USING COGNITIVE TASK ANALYSIS TO CAPTURE EXPERT INSTRUCTION IN DIVISION OF FRACTIONS

By

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Dedication

This dissertation is dedicated, in no particular order, to my father Bill, my wife Tamiko, and my mother Carolyn.

To my father, a 1932 graduate of this university, who once patiently explained to me that trout rest facing the current, and thus, when angling, it is important to wade in an upstream direction. This, he said, increases one's chances of sneaking up on them from behind.

To my wife, who understands me implicitly, and accordingly, understood going in that putting up with me during this process would involve incredible patience, support and understanding on her part. Tamiko, you provided all of that and more, and I owe you a debt of gratitude.

And to my mother, who encouraged me to attempt this task and who provided invaluable support. Mom, your unmitigated disdain for suffering fools is matched only by your immoderate fondness for leisurely dining. The next meal's on me.

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List of Abbreviations

CCSS:	Common Core State Standards
CDM:	Critical Decision Method
CPP:	Concepts, Processes, and Principles
CTA:	Cognitive Task Analysis
GSP:	Gold Standard Protocol
IRB:	Institutional Review Board
K–12:	Kindergarten through Twelfth Grade
NAEP:	National Assessment of Educational Progress
NCTM	National Council of Teachers of Mathematics
OECD:	Organisation for Economic Co-operation and Development
PGSP:	Preliminary Gold Standard Protocol
SME:	Subject Matter Expert
TIMSS	Trends in International Mathematics and Science Study
3i + 3r:	Three Independent Interviews + Three Reviews

Abstract

This study applies Cognitive Task Analysis (CTA), a method for eliciting the automated, unconscious knowledge and skills of experts, to capture expertise in teaching K-12 mathematics. The purpose of this study was to conduct a CTA with middle school teachers who have been identified as experts, to capture the knowledge and skills they use when providing instruction in the division of fractions by fractions. Also, this study investigated whether experts' knowledge omissions in this field would conform to those in other fields, which can approach 70%. CTA methods in this study included semi-structured interviews with three middle school mathematics teachers. The study's findings indicated that these experts recalled, on average, 39.14% of the action and decision steps compared to the gold standard protocol, while omitting, on average, 60.86% of such steps. The study's implications are that the degree of omissions among expert middle school teachers are similar to those of experts in other fields. Additionally, the greater degree of knowledge capture provided by the use of multiple experts, compared to that for a single expert, indicates that the use of CTA for the development of teacher preparation and professional development programs shows promise when compared to current models, which rely primarily on individual experts.

CHAPTER ONE: OVERVIEW OF THE STUDY

Statement of the Problem

Compared to their peers in foreign countries, American K-12 students display less than stellar mathematical achievement. The 2009 Program for International Student Assessment (PISA) assessed fifteen year-olds from Organization for Economic Cooperation and Development (OECD) countries in broad math content measures such as space and shape, quantity, uncertainty and data, and change and relationships (National Center for Education Statistics (NCES), 2013a). American students that year had an average score in math of 487, compared to an average of 496 for all participating OECD countries (NCES, 2009). In 2012, U.S. students fared no better, with an average math score of 481 (NCES, 2013a). This compares to an OECD average of 494, and falls far below the scores of the highest achieving nations, such as Shanghai, China, with a 613, and Singapore, with a score of 573 (NCES, 2012). U.S. students even trailed typical low-performers such as Slovenia, with an average of 501, and The Czech Republic, which scored 499 (NCES, 2012).

An international assessment that focuses more specifically on students' ability in specific content areas is the Trends in International Mathematics and Science Study (TIMSS), which assessed eighth graders from more than forty countries in numbers, algebra, geometry, and data and chance (NCES, 2007). In 2007 American students achieved an average overall math score of 508, just slightly above the international average of 500, and well below traditional math powerhouse countries, such as Singapore, with an average score of 593, and Honk Kong which averaged 572 (NCES, 2007). Scores of American eighth graders on the 2011 TIMSS were little better, with American students averaging a score of 509 in math overall, compared to the international average of 500 (NCES, 2011). Again, Asian countries outperformed all others, with Korea achieving an overall score of 613, and Taiwan a score of 609 (NCES, 2011).

In terms of specific content, one area of math where U.S. students replicate their *overall* mediocrity is in the domain of algebra. In 2009, U.S. students' average score in algebra on the TIMSS was 507, compared to the international average of 500, and well below Korea (608), Japan (567), Hong Kong (575), and Singapore (614) (NCES, 2011). American students' algebra performance on the 2011 TIMSS was slightly better, with a score of 512, but again, well below Korea at 617, Japan at 570, Hong Kong at 583 and Singapore at 614 (NCES, 2011).

The importance of algebra cannot be underestimated. The National Council of Teachers of Mathematics has categorized algebra as a towering accomplishment that is critical to mathematical work (2000). The National Math Advisory Panel (NMAP) identified algebra as a gateway to and necessity for more advanced math course work in high school (2008). The Common Core State Standards (National Governors' Council (NGA), 2010) stress the importance of students' ability to comprehend abstract situations and represent them symbolically. And perhaps most significantly, the authors of the Principles and Standards for School Mathematics (NCTM, 2000) posit that, algebra constitutes a major component of the school mathematical curriculum and serves to unify it.

One domain of elementary and middle school math that is considered foundational for learning algebra is the study of fractions (Fuchs et al., 2014). The National Mathematics Advisory Panel (NMAP, 2008) identified fluency with fractions as a critical steppingstone to algebra. An analysis of two nationally representative data sets, one in Britain and the other in the United States, indicated that elementary students' understanding of fractions predicted their knowledge of algebra in high school (Siegler & Pike, 2013). Kieren (1976) concluded that students must master fractional number concepts to sufficiently learn algebraic concepts. Finally, a survey of 1,000 U.S. Algebra 1 teachers identified fractions as one of the most significant weaknesses in students' preparation for algebra coursework (Hoffer, Venkataraman, Hedberg, & Shagle, 2007).

For students, one particularly vexing component of understanding fractions involves division of fractions. Sharp and Welder (2014) identify division of fractions as a common area of struggle in seventh grade math. Trafton and Zawojewski (1984, p. 20) describe division of fractions as a "troublesome" endeavor for many students. Coughlin (2010/2011) labels division of fractions one of the most complex tasks in elementary math, while Cengiz and Rathouz (2011) consider the procedure one of the most rote-like and least understood elementary math concepts.

Many of the difficulties that students face with division of fractions can be traced to deficiencies in teachers' mathematical knowledge (Koichu, Harel, & Manaster, 2013). Many teachers make use of the *invert and multiply* algorithm, but are unable to explain the principle involved (Borko et al., 1992). Ma (1999) found that many U.S. teachers were unable to create division of fraction word problems, due to insufficient conceptual understanding. Triosh (2000) found that among pre-service teachers of mathematics, there was incomplete understanding of students' fraction misconceptions. If it were possible to remediate these deficiencies in instruction and understanding, and capture the conceptual and procedural knowledge of teachers expert in teaching division of fractions, a schematic representation could be developed (Crandall, Klein &Hoffman, 2006) that could be used to train teachers.

Unfortunately, because the procedural knowledge possessed by experts is largely automated, and thus unconscious (Clark & Estes, 1996), highly skilled practitioners can omit up to 70% of such knowledge when asked to describe the complex tasks in which they are expert

(Clark, Feldon, van Merrienboer, Yates, & Early, 2008). One method of eliciting the knowledge and processes used by experts is Cognitive Task Analysis (CTA) (Tofel-Grehl & Feldon, 2013). CTA is grounded in data that suggest interviewing *multiple experts* increases the body of knowledge and process data from roughly 30%, when one expert is interviewed, to 75% or more when three experts are interviewed (Clark et al., 2008). Researchers can then translate this information into instructional guides that can be used to train novices (Clark et al., 2008). As a result this study proposes to adopt this methodology in capturing and synthesizing the expert declarative and procedural knowledge necessary to create a training guide for teaching division of fractions.

Purpose of the Study

The purpose of this study is to conduct a CTA with middle school teachers who have been identified as experts, to capture the knowledge and skills they use when providing instruction in the division of fractions by fractions.

