algorithm computing matrix rank over $\mathbb{F}_2$

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Abstract

1 Introduction

2 Fourier Analysis on Finite Groups

Let G be a finite abelian group. a character of G is simply a homomorphism ψ from G to the multiplicative group of the complex numbers $\mathbb{C}^*$: $\psi(a+b) = \psi(a)\psi(b)$, and $\psi(-a) = \frac{1}{\psi(a)}$. Since G is finite, we have that every element in the image of ψ is a root of unity, and thus $\frac{1}{\psi(a)} = \overline{\psi(a)}$.

Characters form a group under multiplication. Define the dual group of G to be the group $\widehat{G}$ of all characters of G. Let ψ_0 be the trivial character, which maps all of G to 1; this is the identity element of $\widehat{G}$.

Examples. Let $G = \mathbb{Z}_p$ (p need not be prime) and $\omega_p = e^{\frac{2\pi i}{p}}$. For $a \in \mathbb{Z}_p$, define $\psi_a : G \to \mathbb{C}$ by:

$$\psi_a(x) = \omega_p^{ax}.$$

Here we can see that ω_p is the primitive p-th root of unity. Then $\widehat{G} = \{\psi_a | a \in \mathbb{Z}_p\}$. In our case, $G = \mathbb{Z}_{2^n}$, and for $a \in \mathbb{Z}_{2^n}$, define $\psi_a : G \to \mathbb{C}$ by:

$$\psi_a(x) = (-1)^{a \oplus x} = (-1)^{\sum_{i=1}^n a_i x_i}.$$

Then $\widehat{G} = \{\psi_a | a \in \mathbb{Z}_{2^n}\}$.

Inner Product. For two complex-valued functions f, g on G, define the inner product to be

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{a \in G} f(a)\overline{g(a)} = \mathbb{E}_{a \in G}[f(a)\overline{g(a)}].$$

Now we can see that every function $f : G \to \mathbb{C}$ can be written as a linear combination of characters of G.

Lemma 2.1. *Every $f : G \to \mathbb{C}$ has the following expression:*

$$f(x) = \sum_{a \in \hat{G}} \hat{f}(a)\psi_a(x),$$

where

$$\hat{f}(a) = \langle f, \psi_a \rangle = \mathbb{E}_{x \in G}[f(x)\overline{\psi_a(x)}].$$

Lemma 2.2 (Plancherel Identity.)**.** *Let $f, g : G \to \mathbb{C}$, then:*

$$\langle f, g \rangle = |G| \langle \hat{f}, \hat{g} \rangle = \sum_{a \in \hat{G}} \hat{f}(a)\overline{\hat{g}(a)}.$$

Proof.

$$\begin{aligned}
\langle f(x), g(x)\rangle &= \mathbb{E}_{x\in G}[\left(\sum_{a_1\in\hat{G}} \hat{f}(a_1)\psi_{a_1}(x)\right)\left(\sum_{a_2\in\hat{G}} \hat{g}(a_1)\psi_{a_2}(x)\right)] \\
&= \mathbb{E}_{x\in G}[\sum_{a_1,a_2\in\hat{G}} \hat{f}(a_1)\psi_{a_1}(x)\overline{\hat{g}(a_2)\psi_{a_2}(x)}] \\
&= \sum_{a_1,a_2\in\hat{G}} \mathbb{E}_{x\in G}[\hat{f}(a_1)\psi_{a_1}(x)\overline{\hat{g}(a_2)\psi_{a_2}(x)}] \\
&= \sum_{a_1,a_2\in\hat{G}} \hat{f}(a_1)\overline{\hat{g}(a_2)}\mathbb{E}_{x\in G}[\psi_{a_1}(x)\overline{\psi_{a_2}(x)}] \\
&= \sum_{a_1,a_2\in\hat{G}} \hat{f}(a_1)\overline{\hat{g}(a_2)}\mathbf{1}_{a_1=a_2} \\
&= \sum_{a\in\hat{G}} \hat{f}(a)\overline{\hat{g}(a)} \\
&= |\hat{G}|\mathbb{E}_{a\in\hat{G}}[\hat{f}(a)\overline{\hat{g}(a)}] \\
&= |G|\langle\hat{f},\hat{g}\rangle.
\end{aligned}$$

