CSE596, Fall 2018 **Prelim II** **Nov. 16, 2018**

Open book, open notes, closed neighbors, 48 minutes. The exam totals 80 pts., subdivided as shown. Do em all three problems in the exam book provided—there is no "choice" option. em Show your work—this may help for partial credit.

(1) (24+6 = 30 pts.)

Prove that the following decision problem "ERASURE" is undecidable.

INSTANCE: A TM M and an input x to M.

QUESTION: Does $M(x)$ at any point overwrite a non-blank character by the blank?

Also answer whether the language of this problem is c.e. Say why or why not.

Answer: To reduce from A_{TM}, we map an instance $\langle M, w\rangle$ to a machine M' (and specify $x = w$) as follows: On any input x, $M'(x)$ first simulates $M(w)$ *without writing any blanks.* If $M(w)$ writes a blank, M' writes a special character (say %) instead. If and when $M(w)$ accepts, M' then—and only then—writes a blank over a non-blank character. The code for M' is easily computed by stringing together the code of M (modified to write % and interpret it too as a blank) and the one blank-writing instruction.

For correctness of the reduction $f(\langle M, w\rangle) = \langle M', w\rangle$, if $\langle M, w\rangle \in A_{TM}$ then $M'(w)$ (indeed, $M'(x)$ for any x since this is an "all-or-nothing switch" situation even though we don't need it) sees the acceptance and overwrites a blank, so $f(\langle M, w\rangle) \in L_{\text{ERASURE}}$. But if $\langle M, w\rangle \notin A_{TM}$ then the lone blank-writing instruction is never reached, so $f(\langle M, w\rangle) \notin L_{\text{ERASURE}}$. Hence L_{ERASURE} is undecidable.

But L_{ERASURE} *is* c.e., basically because the behavior of erasing a non-blank char is immediately detectable: We run $M(w)$ via universal simulator and accept if and when it happens.

(2) (5 × 4 = 20 pts. total) *True/False.*

Please write out the words *true* and *false* in full. This time a brief justification is *required*—for 1 pt. *Please write in exam books only.*

(a) Every infinite subset of an undecidable language must be undecidable.

(b) There are languages in $\mathsf{DSPACE}[O(n^2)]$ that do not belong to $\mathsf{DSPACE}[O(n)]$.

(c) If $A \leq_m^p B$ and $B \in \mathsf{co\text{-}NP}$ then $A \in \mathsf{co\text{-}NP}$.

(d) If $A \leq_m^p B$ and A is c.e. then B is c.e.

(e) If 3SAT is in NL then $\mathsf{NP} = \mathsf{P}$.

Answer:

(a) *False*—consider that we can start with a decidable infinite language like 0^* and add strings beginning with 1 to make the whole language undecidable.

(b) *True*—by the Space Hierarchy Theorem since $n = o(n^2)$.

(c) *True*—because then $\tilde{A} \leq_m^p \tilde{B}$ and $\tilde{B} \in \mathsf{NP}$ and the "reduction theorem" applies.

(d) *False*—$A_{TM} \leq_m^p ALL_{TM}$ but ALL_{TM} is not c.e.

(e) *True*—since $\mathsf{NL} \subseteq \mathsf{P}$ the "if" part implies 3SAT $\in \mathsf{P}$ which implies $\mathsf{NP} = \mathsf{P}$ since 3SAT is NP-complete.

(3) 6 + 24 = 30 pts.

The 2018 US election saw great diversity of winning candidates across demographic and social groups. Note that one candidate can represent multiple groups, e.g., a male Latino millennial vegan. State the language of the following decision problem and show that it is NP-complete.

INSTANCE: A list of n election races, each with a Republican and Democratic candidate, and the demographic groups each candidate belongs to out of an overall list $D_1, \ldots, D_m$ of such groups.

QUESTION: Can the election possibly have an outcome in which each group is represented by at least one winning candidate?

Answer: Let the candidates be labeled r_i, d_i for each election race i, $1 \leq i \leq n$. This gives us a set C of $2n$ candidates and each D_j, $1 \leq j \leq m$, is a subset of C. Then $L = \{\langle C, D_1, \ldots, D_m\rangle$: each $D_j \subseteq C$ and there is a set $W \subset C$ such that for each i, W contains exactly one of r_i, d_i and for each j, $W \cap D_j \neq \emptyset\}$. The words "there is" are a tipoff that an NTM can guess W, and then verifying all the subset conditions can be done in time linear in $n + m$, hence linear in the size of $\langle C, D_1, \ldots, D_m\rangle$. Thus $L \in \mathsf{NP}$.

To show completensss, we reduce 3SAT to L. Let any 3CNF formula $\phi(x_1, \ldots, x_n) = C_1 \wedge \cdots \wedge C_m$ be given. For each i, $1 \leq i \leq n$, declare x_i to be the "Democrat d_i" and $\bar{x}_i$ to be the "Republican r_i" (or vice-versa) in the i-th election race. For each clause C_j, take D_j to be the simple set of its (up to) three literals and declare that to be the j-th "demographic group." The resulting set C of candidates (= the literals) plus list $\langle D_1, \ldots, D_m\rangle$ is easily computed—the reduction is almost the identity function—so this is a polynomial-time computable construction.

For correctness, a truth assignment to ϕ is the same as an election outcome in which the winner of each race i was the literal (i.e., candidate) getting the value 1 in the assignment. And the assignment satisfies all the clauses if and only if the set W of winners includes at least one member of each D_j. Going the other way, a set W of winners of each race corresponds to the truth assignment in which the winner got 1, the loser 0, and it satisfies ϕ if and only if each D_j had a winner. So 3SAT $\leq_m^p L$, so L is NP-complete.