The research questions that guide this study are:

1. What are the action and decision steps that expert middle school math teachers recall when they describe how they teach the division of fractions by fractions?
2. What percent of action and/or decision steps, when compared to a gold standard, do expert middle school math teachers omit when they describe how they teach the division of fractions by fractions?

Methodology of the Study

This study's methodology involved conducting a Cognitive Task Analysis to capture the knowledge and skills of middle school teachers from school districts in Southern California who were identified as subject matter experts (SMEs) in providing instruction in division of fractions by fractions. Three SMEs were selected; each participated in the interviews while all three also verified the aggregate data collected. The CTA followed a five-step process:

1) a preliminary phase for building general familiarity with the instructional process;
2) the identification of declarative and procedural knowledge, in addition to any organizational schemes used in applying these knowledge types;
3) a knowledge elicitation phase employing semi-structured interviews;
4) a data analysis phase, involving coding of interview transcripts, determining inter-rater reliability, and individual SME protocol verification;
5) the development of a gold standard protocol that was used to identify expert omissions and that can serve as a training guide for novice teachers.

Definition of Terms

The following are definitions of terms related to Cognitive Task Analysis as suggested by Zepeda McZeal (2014).

Adaptive expertise: When experts can rapidly retrieve and accurately apply appropriate knowledge and skills to solve problems in their fields or expertise; to possess cognitive flexibility in evaluating and solving problems (Gott, Hall Pokorny, Dibble, & Glaser, 1993; Hatano & Inagaki, 2000).

Automaticity: An unconscious fluidity of task performance following sustained and repeated execution; results in an automated mode of functioning (Anderson, *1996; Ericsson, 2004).*

Automated knowledge: Knowledge about how to do something; operates outside of conscious awareness due to task repetition (Wheatley & Wegner, 2001).

Cognitive load: Simultaneous demands placed on working memory during information processing that can present challenges to learners (Sweller, 1988).

Cognitive tasks: Tasks that require mental effort and engagement to perform (Clark & Estes, 1996).

Cognitive task analysis: Knowledge elicitation techniques for extracting implicit and explicit knowledge from multiple experts for use in instruction and instructional design (Clark et al., 2008; Schraagen, Chipman, & Shalin, 2000).

Conditional knowledge: Knowledge about why and when to do something; a type of procedural knowledge to facilitate the strategic application of declarative and procedural knowledge to problem solve (Paris, Lipson, & Wixson, 1983).

Declarative knowledge: Knowledge about why or what something is; information that is accessible in long-term memory and consciously observable in working memory (Anderson, 1996a; Clark & Elen, 2006).

Expertise: The point at which an expert acquires knowledge and skills essential for consistently superior performance and complex problem solving in a domain; typically develops after a minimum of 10 years of deliberate practice or repeated engagement in domain-specific tasks (Ericsson, 2004).

Procedural knowledge: Knowledge about how and when something occurs; acquired through instruction or generated through repeated practice (Anderson, 1982; Clark & Estes, 1996).

Subject matter expert: An individual with extensive experience in a domain who can perform tasks rapidly and successfully; demonstrates consistent superior performance or ability to solve complex problems (Clark et al., 2008).

Organization of the Study

Chapter Two of this study reviews the literature in two sections: the first section of the review examines the literature relevant to instruction in division of fractions by fractions, while the second section examines the literature relevant to Cognitive Task Analysis and its use as a knowledge elicitation technique. Chapter Three describes the study methodology and the manner in which the research approach addresses the research questions. Chapter Four is a review of the study results, and compares these results to each of the research questions. Chapter Five includes a discussion of the findings, an analysis of the implications of the results vis-à-vis instruction in the division of fractions by fractions and CTA, a discussion of the study's limitations, and a consideration of the implications for future research.

CHAPTER TWO: LITERATURE REVIEW

United States K-12 Mathematics Achievement: An International Context

An examination of one internationally administered assessment reveals that, compared to their grade level peers in other Organisation for Economic Co-operation and Development (OECD) countries, American students score at or below international averages for mathematics achievement. This assessment, the Program for International Student Assessment (PISA), has been administered every three years since 2000 by the OECD in over 60 OECD and non-OECD countries (National Center for Education Statistics (NCES), 2013a). The PISA assesses fifteen year-olds in mathematics, reading and science fluency (NCES, 2013a). The mathematics portion of the PISA measures achievement in four, broad content areas: space and shape, quantity, uncertainty and data, and change and relationships (NCES, 2013a). In addition it provides a measure of students' proficiency in three mathematical process skills: employing, formulating, and interpreting (NCES, 2013a). Overall scores reported represent an average of these four content and three process measures, expressed on a scale of from 0 to 1000 (NCES, 2013a). In 2009, a nationally representative sample of 5,233 U.S. students had an average overall math score of 487, which compares to an OECD average of 496, and falls far below Korea at 546, Finland at 541, and Switzerland at 534 (NCES, 2009). Overall, in 2009 the overall score for U.S. students was surpassed by 25 of the 34 participating OECD countries (NCES, 2009). These results are presented in Figure 1.

PISA scores in 2012, the most recent administration of the test, were equally disappointing, with a nationally representative sample of 6,111 U.S students posting an overall average of 481, which compares to an OECD overall average of 494 (NCES, 2012). Again, U.S. scores were well below those of higher-achieving countries, such as Korea with 554, Japan with 536, and Switzerland with 531 (NCES, 2012). And again, U.S. students' were outperformed by most of the participating countries: 23 of the 34 OECD nations had higher average scores than the U.S. (NCES, 2012). These results are presented in Figure 1.

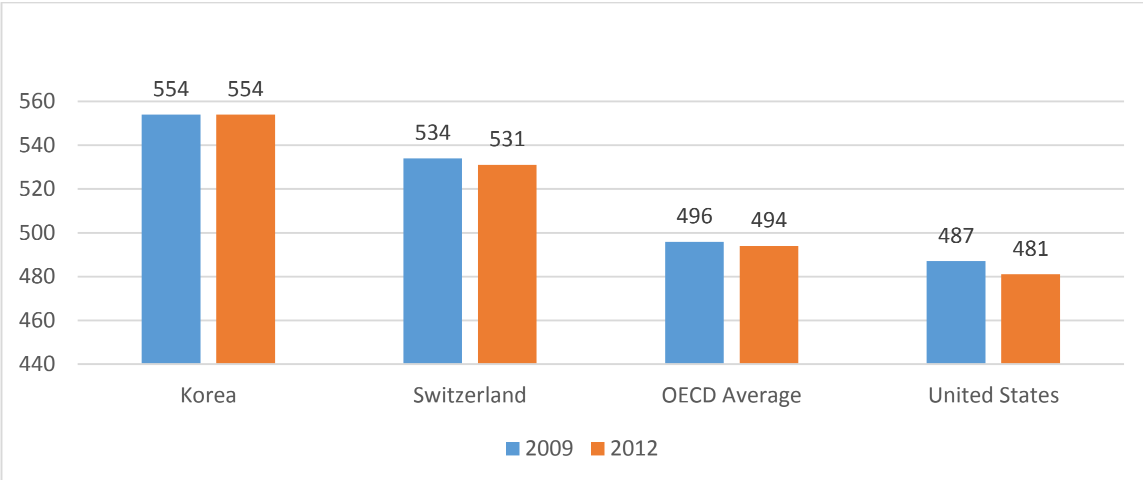

Figure1. PISA overall mathematics scores for Korea, Switzerland, and OECD average, as compared to the United States. Scale: 0 to 1,000.

Another major international assessment, The Trends in International Mathematics and Science Study (TIMSS), is administered by the International Association for Evaluation of Educational Achievement (IAE), an independent international research cooperative with approximately 70 member countries (NCES, 2013b). The TIMSS has been administered every four years since 1995 to fourth and eighth graders, and unlike the PISA, the TIMSS assesses students in more traditional, specific content areas: fourth graders are assessed in numbers, geometric shapes and measures, and data display, while eighth graders are assessed in numbers, algebra, geometry, and data and chance (NCES, 2013b). Results that combine the various content area scores into a combined average are reported on a scale of 0 to 1000. Focusing in on eighth graders reveals a picture of U.S. underachievement similar to that depicted by PISA assessments. A nationally representative sample of 7,377 U.S. eighth graders in 2007 averaged

508 in math, slightly above the international average of 500 (NCES, 2007), yet well below high performing nations such as Taipei at 598, Korea at 597, Singapore at 593, Hong Kong at 572, and Japan at 570 (NCES, 2007).