□

Lemma 2.3 (Parseval Identity.)**.** *Let* $f : G \to \mathbb{C}$*, then*

$$\langle f, f\rangle = \|f\|_2^2 = \sum_{a\in\hat{G}} |\hat{f}(a)|^2.$$

3 Simulation of Stabilizer Circuits

It is straightforward that

$$a(G) = \frac{1}{2^n}\sum_{x\in\mathbb{F}_2^n} f(x) = \langle f, \psi_0\rangle = \hat{f}(0)$$

Note that the fourier coefficient $\hat{f}(0)$ is exactly the amplitude $\langle 0^n|C_G|0^n\rangle$ for 0^n on the stabilizer circuit corresponding to G. To extend it to arbitrary output b, we know that the corresponding quadratic form becomes

$$q_b(x) = q(x) + 2\sum_{j:b_j=1} x_j \mod 4$$

with $b = (b_1, \cdots b_n)$. We obtain

$$\langle \mathbf{b}|C_G|0^n\rangle = \frac{1}{2^n}\sum_{x\in F_2^n} f(x)\psi_b(x) = \hat{f}(b).$$

Now consider a graph $G = (V, E)$ and its adjacency matrix $\mathbf{A}_G$. Assume we are also given matrix $\mathbf{P}$ such that $\mathbf{P}^\top\mathbf{A}_G\mathbf{P} = \mathbf{D}$ where $\mathbf{D}$ is in normalized form. Let $\mathbf{E}_i$ be the matrix with only the (i, i)-th entry being 1 and others 0. Also let $\mathbf{E}_\mathbf{v} = diag(v_1, \cdots, v_n)$ with $\mathbf{v} = (v_1, \cdots, v_n)$. It is easy to check that $\mathbf{P}^\top 2\mathbf{E}_i\mathbf{P}$ is again a diagonal matrix. More precisely,

$$\mathbf{P}^\top 2\mathbf{E}_i\mathbf{P} = 2diag(P_{i,1}\cdots, P_{i,n}) \mod 4,$$

and we have

$$\mathbf{P}^\top 2\mathbf{E_v}\mathbf{P} = 2\sum_{i:v_i=1} diag(P_{i,1}\cdots, P_{i,n}) \mod 4.$$

Suppose the vector set $\{\mathbf{v_i}\}$ with cardinality $\mathsf{rank}(\mathbf{A}_G)$ over $\mathbb{F}_2$ such that

$$\mathbf{P}^\top 2\mathbf{E}_{\mathbf{v_i}}\mathbf{P} = 2\mathbf{E}_i.$$

With lemma 4.1, 4.2 and 4.3 in [], we can check that the vector space spanned by $\{\mathbf{v_i}\}$ over $\mathbb{F}_2$ gives all the vector $\mathbf{b}$ such that $\langle \mathbf{b}|C_G|0^n\rangle \neq 0$. The proof can be extended directly from the proof for those lemmas. Hence we have the following:

Lemma 3.1. *Given an adjacency matrix* $\mathbf{A}$ *for a stabilizer circuit, the set of all outputs of non-zero amplitudes form a vector space of dimension* $\mathsf{rank}(\mathbf{A})$ *over* $\mathbb{F}_2$*.*

4 Algorithm Description

For rank computing, we just need to focus on the cases of bipartite graphs. Pictorially, we can look at the nodes of a given bipartite graph as two separate node sets: one on the left side (LHS) and one on the right right (RHS). Then each edge only connects one node on the left to one on the right.

