now, still ignoring the constant factor, we get the matrix

$$B = \left(1 \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} \;\; - i \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}.$$

Now $B^T = \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ but we have to remember that when we then complex-conjugate, the signs of the imaginary parts change again, so we get $\begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}$ which is B back again. So $B^* = B$, which is the definition of B being **Hermitian**. Is it unitary? See,

$$BB^* = \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} = \begin{bmatrix} 1 + i \cdot -i & -i + i \\ i - i & i \cdot -i + -1 * -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The off-diagonal entries duly canceled. So when we restrore the constant factor, do we get a unitary matrix? Well, we thus actually have

$$B = \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}.$$

When we multiply BB^*, the factor outside becomes $\frac{1}{4}$. One-quarter. Not one-half. So $BB^* = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$. This is not the identity matrix, so B is **not** unitary. *See end note for some further chitchat, FYI.*

(4) Now let A be any 2×2 matrix and A^2 its square under ordinary matrix multiplication (not tensor product). Does $A \otimes A^2$ always equal $A^2 \otimes A$? Try it when A is the Hadamard matrix, and diagram a little two-qubit quantum circuit to interpret what $A \otimes A^2$ and/or $A^2 \otimes A$ winds up being in this case. (15 pts. total)

Answer: When A is the Hadamard matrix H, then via the fact $H^2 = I$, we get:

$$\begin{aligned} A \otimes A^2 &= H \otimes I, \qquad \text{but} \\ A^2 \otimes A &= I \otimes H. \end{aligned}$$

These are **not** the same. Visualized as two-qubit quantum circuits, the former is a single Hadamard gate on line 1, whereas the latter is a single Hadamard gate on line 2. These have different actions—e.g., when followed by a CNOT gate between the lines, one causes entanglement and the other does not.

(5) Show that the vector $w = \frac{1}{2}(1, 1, 1, -1)$ cannot be written as a tensor product of two smaller vectors. That is, it represents an entangled quantum state. Show this by writing out the equations you get if $w = (a, b) \otimes (c, d)$ and proving that they cannot be solved for this w. (You can if you want ignore the $\frac{1}{2}$ factor in w. 12 pts.)

Answer: Suppose it could be a tensor product $(a, b) \otimes (c, d) = (ac, ad, bc, bd)$. Then we would have the equations $ac = 1$, $ad = 1$, $bc = 1$, and $bd = -1$. But the first two equations entail $d = c$, whereupon the latter two become $bc = 1$ and $bc = -1$. This is impossible. So w is entangled.

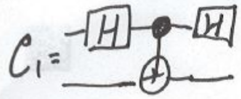
(6) Let C_1 be the two-qubit quantum circuit consisting of one Hadamard gate on line 1, then a CNOT gate with control on line 1 and target on line 2, and then another Hadamard gate on line 1. Let C_2 be the circuit that looks like C_1 "upside down": it has Hadamard on line 2, then CNOT with control on line 2 and target on line 1, and finally another Hadamard gate on line 2.

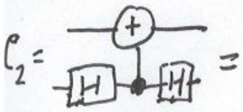
(a) Draw these two quantum circuits. (3 pts.)

(b) Use matrices to show that these circuits are equivalent. (OK, multiplying 4×4 matrices is tedious but this is "good for you." Good for 12 pts., anyway)

(c) Draw the "maze diagrams" for these two circuits, and trace using "signed mice" the result of running each on the input state $|10\rangle$. Check that you get the same results, as part (b) mandates. (12 pts., for 27 on the problem and 108 on the set)

Answer (hand-drawn):

$C_1 =$  $= \dfrac{\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}}{\sqrt{2}} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix} \dfrac{\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}}{\sqrt{2}} = \frac{1}{2}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&1&0&-1\\1&0&-1&0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1&1&1&-1\\1&1&-1&1\\1&-1&1&\underline{1}\\-1&1&\underline{1}&1\end{bmatrix}$

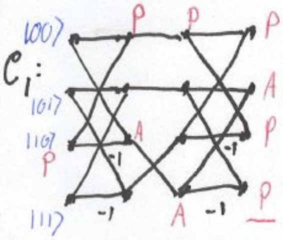
$C_2 =$  $= \dfrac{\begin{bmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{bmatrix}}{\sqrt{2}} \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix} \dfrac{\begin{bmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{bmatrix}}{\sqrt{2}} = \frac{1}{2}\begin{bmatrix}1&1&0&0\\1&-1&0&0\\0&0&1&1\\0&0&1&-1\end{bmatrix}\begin{bmatrix}1&1&0&0\\0&0&1&-1\\0&0&1&1\\1&-1&0&0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1&1&1&-1\\1&1&-1&\underline{1}\\1&-1&1&1\\-1&\underline{1}&1&1\end{bmatrix}$

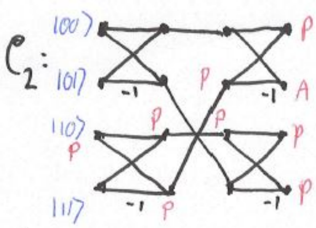
Here I first composed the two matrices on the right. The two underlined 1s in each final matrix resulted as $(-1)\cdot(-1)$. Anyway, the two final matrices are the same. Maze diagrams:

$H \otimes I$ gives 

$I \otimes H$ gives 

CNOT =  "CNOL" =  So we get

C_1: 

C_2: 

In both cases, "PAPP" stands for $(1, -1, 1, 1)$ which is the third column of the final matrix. This gives the action on the basis vector $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ which is $|10\rangle$.

[The underlined "Phil" is a case of A = "Anti-Phil" crossing another -1 edge and turning back into P = "Phil." Well, this is just a silly way of visualising $(-1)\cdot(-1) = +1$. This corresponds to the underlined 1 at the bottom of column 3 of the final matrix for circuit C_1. Whereas, the third column of the matrix for C_2 has no underline, and no "double-cheese" occurs in the maze trace for C_2.]

End Note on problem (3): That we got only $\frac{1}{2}I$ not I may seem loke a nasty trick, but who is the trickster? Since v is a column vector, we can think of it as a "ket": $|\,v\,\rangle$. The associated "bra" vector is then written $\langle\,v\,|$. But this is not the row vector in problem (2) even though it has the same entries. The imaginary part must be conjugated. Thus $\langle\,v\,|$ is the row vector $\frac{1}{\sqrt{2}}(1, -i)$ given here. The tensor product is then the same as the **outer product** which is written $|\,v\,\rangle\langle\,v\,|$. Visually this is $v \otimes u$ not $u \otimes v$ but we saw the order didn't matter. This is defined on page 134 of the supplementary physics-based reading and applied on the top of page 146. When we wrote $|\,v\,\rangle$ with a ket, we intended it to be part of an orthonormal basis. In 2-dimensional space, that needs one more unit vector, which in this case can be $w = \frac{1}{\sqrt{2}}\binom{i}{1}$. A similar calculation shows that

$$w \otimes \bar{w} = |\,w\,\rangle\langle\,w\,| = \frac{1}{2}\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix},$$

which is the transpose of B without conjugating. Let's call it C. Then we again get that $CC^* = \frac{1}{2}I$. The drumroll is that if we add the two outer-product matrices together:

$$D = |\,v\,\rangle\langle\,v\,| + |\,w\,\rangle\langle\,w\,| = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

which already is the identity matrix, hence is unitary. So getting only half the identity matrix from $|\,v\,\rangle\langle\,v\,|$ is because v was only half the basis.