In 2011, the most recent administration of the TIMSS, 10,477 nationally representative U.S. eight graders again failed to make headway against the international math average of 500, scoring a mere 9 points above it (NCES, 2011). Again, U.S. eighth graders were significantly outperformed by Asian school systems: Korea averaged 613, Singapore 611, Taipei 609, Hong Kong 586, and Japan averaged 570 (NCES, 2011). An examination of these results leads to an obvious conclusion: in the mathematics portions of both the PISA and the TIMSS, American middle and high school students achieve at levels just below or only slightly above international averages, while at the same time, U.S. average math scores are *far* below the scores of the highest performing school systems. Figures for both 2007 and 2011 are presented in Figure 2.

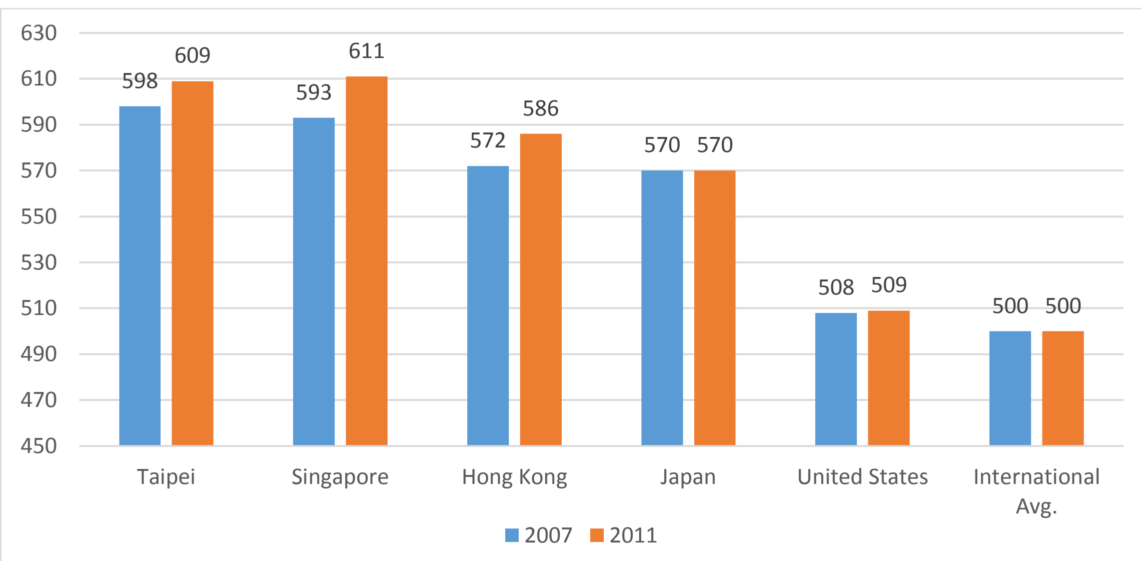

Figure 2. TIMSS combined average math scores, Taipei, Singapore, Hong Kong, Japan, and the international average, as compared to the United States, 2007 and 2011.

Considering the lackluster overall math achievement of American students on these international assessments, it is not surprising then that U.S. students also display mediocre performance in several of the individual content areas that are assessed. Using the traditional content clusters that are disaggregated by the TIMSS, one domain in particular, algebra, bears examination. In the 2007 TIMSS, U.S. eighth graders produced results in algebra that mirror their *overall* math achievement for that year, scoring an average of 507, compared to the international algebra average of 500, while again, they fell well behind many Asian school systems, such as Taipei at 629, Korea at 608, and Singapore at 591 (NCES, 2007). Results for 2011 were only slightly better: U.S. students averaged 512 in algebra, while the international average was again 500 (NCES, 2011). And, once more, Asian systems far outpaced American students, with Korea, Singapore, Taipei, and Hong Kong averaging 613, 611, 609, and 586, respectively (NCES, 2011). In terms of algebra achievement, American eighth-graders replicate their *overall* math performance, being barely able to distinguish themselves from the international average and unable to match the results of the highest performing school systems. The TIMSS algebra results for Taipei, Singapore, Korea and the United States are summarized in Figure 3.

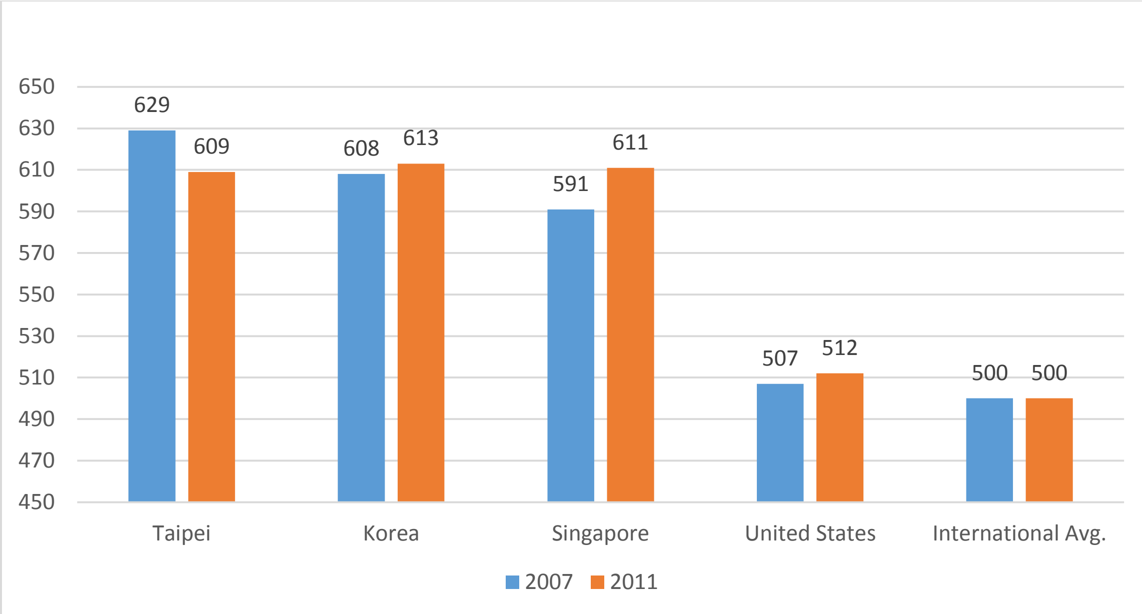

Figure 3. TIMSS algebra results for Taipei, Korea, Singapore, and the international average, as compared to the United States, 2007 and 2011.

A finer-grained examination of K-12 algebra performance, within the United States, and, compared across multiple grade levels, helps further define the scope of this deficiency in algebra achievement.

United States Algebra Achievement: A National Context

In the U.S. the National Assessment of Educational Progress (NAEP) is the largest ongoing, nationally representative study of student achievement in a variety of subjects, including math, reading, science and writing (NCES, 2013c). Known as the Nation's Report Card, the NAEP was first administered nationally in 1969 (NCES, 2013c). However, it wasn't until 1990 that administration of the NAEP took its present form. It is now administered continually, roughly every two years, to fourth, eighth and twelfth-graders during a January to March testing window (NCES, 2013c). In the most recent administration in 2013, roughly 376,000 fourth graders, 341,000 eighth graders and 92,000 twelfth graders from across the nation took part (NCES, 2013c).

The math portion of the NAEP assesses students in five content areas: number properties and operations; measurement; geometry; data analysis, statistics and probability; and algebra (NCES, 2013c). Scores are disaggregated for each of the content areas, and for each of the three tested grade levels, and reported on a scale of 0 to 500 in fourth and eighth grade, and 0 to 300 in twelfth grade (NCES, 2013c). The NCES further assigns these scaled scores to one of three performance bands: *basic*, *proficient*, or *advanced* (NCES, 2013c).