4.1 Base Example

Consider a base case **??**: an n-node bipartite graph $G = (V, E)$ with two nodes on LHS and arbitrary many nodes on RHS. Then the corresponding quadratic form will be

$$q(x) = 2x_1 \sum_{(1,i)\in E} x_i + 2x_2 \sum_{(2,i)\in E} x_i \mod 4.$$

Moreover in bipartite cases, it is equivalent to

$$q(x) = x_1 \sum_{(1,i)\in E} x_i + x_2 \sum_{(2,i)\in E} x_i \mod 2.$$

Hereafter, we identify $q(x)$ with the one over $\mathbb{F}_2$. Define

$$f(x) = (-1)^q(x).$$

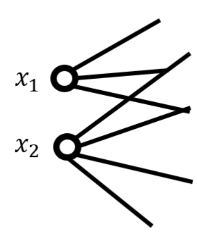

Figure 1:

Now let

$$g_1(x) = (-1)^{x_1\sum_{(1,i)\in E} x_i}, \quad g_2(x) = (-1)^{x_2\sum_{(2,i)\in E} x_i}.$$

We have $f(x) = g_1(x)g_2(x)$.

Note that the amplitude $\langle 0^n|C_G|0^n\rangle = \hat{f}(0)$ now equals $\frac{1}{2^n}\sum_{x\in\mathbb{F}_2^n} g_1(x)g_2(x)$, and more precisely, it is the following:

$$\hat{f}(0) = \frac{1}{2^n}\sum_{x\in\mathbb{F}_2^n} g_1(x)g_2(x) = \langle g_1, g_2\rangle = \sum_{a\in\mathbb{F}_2^n} \hat{g}_1(a)\hat{g}_2(a),$$

where the last equality follows Plancherel Identity. More generally, we have

$$\begin{aligned}\hat{f}(b) &= \frac{1}{2^n}\sum_{x\in\mathbb{F}_2^n} g_1(x)g_2(x)\psi_b(x)\\ &= \langle g_1, g_2\cdot\psi_b\rangle\\ &= \sum_{a\in\mathbb{F}_2^n} \hat{g}_1(a)\widehat{g_2\cdot\psi_b}(a)\\ &= \sum_{a\in\mathbb{F}_2^n} \hat{g}_1(a)\langle g_2\cdot\psi_b, \psi_a\rangle\\ &= \sum_{a\in\mathbb{F}_2^n} \hat{g}_1(a)\langle g_2, \psi_{b+a}\rangle\\ &= \sum_{a\in\mathbb{F}_2^n} \hat{g}_1(a)\widehat{g_2}(b+a).\end{aligned}$$

Apparently, we also have $\hat{f}(b) = \sum_{a\in\mathbb{F}_2^n} \hat{g}_1(b+a)\widehat{g_2}(a)$ from commutativity.

Note that the rank of the adjacency matrix for component defined by g_1 is 2, and so is for g_2. Now suppose $V = \mathsf{span}(\{\mathbf{v_1}, \mathbf{v_2}\})$ (as defined and discussed in Section 3) for g_1 such that $\hat{g}_1(c_1\mathbf{v_1} + c_2\mathbf{v_2}) \neq 0$ with $c_i \in \mathbb{F}_2$, and $W = \mathsf{span}(\{\mathbf{w_1}, \mathbf{w_2}\})$ for g_2. Without loss of generality, assume $\hat{g}_1(0) = \hat{g}_1(\mathbf{v_1}) = \hat{g}_1(\mathbf{v_2}) > 0$ and $\hat{g}_1(\mathbf{v_1}+\mathbf{v_2}) < 0$. Same for g_2. Now we have

$$\begin{aligned}\hat{f}(b) &= \sum_{a\in\mathbb{F}_2^n} \hat{g}_1(a)\widehat{g_2}(b+a)\\ &= \frac{1}{2}\hat{g}_2(\mathbf{b}) + \frac{1}{2}\hat{g}_2(\mathbf{b}+\mathbf{v_1}) + \frac{1}{2}\hat{g}_2(\mathbf{b}+\mathbf{v_2}) - \frac{1}{2}\hat{g}_2(\mathbf{b}+\mathbf{v_1}+\mathbf{v_2}).\end{aligned}$$