NCES descriptions of what students should know and be able to do at each of the performance bands evidence a broad continuum of mathematical performance. Generally, students performing at the *basic* level show *some* evidence of understanding the mathematical concepts and procedures for a particular content strand, while students scoring at the *proficient* level can more consistently *apply* conceptual and procedural knowledge when solving problems (NCES, 2013c). Meanwhile students performing at the *advanced* level can both *apply* such knowledge, and also *integrate, synthesize*, and/or *make generalizations* about their mathematical understanding (NCES, 2013c).

Based on the marked differences in mathematical understanding depicted by these descriptions, it is somewhat disheartening then to examine the disaggregated algebra scores for American fourth, eighth and twelfth graders. In 2009, for algebra, U.S. fourth-graders scored *basic* with 244 (NCES, 2013c). In 2011 they scored *basic*, also with 244, and in 2013 they scored *basic* again, with 245 (NCES, 2013c). The *cut score* (the minimum qualifying result) for *proficient*, meanwhile, was 249 (NCES, 2013c), meaning that for those three administrations of the NAEP, American fourth-graders were performing at the higher ranges of the *basic* performance band, yet still not scoring high enough to evidence mathematical understanding that could be categorized as *proficient* – namely, the ability to consistently *apply* their conceptual and procedural algebraic knowledge.

Algebra scores for eighth and twelfth-graders were roughly comparable. In 2009, U.S. eighth-graders scored *basic* at 287, in 2011 they scored *basic* at 289, and in 2013 they scored *basic* again at 290, with a *cut score* for *proficient* of 299 (NCES, 2013c). Twelfth-graders in 2009 scored *basic* with a 155, did not participate in 2011, and in 2013 scored *basic* as well, with 155, compared to a *cut score* for *proficient* of 176 (NCES, 2013c). However, upon closer examination, the scores for all three grade levels and the corresponding *proficient cut scores* reveal a worrisome trend: U.S. students' algebra scores tend to gravitate from the higher reaches of the *basic* performance band in the elementary grades, to the middle and lower portions as they matriculate into middle and high school. To wit, in fourth grade, U.S. students are scoring approximately 5 points below the *proficient* cut line, in eighth grade they are scoring about 10 points below, and by the time they reach twelfth-grade, they are performing at roughly 20 points below the line. Not surprisingly, these results are in line with the conclusions of the National Mathematics Advisory Panel, which reported that the decline in overall math achievement among U.S. students begins in late middle school, where for a majority of students, algebra courses are introduced into the curriculum (2008).

What implications, then, can be drawn from the levels of performance outlined above? How does the underperformance of U.S. students in this one content domain affect their ability to make sense of the mathematics that make up the other content domains? And how does below-average achievement in algebra affect students' ability to be successful in college and/or career? To begin to explore the answers to these questions, it is necessary to examine the role that algebra plays in the K-12 mathematics curriculum, and its importance in the wider sphere of academic and professional aspirations.

Overall Significance of Algebra

As defined by the Oxford English Dictionary algebra is, "The part of mathematics which investigates the relations and properties of numbers or other mathematical structures by means of general symbols" ("Algebra", 2002, p.52). Unlike arithmetic, which involves performing numeric operations that generally ignore the features of and relations between associated mathematical expressions, algebra takes as its focus the abstract and structural representations of numbers and the relations between and among them (Tolar, Lederberg, & Fletcher, 2008). According to the National Council of Teachers of Mathematics (NCTM) algebra is a "…way of thinking and a set of concepts and skills that enable students to generalize, model, and analyze mathematical relationships" (2000, p.1). The NCTM further characterizes algebra as critical to mathematical work, because it is a major content component and serves to unify the school mathematics curriculum (2000). When the National Mathematics Advisory Panel was created by executive order in 2006, its mission was to foster greater understanding and achievement in math by American students (2008). Undergirding this mission was a mandate to support talent and creativity, ensure continued American competitiveness, encourage innovation and help government give students the education they need (NMAP, 2008). It is not surprising, then, that given the importance of algebra to mathematics in general, the NMAP (2008) included as a major focus of its research endeavor the essential components that constitute the K-12 algebra curriculum and the course work that serves as prerequisite.

Taking algebra in middle and high school confers a number of benefits on students. It is widely recognized as a gateway course for student access to, increased rates of enrollment in, understanding of, and overall success in higher levels of study in mathematics, as well as science (American Institute for Research, 2006; Haas, 2005; Kaput, 2000; Matthews & Farmer, 2008; U.S. Department of Education, 1997; Walker & Senger, 2007; Wang & Goldschmidt, 2003). It is seen as a predictor of high school graduation (American Institutes for Research, 2006). Middle school Algebra I and high school Algebra II are also recognized as foundational preparation courses for college entrance exams and, thus, as gateway courses to higher education (Evan, Gray & Olchefske, 2006; Johnson, 2010; Schiller & Hunt, 2003; Swail et al., 2004). Specifically, students who take algebra are more likely than those who do not to enroll in four-year colleges (Adelman, 1994; Schneider et al., 1994) and to continue on to graduation (American Institutes for Research, 2006). Furthermore, the NCTM concluded that competence in algebra is an important adult skill, both for academic and vocational pursuits (2000).

Given then the importance of algebra, as a cornerstone of the overall math curriculum, as a key to success in advanced math, as a predictor of achievement in high school, and as crucial to college and career success, it would behoove one to investigate the underlying components of the algebra curriculum itself. To wit, what are the major topics and content strands that constitute middle and high school algebra?

Components of Secondary Algebra

The NCTM recommends that a prekindergarten through twelfth grade mathematics curriculum enable students to function proficiently in four areas of algebra (NCTM, 2000). These are, 1) understanding patterns, relations, and functions; 2) using algebraic symbols to represent and analyze mathematical structures and situations; 3) representing and understanding quantitative relationships through the use of mathematical models; and 4) the ability to analyze change in various contexts (NCTM, 2000).

The Common Core State Standards for mathematics specify four overarching areas of high school algebra (NGA, 2010). The first involves identifying structure in expressions, which includes the ability to interpret and create expressions to solve problems. The second is fluency with polynomial arithmetic and rational functions. This relates to the ability to understand the relations between zeroes and factors of polynomials, and using polynomials to solve problems. The third area is the ability to create equations that describe numbers or relationships. The fourth area involves reasoning with equations and inequalities. This covers the ability to solve inequalities, equations and systems of equations, and proficiency in using reasoning in solutions and the ability to justify that reasoning (NGA, 2010).

The National Mathematics Advisory Panel has created a list of what it deems to be the major topics of school algebra. These are, 1) symbols and expressions, a topic which includes polynomials and rational expressions; 2) linear equations, encompassing number lines and graphing; 3) quadratic equations; 4) six major kinds of functions, (linear, quadratic, polynomial, nonlinear, logarithmic and trigonometric); 5) the algebra of polynomials, including roots, complex numbers, and binomial coefficients; and 6) combinations and finite probability (NMAP, 2008).

Given the breadth and scope of these descriptions of what constitutes secondary algebra, it is important then to gain an understanding of the prerequisite knowledge and conceptual understanding students in elementary mathematics must possess before they first encounter algebra in middle school.

Foundations of Secondary Algebra

Stacey and MacGregor (1997) describe a number of specific, rather narrowly defined conceptual understandings students must possess to be successful in middle and high school algebra. These include an understanding of the quantity zero, and its relation to both addition and multiplication, as well as an understanding of the quantity one, and its role in multiplication (Stacey & MacGregor, 1997). The authors also posit that success in algebra requires that students have a sound understanding of the concept of reciprocals.

A number of authors address the prerequisite understanding necessary before students are ready to tackle algebra from a somewhat broader conceptual perspective (Bay-Williams, 2001; Edwards, 2000; NCTM, 2008; Stacey & MacGregor, 1997). These include an understanding of the properties of numbers (Stacey & MacGregor, 1997), and specifically, the commutative, distributive, and associative properties (Edwards, 2000). In addition, students' understanding of numbers must extend beyond positive whole numbers to negative integers and positive and negative rational numbers (Stacey & MacGregor, 1997). Stacey & MacGregor (1997) further contend that students must have conceptual understanding of the equal sign and its connection to the notion of equality. Also deemed important is a basic understanding of numerical patterns (Bay-Williams, 2001), which includes the ability to describe and generalize about them (NCTM, 2008). Furthermore, the NCTM (2008) holds that as a precursor to algebra coursework, students need to be able to identify mathematical relationships.

Viewing the conceptual understanding critical to success in secondary algebra from an even broader, almost overarching perspective, the NCTM (2008) and the NMAP (2008) have distilled what constitutes a host of specific knowledge and understandings into three main competencies. These are fluency with whole numbers (NCTM, 2008; NMAP, 2008), selected aspects of measurement and geometry, and fluency with fractions (NMAP, 2008).

Taking as an example just one of these core competencies, fluency with fractions, one can see encapsulated in it many of the more specific competencies outlined previously. For example, competence in fractions draws upon: the importance of understanding the relation of the quantity one to multiplication (Edwards, 2000), the necessity of being able to grasp the significance of the equal sign (Stacey & MacGregor, 1997), the need for an understanding of the properties of numbers (Stacey & MacGregor, 1997; Edwards, 2000), the importance of an understanding of and an ability to describe and make generalizations about patterns (Bay-Williams, 200; NCTM, 2008), the need for an ability to understand and manipulate reciprocals (Edwards, 2000), the necessity of being able to identify relationships (NCTM, 2008), and the importance of an understanding of whole numbers and integers that extends to rational numbers (Stacey & MacGregor, 1997). As a result, because so many requisite conceptual understandings are components of fractional fluency, it is no wonder then that the NMAP regards it as a core foundation for success in algebra.

Fluency with Fractions as Critical Foundation for Success in Algebra

A number of other researchers also point to competency in fractions as a critical precursor to the study of algebra (Fuchs et al., 2014; Hoffer, Venkataraman, Hedberg, & Shagle, 2007; Kieren, 1976; NMAP, 2008; Siegler & Pike, 2013). According to Kieren (1976), students must master the concepts of fractional numbers in order to be prepared to learn algebraic concepts. The NMAP (2008) contends that the teaching of fractions introduces students to two of the integral aspects of algebra: manipulating numbers through symbolic notation and the concept of generality. An analysis of two nationally representative data sets, one in Britain and the other in the United States, indicated that elementary students’ understanding of fractions predicted their knowledge of algebra in high school (Siegler & Pike, 2013). A recent survey of 1,000 Algebra I teachers identified fractions as one of the most significant weaknesses in students’ preparation for algebra coursework (Hoffer, et al., 2007). Furthermore, Fuchs et al.

(2014) characterized the study of fractions in elementary and middle school as foundational for learning algebra.

So given then the consensus that exists as to the importance of a well-grounded understanding of fractions for the study of secondary algebra, what is the recommended grade-by-grade framework for instruction in fractions, from its introduction in elementary school to the onset of formal algebra in eighth grade? An examination of two key artifacts, the CCSS for Mathematics and the NMAP Benchmarks for the Critical Foundations, provides a roadmap for answering that question.

A Grade-Level Framework for Instruction in Fractions

Both the NMAP and the CCSS delineate, by grade level, the content strands for teaching fractions.

Third grade. Traditionally, students are introduced to formal instruction in fractions in the third grade (NGA 2010; NMAP, 2008). Instruction focuses on the concepts of parts of a whole, simple equivalence, and visual modeling (NGA, 2010). Later in the year, students learn to understand whole numbers as fractions, locate fractions on a number line, and to compare fractions with like denominators (NGA, 2010).

Fourth grade. In fourth grade, students build on the concepts from the previous year, such as how to identify and represent fractions, both with models and on number lines (NMAP, 2008). Students begin to generate equivalents through multiplication, and learn to compare fractions with unlike numerators or denominators (NGA, 2010). Students are introduced to mixed numbers, and begin addition and subtraction of both fractions and mixed numbers with like denominators (NGA, 2010). The second semester of fourth grade finds students multiplying fractions by whole numbers, and beginning to grapple with decimal-fraction equivalence (NGA, 2010).

Fifth grade. Students continue their work comparing, adding and subtracting fractions (NMAP, 2008), and move on to addition and subtraction with unlike denominators (NGA, 2010). In fifth grade, students are introduced to the concept of fractions as division, and use this understanding to yield fractions from the division of whole numbers (NGA, 2010). Whereas the CCSS (2010) recommends that students in this grade are also introduced to multiplication of fractions by fractions, the NMAP reserves this strand until sixth grade (NMAP, 2008).

Sixth grade. In sixth grade, instruction focuses on the multiplication of fractions by fractions, as well as the beginning of instruction of the concept of division of fractions by fractions, with learning activities that emphasize the underlying meaning of these two operations (NGA, 2010; NMAP, 2008).

Seventh grade. In the last year before the introduction of formal algebra, instruction focuses on consolidating understanding of the four operations as applied to fractions, and features the introduction of negative fractions (NMAP, 2008). In addition, students extend their understanding of addition, subtraction, multiplication and division to rational numbers, including decimals (NGA, 2010).

From among this progression of recommended instruction involving fraction concepts, perhaps one area stands out as most challenging for a large percentage of students. This involves the division of fractions by fractions.

Division of Fractions

A significant body of research points to the division of fractions by fractions as a particularly problematic endeavor for many students (Cengiz and Rathouz, 2011; Coughlin

2010/2011; Fendel, 1987; Ma, 1999; Ott, Snook, & Gibson, 1991; Payne, 1976; Sharp & Welder 2014; Tirosh, 2000; Trafton & Zawojewski, 1984). Sharp and Welder (2014) point to the division of fractions by other fractions as a common area of struggle in seventh grade math. Trafton and Zawojewski describe the understandings involved in division of fractions as a "troublesome" endeavor for many students (1984, p. 20). Coughlin (2010/2011) labels the division of fractions by fractions one of the most complex tasks in elementary math, while multiple authors consider the procedure one of the most rote-like, mechanistic, and least understood elementary math concepts (Cengiz and Rathouz, 2011; Fendel, 1987; Payne, 1976; Tirosh, 2000). Ott, Snook, & Gibson (1991) posit that many students lack a clear understanding of the meaning of division involving fractions, and that, furthermore, most students are incapable of correctly interpreting and articulating the results of such calculations. Finally, Ma (1999, p.55) contends that, "[d]ivision by fractions, the most complicated operation with the most complex numbers, can be considered as a topic at the summit of arithmetic."

An analysis of the cognitive mechanisms that must come into play when students are confronted with problems involving division of fractions by fractions reveals the difficulties inherent in this particular mathematical task, and helps to explain why it is so challenging for such a significant percentage of U.S. students.

Cognitive Components of Fractions and Division

Generally, there is a developmental progression by which children are able to make sense of fraction concepts. This begins with the concept of *partitioning* a unit whole or region into component pieces of the same size (Sharp & Adams, 2002). Initially, students find success with the skill of *halving*, wherein they seek to partition a whole into two, or any power of two (e.g. 4, 8, 16 etc.) pieces/denominators (Sharp & Adams, 2002). Halving leads directly to the next stage, *evenness,* where students become comfortable with partitioning unit wholes into other even pieces/denominators (Sharp & Adams, 2002). By the time children have developed the ability to grapple with *oddness*, they are able to partition unit wholes or regions into any number of pieces, which includes odd numbers of pieces/denominators (Sharp & Adams, 2002). In sum, children begin work with fractions by contemplating values such as $\frac{1}{2}$, $\frac{1}{4}$, or $\frac{1}{8}$, and progressing to the stage where they have become comfortable understanding and representing fractions such as $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{9}$ (Sharp & Adams, 2002). More importantly, what ultimately emerges from this developmental process is the key understanding that, as the number of parts/denominators increases, the smaller each becomes (Gabriel et al., 2013; Siegler & Pyke, 2013).

In addition to partitioning, there are several other cognitive sub-constructs involved in making sense of fractions (Bottge, Ma, Gassaway, Butler, & Toland, 2014). One of these, *ratio*, involves the idea that fractions represent a comparison between the numerator and denominator, and if each is multiplied by the same quantity, that comparative relationship does not change (Bottge, et al, 2014). The *operator* principle introduces students to the idea that a fraction such as $\frac{3}{5}$ can be seen as 3 x $\frac{1}{5}$ of a unit whole, or as $\frac{1}{5}$ x 3 unit wholes (Bottge, et al., 2014). The *quotient* sub-construct requires students to understand that a fraction is not two separate values (numerator and denominator), but rather a single value: the quotient that results from dividing those two separate values (Bottge et al., 2014). Finally, *measure* involves the concept that a fraction is both a number, and an interval between one point, usually zero, and another (Bottge et al., 2014).