There will be three different cases to consider:

1. $V \perp W$
2. $\dim(V\cap W) = 1$
3. $\dim(V\cap W) = 2$, i.e., $V = W$

Case (1): $V \perp W$.

$$\hat{f}(\mathbf{0}) = \frac{1}{2}\hat{g}_2(\mathbf{0}) + \frac{1}{2}\hat{g}_2(\mathbf{0}+\mathbf{v_1}) + \frac{1}{2}\hat{g}_2(\mathbf{0}+\mathbf{v_2}) - \frac{1}{2}\hat{g}_2(\mathbf{b}+\mathbf{v_1}+\mathbf{v_2}) = \frac{1}{4} = \frac{1}{2^{r/2}},$$

because $\hat{g}_2(\mathbf{v_1}) = \hat{g}_2(\mathbf{v_2}) = \hat{g}_2(\mathbf{v_1}+\mathbf{v_2}) = 0$ and $\hat{g}_2(\mathbf{0}) = \frac{1}{2}$. Hence, the total rank is 4. We can also see that the basis set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{w_1}, \mathbf{w_2}\}$ gives the vector space $(V\cup W)$ such that $\hat{f}(\mathbf{b}) \neq 0, \forall\mathbf{b} \in V\cup W$.

Case (2): $\dim(V\cap W) = 1$. Let $\mathbf{t}$ be the basis vector of the intersection space. We do a case-by-case analysis to all the possible situations as listed below:

(a) $\mathbf{t} = \mathbf{v_1} = \mathbf{w_1}$: rank $= 2$, basis $\{\mathbf{w_1}, \mathbf{w_2} + \mathbf{v_2}\}$

(b) $\mathbf{t} = \mathbf{v_2} = \mathbf{w_2}$: rank $= 2$, basis $\{\mathbf{w_2}, \mathbf{w_1} + \mathbf{v_1}\}$

(c) $\mathbf{t} = \mathbf{v_1} = \mathbf{w_2}$: rank $= 2$, basis $\{\mathbf{w_2}, \mathbf{w_1} + \mathbf{v_2}\}$

(d) $\mathbf{t} = \mathbf{v_2} = \mathbf{w_1}$: rank $= 2$, basis $\{\mathbf{w_1}, \mathbf{w_2} + \mathbf{v_1}\}$

(e) $\mathbf{t} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1}$: not possible

(f) $\mathbf{t} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_2}$: not possible

(g) $\mathbf{t} = \mathbf{v_1} = \mathbf{w_1} + \mathbf{w_2}$: not possible

(h) $\mathbf{t} = \mathbf{v_2} = \mathbf{w_1} + \mathbf{w_2}$: not possible

(i) $\mathbf{t} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1} + \mathbf{w_2}$: rank $= 2$, basis $\{\mathbf{w_1} + \mathbf{w_2}, \mathbf{w_1} + \mathbf{v_1}\}$ or basis $\{\mathbf{w_1} + \mathbf{v_1}, \mathbf{w_1} + \mathbf{v_2}\}$

Consider the situation: (2.a) $\mathbf{t} = \mathbf{v_1} = \mathbf{w_1}$.

$$\hat{f}(\mathbf{b}) = \frac{1}{2}\hat{g}_2(\mathbf{b}) + \frac{1}{2}\hat{g}_2(\mathbf{b} + \mathbf{v_1}) + \frac{1}{2}\hat{g}_2(\mathbf{b} + \mathbf{v_2}) - \frac{1}{2}\hat{g}_2(\mathbf{b} + \mathbf{v_1} + \mathbf{v_2}).$$

By plugging in $\mathbf{b} = \mathbf{0}$, we get $\hat{f}(\mathbf{0}) = \frac{1}{2}$ since $\hat{g}_2(\mathbf{v}_2) = \hat{g}_2(\mathbf{v}_1 + \mathbf{v}_2) = 0$ by the fact that $\mathbf{v}_2 \in W$. Hence, rank remains to be 2.