Through introduction to two of these sub-constructs, quotient and measure, children grapple for the first time with the notion of rational numbers, defined as any numbers that can be expressed as the quotient of two integers (positive and negative whole numbers), in the form $\frac{a}{b}$ (Gabriel et al., 2013). Perhaps the major difficulty for students at this juncture in their cognitive development is the notion of *whole number bias* (Ni & Zhou, 2005), which reflects the fact that, up to this point, students in elementary mathematics have worked almost exclusively with whole numbers (Siegler & Pyke, 2013). As a result they tend, naturally, yet often mistakenly, to ascribe to rational numbers many of the properties of whole numbers (Siegler & Pyke, 2013).

This highlights an important distinction. Whole numbers and integers (which, unlike whole numbers, also include negative numbers) represent *discrete* values (Gabriel et al., 2013). This means that whole numbers and integers have unique successors: between any two consecutive whole numbers or integers there is no other value (Gabriel et al., 2013). Rational numbers, on the other hand, represent *continuous*, or *dense*, values (Vamvakoussi & Vosniadou, 2011). In other words, no rational number has a unique successor (Vamvakoussi & Vosniadou, 2011). Therefore, between any two rational numbers, there is an infinitude of other rational numbers (Gabriel et al., 2013). In addition, any rational number can be represented by an infinitude of other rational numbers (Gabriel et al., 2013). In other words the value represented by $\frac{1}{2}$ can also be expressed as $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}$, and so on, to $\frac{\infty}{\infty}$ (Gabriel et al., 2013).

All of these notions come into play when students begin to grapple with operations involving fractions. The result is a multiplicity of often counterintuitive relationships. Common denominators are required to add and subtract fractions, but, generally, are not required to multiply and divide them (Siegler & Pyke, 2013). Only the numerator comes into play in fraction addition and subtraction, while with division and multiplication, both numerator and denominator are operated upon (Siegler & Pyke). And perhaps most challenging of all for students, multiplication of fractions does not necessarily result in a greater value, while fraction division does not necessarily yield a smaller value (Vamvakoussi & Vosniadou, 2011).

From a cognitive perspective then, and taking into consideration each of these aspects of rational numbers, division of fractions by fractions may be the most counterintuitive, confusing and challenging undertaking for elementary students in working with this subset of numbers. Returning to whole numbers, models of division typically adopt either a *partitive* or a *quotative* orientation (Koichu, Harel, & Monaster, 2013). In a partitive approach, also sometimes referred to as *sharing*, a learner partitions the dividend into the number of groups indicated by the divisor, and then counts the number of items in each group (Koichu et al., 2013). So in the problem 8÷4, one would distribute the dividend, 8, among the 4 groups comprising the divisor, yielding 2 items in each group. In a quotative approach, commonly known as *long division*, the learner simply counts the number of times the divisor, 4, can be subtracted from the dividend 8 (Koichu et al., 2013). Thus, in the same problem, 8÷4, one would determine how many times the divisor 4 could be subtracted from the dividend 8, the result of which, again, is 2.

When applied to division of fractions, each of these approaches becomes much more complicated (Li, 2008) and can, for many learners, appear nonsensical (Rizvi & Lawson, 2007). For the problem $\frac{1}{8} \div \frac{1}{2}$, a partitive approach would require one to partition, or share, the quantity $\frac{1}{8}$ among $\frac{1}{2}$ groups, which, at face value, appears to be a *non sequitur*. In a quotative approach, one would need to determine how many times $\frac{1}{2}$ could be subtracted from $\frac{1}{8}$, an operation that would appear to result in a negative number, whilst the correct answer must be positive. To move beyond these apparent contradictions and fully grasp the conceptual nature of fraction division, one needs to contemplate a fundamental difference between whole numbers and rationals. To wit, whereas whole numbers and integers answer the question "*how many*", rational numbers are concerned with the question "*how much*" (Vamvakoussi & Vosniadou, 2011). Thus in the problem $\frac{1}{8} \div \frac{1}{2}$, it is much harder to conceptualize *how many* times $\frac{1}{2}$ can be subtracted from, or *fit* into, $\frac{1}{8}$, than it is to conceptualize *how much* of $\frac{1}{2}$ can fit into $\frac{1}{8}$. This latter approach leads to the solution, $\frac{1}{4}$. In other words, $\frac{1}{4}$ of $\frac{1}{2}$ *fits into* $\frac{1}{8}$, because $\frac{1}{2}$ is the same as $\frac{4}{8}$, and $\frac{1}{4}$ of those four-eighths (four parts, each part comprising one-eighth) is indeed one-eighth.

Rizvi and Lawson (2007), further distill these distinctions into a greater, overarching framework – namely, that a conceptual understanding of fraction division requires students to make connections between the concept of division *and* the concept of ratio, or rate. This involves being able to perceive the multiplicative relationship that exists among the dividend, the divisor, and the quotient (Rizvi & Lawson, 2007). This means that learners need to understand that a problem such as 15 ÷ 3 represents a ratio between the dividend, 15, and the divisor, 3, and that those two values are just one pair in an infinite set of other pairs connected by the same ratio (Rizvi & Lawson, 2007). To understand what that ratio is, one needs to ask, "If 15 *is for* 3, then how many *are for* 1?" (Rizvi & Lawson, 2007). This leads to the conclusion that, according to the underlying ratio, or multiplicative relationship, 10 would correspond to 2, and 5 would correspond to 1. In this manner, learners can conceptually come to an understanding that 15 ÷ 3 = 5. Extending this orientation to fraction division, $\frac{1}{8} \div \frac{1}{2}$ can be analyzed as, " $\frac{1}{8}$ is for $\frac{1}{2}$ so how much is for 1?" Doubling $\frac{1}{2}$ would yield 1, so one would then simply also double $\frac{1}{8}$ which results in $\frac{2}{8}$, or $\frac{1}{4}$.

To conclude, in light of the level of complexity inherent in developing conceptual understanding of fraction division, it is little wonder that this domain of elementary mathematics poses such a challenge to a significant percentage of students. Obviously, instructional practice vis-à-vis this important topic must go beyond superficial and mechanistic approaches so that students can build the deep, conceptual understanding necessary to have success with these concepts, fundamental as they are to algebraic understanding. One conclusion, then, is obvious: an examination of current instructional practices vis-à-vis fraction division is necessary to analyze their efficacy in building conceptual understanding.

How Division of Fractions is Taught

Among a number of instructional approaches for teaching division of fractions, perhaps three are the most commonly used. These are the *invert and multiply* approach, the *complex fraction* approach, and the *common denominator* approach (Li, 2009).

The invert and multiply approach. One common approach to division of fractions by fractions is the invert and multiply procedure. In this approach, students are taught to invert the divisor and then multiply it by the dividend. In other words, $\frac{a}{b} \div \frac{c}{d}$ becomes $\frac{a}{b} \times \frac{d}{c}$. The rationale for this approach is that division and multiplication are inverse properties (Li, 2008). Additionally, the inverse of any number, *a*,

is $\frac{1}{a}$. Thus, in the invert and multiply procedure, both the dividend *and* the operation are inversed, which serves to retain the mathematical balance of the original expression (Chabe, 1963; Li, 2008). The logic of this approach can be seen when dividing whole numbers. For example, in the equation, $6 \div 3 = 2$, one can obtain the same quotient, 2, by inverting the dividend and inversing the operation. In other words, $6 \div 3$ becomes $6\times \frac{1}{3} = \frac{6}{3} = 2$. This same rationale extends to fractions. For example, $\frac{a}{b} \div \frac{c}{d}$ is the same as $\frac{a}{b} \times \frac{1}{\frac{c}{d}}$, an expression in which the dividend is $\frac{a}{b}$ and the divisor is $\frac{1}{\frac{c}{d}}$. If we then multiply both numerator and denominator of the expression's divisor, $\frac{1}{\frac{c}{d}}$, by the same value, $\frac{d}{c}$, the denominator's inverse, we get an equivalent fraction with a denominator of 1: $\frac{a}{b} \times \frac{1 \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} = \frac{a}{b} \times \frac{\frac{d}{c}}{1} = \frac{a}{b} \times \frac{d}{c}$ (Li, 2008). Using a concrete example, $\frac{2}{3} \div \frac{4}{5}$, the procedure would be the same. Thus, $\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{1}{\frac{4}{5}} = \frac{2}{3} \times \frac{1 \times \frac{5}{4}}{\frac{4}{5} \times \frac{5}{4}} = \frac{2}{3} \times \frac{\frac{5}{4}}{1} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12}$.

The complex fraction approach. The complex fraction approach for dividing fractions draws on the division properties of whole numbers (Li, 2008; Novillis, 1979). Just as 6 ÷ 3 is the same as $\frac{6}{3}$, so can $\frac{a}{b} \div \frac{c}{d}$ be expressed as $\frac{\frac{a}{b}}{\frac{c}{d}}$. And much like the invert and multiply approach, the complex fraction procedure makes use of equivalent fractions to transform elements of the expression. Thus, the expression $\frac{a}{b} \div \frac{c}{d}$ becomes $\frac{\frac{a}{b}}{\frac{c}{d}}$. And, multiplying both the numerator and denominator by the same value, $\frac{d}{c}$, the denominator's inverse, yields an equivalent expression: $\frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}}$. Since the two fractions in the denominator are inverses, they yield a product of 1. Thus, the expression becomes $\frac{\frac{a}{b} \times \frac{d}{c}}{1}$ which is equal to $\frac{a}{b} \times \frac{d}{c}$. Turning again to a concrete example, $\frac{2}{3} \div \frac{4}{5} = \frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3} \times \frac{5}{4}}{\frac{4}{5} \times \frac{5}{4}} = \frac{\frac{2}{3} \times \frac{5}{4}}{1} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12}$.

The common denominator approach. The common denominator approach builds on students' familiarity with *least common multiples,* a strategy commonly taught in elementary math for adding and subtracting fractions with unlike denominators. In general, the common denominator approach can be expressed as:

$\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times d} \div \frac{c \times b}{d \times b} = \frac{ad}{bd} \div \frac{cb}{bd} = \frac{ad \div cb}{1} = ad \div bd = \frac{ad}{bd}$ (Gregg & Gregg, 1983; Li, 2008). Unlike the previous two approaches, this technique does not rely on inverting the operation from division to multiplication. Using, once again, a concrete example, $\frac{2}{3} \div \frac{4}{5}$, this approach would involve identifying the least common multiple of the two denominators, 3 and 5, which would be 15. Thus, $\frac{2}{3} \div \frac{4}{5} = \frac{2 \times 5}{3 \times 5} \div \frac{4 \times 3}{5 \times 3} = \frac{10 \div 12}{15 \div 15} = \frac{10 \div 12}{1} = 10 \div 12 = \frac{10}{12}$.

In the hands of a competent instructor, any of these three techniques could serve as a framework within which to build the deep conceptual understanding necessary for students to make meaning with respect to division of fractions. Unfortunately, a number of studies point to inadequacies on the part of teachers as a leading cause for the difficulties many students face when dealing with the division of fractions by fractions (Gearhart & Saxe, 2004; Holmes, 2012; LeSage, 2012; Ma, 1999; Matthews & Ding, 2011; Shulman, 1987; Tirosh, 2000).

Teacher Deficiencies: Instruction in Division of Fractions

Studies detailing inadequacy on the part of math teachers describe three underlying problems: inadequate teacher subject matter knowledge, teachers' failure to grasp student understanding, and poor instructional practice.

Inadequate teacher subject matter knowledge. LeSage (2012) found that many teachers of fraction division lack conceptual understanding of rational numbers, and of the challenges these pose to students, and as a result, teachers pass these misunderstandings on to students. Holmes (2012) found a lack of deep subject matter knowledge among teachers of division of fractions, while Shulman (1987) found teachers of fractions lack not only subject matter knowledge, but also what he termed *pedagogical content* knowledge, which is a mental storehouse of topic-specific examples and clarifications. Ball, Thames, and Phelps (2008) further delineated such knowledge into two sub-categories: one of which they term *knowledge of content* *and teaching*. A significant number of mathematics teachers exhibit deficiencies in this type of knowledge, which is a measure of the degree to which teachers' knowledge about mathematics effectively interacts with their design of instruction (Ball et al., 2008).

Teacher inability to gauge student understanding. Shulman (1987) also addressed the inability of teachers of math to consider the needs and interests of their students. He posited that ineffective teachers of subjects such as the division of fractions are unable to blend content and pedagogy into representations that conform to the idiosyncratic abilities of their students (Shulman, 1987). Tirosh (2000) found that many teachers of division of fractions lack knowledge of their students' misconceptions, which can include mistakes that are algorithm-based, student beliefs about rational numbers that draw mistakenly from the properties of whole numbers, or lack of prior knowledge. Ball et al. (2008), again drawing on and refining Shulman's (1987) notion of pedagogical content knowledge, posited that many teachers lack *knowledge of content and students*, an ability to understand student interests and motivations, as well as the ways in which student thinking emerges, is incomplete, or is marked by misconceptions. .

Inappropriate instructional practice. Matthews and Ding (2011) described misconceptions on the part of teachers in formulating problems. Specifically, they found that many teachers adopted a *measurement* (how many groups) perspective when creating division of fractions word problems, when a *partitive* (how many in each group) approach was called for (Matthews & Ding, 2011). According to Ma (1999), most U.S. math teachers of fraction division fail to even use story problems, which are deemed critical for helping students make connections between the underlying concepts and their own lives. Ma (1999) attributed this failure to incomplete understanding on the part of teachers as to the meaning of fraction division, and an inability to connect this topic to other models in math.

Such teacher deficiencies become an even more critical issue when considered from the perspective of the current movement for the implementation of standards-based curricula, such as those embodied in the Principles and Standards for School Mathematics and the Common Core State Standards. As such, it is instructive to consider current models for teacher training.

How Teachers are Trained

To enable teachers to implement standards-based curricula, school districts are increasingly required to pay greater attention to the manner in which they support all teachers, whether they are new to the profession or seasoned veterans (Darling-Hammond, 2004; Polly et al., 2014). Crucial to this endeavor has been an emphasis on providing ongoing professional development that builds teachers' capacity (Darling-Hammond, 2004). Such professional development has been found to be most effective when it develops teachers' knowledge of content and pedagogy (Garet, Porter, Desimone, Birman, & Yoon, 2001; Heck, Banilower, Weiss, & Rosenberg, 2008), provides teachers with a sense of ownership of their professional learning (Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010), and features ongoing support in the form of both a workshop model as well as classroom-centered experiences (Loucks-Horsley et al., 2010; Polly & Hannafin, 2010). Because it has been found that adults are largely incapable of assuming responsibility for and enacting their own professional learning (Downey, Steffy, English, Frase, & Poston, (2004), school districts often rely on subject matter experts in content knowledge and pedagogy to conduct their professional development initiatives.

However, according to Feldon and Clark (2006), when experts self-report on the knowledge and skills critical to their expertise, the results are often incomplete, inaccurate, and contain errors and omissions that can impede the ability of trainees to perform a target task. It has been found that the extent of such expert omissions can reach 70% of the knowledge and skills crucial for replicating expert performance (Feldon & Clark, 2006). As a result, trainees often seek to fill the voids created by such incomplete information with information of their own, which is often riddled with fallacies and misconceptions (Feldon & Clark, 2006).

Summary

There is a record of underachievement among U.S. students in algebra, with origins that can be traced to the middle school grades. Furthermore, the difficulty inherent in understanding the conceptual nature of mathematics in general, and algebra and fractions in particular, makes teaching these concepts problematical for a significant number of teachers. Because the current model for teacher professional development relies on subject matter experts, and due to the extent of knowledge omissions by such experts during training of novices, this study used cognitive task analysis (CTA), a method of eliciting expert knowledge, to capture the expertise of teachers proficient in teaching division of fractions, a foundational competency for algebra achievement.