Now some insights for finding the basis for the space S such that $\hat{f}(\mathbf{b}) \neq 0\ \forall \mathbf{b} \in S$ are (1) if $\mathbf{b} \in W$, both terms $\hat{g}_2(\mathbf{b} + \mathbf{v_2})$ and $\hat{g}_2(\mathbf{b} + \mathbf{v_1} + \mathbf{v_2})$ will be zero; (2) if $\mathbf{b} \notin W$, we need $\hat{g}_2(\mathbf{b} + \mathbf{v_2}) = -\hat{g}_2(\mathbf{b} + \mathbf{v_1} + \mathbf{v_2})$ and not equal to zero. It is easy to check that $\{\mathbf{w_1}, \mathbf{w_2} + \mathbf{v_2}\}$ is one valid basis for the space S such that $\hat{f}(\mathbf{w_1} + \mathbf{w_2} + \mathbf{v_2})$ has negative amplitude.

The same analysis can be applied to cases (2.b), (2.c) and (2.d) and derive what it shows above. In more general words, the basis consists of (1) one member from the intersection space and (2) one member produced by summing up the basis outside this intersection space.

Cases (2.e), (2.f), (2,g) and (2.h) are impossible because, for instance in (2.e),

$$\hat{f}(\mathbf{0}) = \frac{1}{2}\hat{g}_2(\mathbf{0}) + \frac{1}{2}\hat{g}_2(\mathbf{v_1}) + \frac{1}{2}\hat{g}_2(\mathbf{v_2}) - \frac{1}{2}\hat{g}_2(\mathbf{v_1} + \mathbf{v_2}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 - \frac{1}{2} \cdot \frac{1}{2} = 0,$$

where contradict the fact that $\hat{f}(\mathbf{0}) > 0$ for bipartite graphs.

As for the last case: $\mathbf{t} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1} + \mathbf{w_2}$. We know the rank is again 2 by $\hat{f}(\mathbf{0}) = \frac{1}{2}$. Following the same insights as above, we can derive a basis set $\{\mathbf{w_1} + \mathbf{w_2}, \mathbf{w_1} + \mathbf{v_1}\}$. However, $\hat{f}(\mathbf{w_1} + \mathbf{w_2})$ gives negative amplitude, and we would want a basis set $\{\mathbf{s}_1, \mathbf{s}_2\}$ such that $\hat{f}(\mathbf{s}_1 + \mathbf{s}_2) < 0$, because this will be consistent with the fact that both $\hat{g}_1(\mathbf{v}_1 + \mathbf{v}_2)$ and $\hat{g}_2(\mathbf{w}_1 + \mathbf{w}_2)$ are negative. Hence, instead, we take the basis set $\{\mathbf{w_1} + \mathbf{v_1}, \mathbf{w_1} + \mathbf{v_2}\}$ associated with f.

Case (3): $\dim(V \cap W) = 2$. Let $\{\mathbf{t_1}, \mathbf{t}_2\}$ be the basis of the intersection space. We do a case-by-case analysis to all the possible situations as listed below:

(a) $\mathbf{t_1} = \mathbf{v_1} = \mathbf{w_1}, \mathbf{t}_2 = \mathbf{v_2} = \mathbf{w_2}$: rank $= 0$

(b) $\mathbf{t_1} = \mathbf{v_1} = \mathbf{w_2}, \mathbf{t}_2 = \mathbf{v_2} = \mathbf{w_1}$: rank $= 0$

(c) $\mathbf{t_1} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1}, \mathbf{t}_2 = \mathbf{v_1} = \mathbf{w_2}$: not possible

(d) $\mathbf{t_1} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1}, \mathbf{t}_2 = \mathbf{v_2} = \mathbf{w_2}$: not possible

(e) $\mathbf{t_1} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_2}, \mathbf{t}_2 = \mathbf{v_1} = \mathbf{w_1}$: not possible

(f) $\mathbf{t_1} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_2}, \mathbf{t}_2 = \mathbf{v_2} = \mathbf{w_1}$: not possible

(g) $\mathbf{t_1} = \mathbf{v_1} = \mathbf{w_1} + \mathbf{w_2}, \mathbf{t}_2 = \mathbf{v_2} = \mathbf{w_1}$: not possible (same as (3.f))

(h) $\mathbf{t_1} = \mathbf{v_1} = \mathbf{w_1} + \mathbf{w_2}, \mathbf{t}_2 = \mathbf{v_2} = \mathbf{w_2}$: not possible (same as (3.d))

(i) $\mathbf{t_1} = \mathbf{v_2} = \mathbf{w_1} + \mathbf{w_2}, \mathbf{t}_2 = \mathbf{v_1} = \mathbf{w_1}$: not possible (same as (3.e))

(j) $\mathbf{t_1} = \mathbf{v_2} = \mathbf{w_1} + \mathbf{w_2}, \mathbf{t}_2 = \mathbf{v_1} = \mathbf{w_2}$: not possible (same as (3.c))

(k) $\mathbf{t_1} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1} + \mathbf{w_2}, \mathbf{t_2} = \mathbf{v_1} = \mathbf{w_1}$: rank $= 0$ (same as (3.a))

(l) $\mathbf{t_1} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1} + \mathbf{w_2}, \mathbf{t_2} = \mathbf{v_1} = \mathbf{w_2}$: rank $= 0$ (same as (3.b))

(m) $\mathbf{t_1} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1} + \mathbf{w_2}, \mathbf{t_2} = \mathbf{v_2} = \mathbf{w_1}$: rank $= 0$ (same as (3.b))

(n) $\mathbf{t_1} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1} + \mathbf{w_2}, \mathbf{t_2} = \mathbf{v_2} = \mathbf{w_2}$: rank $= 0$ (same as (3.a))

Consider (3.a): $\mathbf{t_1} = \mathbf{v_1} = \mathbf{w_1}, \mathbf{t}_2 = \mathbf{v_2} = \mathbf{w_2}$.

$$\begin{aligned}
\hat{f}(\mathbf{0}) &= \frac{1}{2}\hat{g}_2(\mathbf{0}) + \frac{1}{2}\hat{g}_2(\mathbf{v_1}) + \frac{1}{2}\hat{g}_2(\mathbf{v_2}) - \frac{1}{2}\hat{g}_2(\mathbf{v_1} + \mathbf{v_2}) \\
&= \frac{1}{2}\hat{g}_2(\mathbf{0}) + \frac{1}{2}\hat{g}_2(\mathbf{w_1}) + \frac{1}{2}\hat{g}_2(\mathbf{w_2}) - \frac{1}{2}\hat{g}_2(\mathbf{w_1} + \mathbf{w_2}) \\
&= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot(-\frac{1}{2}) \\
&= 1.
\end{aligned}$$

This means that the rank becomes 0. We can see that this actually corresponds to the scenario that components g_1 and g_2 have identical structure and hence cause cancellation in the combined stabilizer circuit. The same argument works for case (3.b).

Now for (3.c): $\mathbf{t_1} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1}, \mathbf{t}_2 = \mathbf{v_1} = \mathbf{w_2}$.

$$\begin{aligned}
\hat{f}(\mathbf{0}) &= \frac{1}{2}\hat{g}_2(\mathbf{0}) + \frac{1}{2}\hat{g}_2(\mathbf{v_1}) + \frac{1}{2}\hat{g}_2(\mathbf{v_2}) - \frac{1}{2}\hat{g}_2(\mathbf{v_1} + \mathbf{v_2}) \\
&= \frac{1}{2}\hat{g}_2(\mathbf{0}) + \frac{1}{2}\hat{g}_2(\mathbf{w_2}) + \frac{1}{2}\hat{g}_2(\mathbf{w_2} + \mathbf{w_1}) - \frac{1}{2}\hat{g}_2(\mathbf{w_1}) \\
&= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot(-\frac{1}{2}) - \frac{1}{2}\cdot\frac{1}{2} \\
&= 0
\end{aligned}$$

contradicting the fact that $\hat{f}(\mathbf{0}) > 0$ for bipartite graphs. I would like to point out that this case is valid for larger bipartite graphs, which will be discussed in Section 4.2. Using this argument on cases (3.d) through (3.f) can lead to the same conclusion on them. Also note that (3.g) through (3.j) are the same as (3.c) through (3.f), respectively. For instance, (3.g) is identical to (3.f) because $\mathbf{v_1} + \mathbf{v_2} = \mathbf{t_1} + \mathbf{t_2} = \mathbf{w_2}$. Hence, they will again lead to zero rank.

Case (3.k): $\mathbf{t_1} = \mathbf{v_1} + \mathbf{v_2} = \mathbf{w_1} + \mathbf{w_2}, \mathbf{t_2} = \mathbf{v_1} = \mathbf{w_1}$.

$$\begin{aligned}
\hat{f}(\mathbf{0}) &= \frac{1}{2}\hat{g}_2(\mathbf{0}) + \frac{1}{2}\hat{g}_2(\mathbf{v_1}) + \frac{1}{2}\hat{g}_2(\mathbf{v_2}) - \frac{1}{2}\hat{g}_2(\mathbf{v_1} + \mathbf{v_2}) \\
&= \frac{1}{2}\hat{g}_2(\mathbf{0}) + \frac{1}{2}\hat{g}_2(\mathbf{w_1}) + \frac{1}{2}\hat{g}_2(\mathbf{w_2}) - \frac{1}{2}\hat{g}_2(\mathbf{w_1} + \mathbf{w_2}) \\
&= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot (-\frac{1}{2}) \\
&= 1.
\end{aligned}$$

Note that this case is exactly (3.a) because $\mathbf{v_2} = \mathbf{t_1} + \mathbf{t_2} = \mathbf{w_2}$. Similarly, we have that (1) (3.l) corresponds to (3.b); (2) (3.m) to (3.b); (3) (3.n) to (3.a).

4.2 Generalization

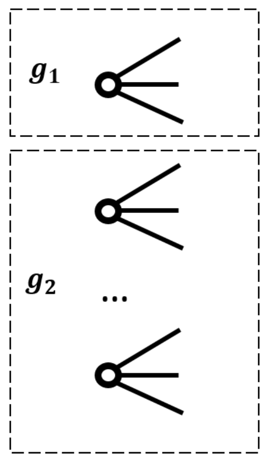

Figure 2:

Note that if we chop off a bipartite graph as in Figure 2, the following formula still works:

$$\begin{aligned}
\hat{f}(b) &= \sum_{a \in \mathbb{F}_2^n} \hat{g}_1(a)\widehat{g_2}(b + a) \\
&= \frac{1}{2}\hat{g}_2(\mathbf{b}) + \frac{1}{2}\hat{g}_2(\mathbf{b} + \mathbf{v_1}) + \frac{1}{2}\hat{g}_2(\mathbf{b} + \mathbf{v_2}) - \frac{1}{2}\hat{g}_2(\mathbf{b} + \mathbf{v_1} + \mathbf{v_2}),
\end{aligned}$$

where the vector space $V = \mathsf{span}(\{\mathbf{v_1}, \mathbf{v_2}\})$ is associated with g_1 such that $\hat{g}_1(c_1\mathbf{v_1} + c_2\mathbf{v_2}) \neq 0$ with $c_i \in \mathbb{F}_2$

5 Conclusions