To more fully understand how cognitive task analysis makes possible the capture of such expert knowledge, it is necessary first to examine the nature of knowledge itself, as well as the manner in which knowledge becomes automated. Furthermore, it is important to explore how expertise is characterized, how expert knowledge is developed, and the implications of expertise vis-à-vis knowledge omissions.

Knowledge Types

The purpose of education is to replicate knowledge (Jackson, 1985). Researchers have categorized knowledge into two types: declarative and procedural (Ambrose, Bridges, DiPietro, Lovett, & Norman, 2010; Anderson & Krathwohl, 2001; Clark & Estes, 1996)

Declarative Knowledge

Declarative knowledge is information about *why*, *what*, and *that*, and is typified by its conscious quality and the speed with which it can be learned and modified (Clark & Estes, 1996). Most knowledge comes into cognition in declarative form, is committed to long-term memory (Anderson & Finchman, 1994), and serves to help humans handle novel tasks (Clark & Estes, 1996).

Procedural Knowledge

Procedural knowledge is knowing *how* and *when* (Anderson & Krathwohl, 2001), consists of IF/THEN propositions (Anderson, 1982), is goal-oriented, and promotes problem solving (Corbett & Anderson, 1995). Conditional knowledge, which is a type of procedural knowledge, involves knowledge of *when*, as well as *why*, provides a rationale for various actions (Paris, Lipson, & Wixson, 1983), and modulates the fact-to-action (declarative-to-procedural) process (Anderson, 1982).

Automaticity

Automaticity is the process by which declarative and procedural task knowledge becomes automated and unconscious in nature, as a result of repeated performance and deliberate practice (Ericsson, Krampe, & Tesch-Romer, 1993). Four stages of automaticity have been identified: a cognitive stage, in which a learner can complete a task with initial instruction, an associative stage, in which the learner works through the procedure and acquires relevant declarative knowledge and requires less cueing, an autonomous stage, in which verbal cueing is no longer necessary, and a fourth stage, in which subject matter experts (SMEs) add their own innovations (Anderson, 1996; Ericsson, Krampe, & Tesch-Romer, 1993).

Automated knowledge helps to reduce cognitive overload, by freeing up limited working memory (Kirschner, Sweller, & Clark, 2006), which enables the expert to attend to novel tasks and deploy strategies to solve problems (Clark, 1999). However, automated processes often initiate without prompting, and then run to completion (Feldon, 2007). This leads to a double-edged sword: although the expert has ample working memory, automated processes are resistant to change, and require considerable monitoring in order to modify or eliminate them (Clark, 2008; Wheatley & Wegner, 2001).

Expertise

An expert is one who is uncommonly accurate and reliable in making judgments, displays superior skill and economy of effort in task completion, and is able to deal effectively with certain types of rare or problematic cases (Chi, 2006).

Characteristics of Experts

Expertise is typified by extensive, highly structured domain knowledge, a command of effective strategies for domain-specific problem solving, and expanded working memory within which elaborated schemas allow for the rapid storage, retrieval and manipulation of information (Chi, 2006). Viewed through the lens of a relative approach, expertise does not require innate talent per se; rather it is a level of proficiency that novices can achieve (Chi, 2006). However, expertise is task specific, and does not transfer from domain to domain (Bedard & Chi, 1992).

Experts outperform novices in the essential skills of a domain (Feldon, 2007). These skills are a product of experience-based domain knowledge (Feldon, 2007), which can include principles, concepts, and connections (Bedard& Chi, 1992). Experts can draw upon knowledge structures that facilitate the recall of problem states and allow them to engage in forward reasoning (Bedard & Chi, 1992).

Experts view problems differently than novices (Bedard & Chi, 1992). They can see beyond function and simple schemas, can identify relevance among a number of cues, instill meaning in ill-defined problems, and are able to select and match strategies to problems (Bedard & Chi, 1992).

In general, experts possess a superior functional capacity of working memory (Feldon, 2007), through selective encoding of relevant information and mechanisms (Ericsson & Lehmann, 1996). They can more rapidly attend to, encode, manipulate, and decode domain-relevant information in working memory than can non-experts (Feldon, 2007). In addition, this superior memory function is found in both short and long-term working memory (Ericsson & Lehmann, 1996).

Building Expertise

Expertise is acquired as a result of continuous and deliberate practice. Alexander's (2003) Model of Domain Learning posits that its development involves a journey from acclimation, whereby learners adapt to an unfamiliar domain and task, to competence, whereby learners demonstrate a foundational body of knowledge, cohesive and principled in structure, to proficiency, whereby the expert has developed a synergy among the various cognitive components of his expertise.

Crucial to the development of expertise is deliberate practice, which is characterized by repeated performance of the task, as well as an innate motivation on the part of the learner to attend to the task and exert effort to improve performance (Ericsson, Krampe, & Tesch-Romer, 1993). In addition, the learner needs immediate informative feedback and knowledge of task performance during practice (Ericsson, Krampe, & Tesch-Romer, 1993). It is also important to limit daily practice time, to avoid exhaustion (Ericsson, Krampe, & Tesch-Romer, 1993).

One hindrance to building expertise presents itself as learners reach the automated phase of task competence, because performance at this point often reaches a stable plateau, and no further improvement in performance occurs (Ericsson, K. A., 2004). The challenge for the learner is to avoid such arrested development through an orderly and deliberate approach to practice that includes task monitoring, planning and analysis, with the aim of identifying changes that can be integrated into one's performance (Ericsson, K.A., 2004).

Consequences of Expertise

As new knowledge becomes automated and unconscious, experts are often unable to completely and accurately recall the knowledge and skills that comprise their expertise (Chi, 2006; Feldon, 2007). Experts often are overly-confident, overlook details, make inaccurate predictions and offer faulty advice (Chi, 2006). In addition, as their skills improve, experts' self-report errors and omissions tend to increase, while their accuracy of introspection decreases (Feldon, 2007). Furthermore, because experts' schemas are adapted to problem solving, they can fail to articulate relevant cues, and can unintentionally fabricate consciously reasoned explanations for their automated behaviors (Feldon, 2007).

Expert Omissions

Because automaticity and the accuracy of self-reporting have been found to be negatively correlated, experts in an instructional role may unintentionally leave out information that learners must master when learning procedural skills (Feldon, 2004). In fact, experts may leave out up to 70% of the critical information necessary to perform a task, forcing novices to fill in the blanks with error-prone trial-and-error methods (Clark et al., 2011). The automated nature of knowledge causes procedural steps to blend together in experts' mind and makes it difficult for them to share the complex thought processes of technical skill execution (Clark et al., 2011).

Cognitive Task Analysis (CTA)

Definition of CTA

Cognitive task analysis refers to a variety of methods for eliciting and representing the knowledge and skills of expert practitioners when they perform complex tasks and solve difficult problems. CTA, an extension of traditional task analysis, uses a variety of interview and observation strategies to identify the knowledge, thought processes and goals that underlie observable task performance, as well as overt and covert cognitive functions (Chipman, 2000; Clark, Feldon, van Merrienboer, Yates, & Early, 2008).

Brief History of CTA

CTA can be traced to the advent of applied psychology in the 1880s, the growth of time and motion studies in the early 20th century, and the study of complex machine systems in the mid-20th century (Militello & Hoffman, 2008). It emerged as a result of the study of social, psychological and cognitive activities in the workplace in the 1960s, and became prominent in the 1980s when the study of knowledge acquisition fueled a demand for expert systems and other applications of artificial intelligence (Hoffman & Woods, 2000).

CTA Methodology

A number of researchers have identified discrete stages through which a typical cognitive task analysis would proceed (Chipman, Schraagen, and Shalin, 2000; Clark, Feldon, van Merrienboer, Yates, & Early, 2008). These are (a) a preliminary, data collection phase; (b) identification of knowledge representations; (c) application of knowledge elicitation methods; (d)) expert review and analysis of elicited knowledge; and (e) formatting of results for the desired application (Chipman, Schraagen, and Shalin, 2000; Clark, Feldon, van Merrienboer, Yates, & Early, 2008).

Taxonomies of Knowledge Elicitation Techniques

Knowledge elicitation, a subset of knowledge acquisition, is the process of extracting the domain specific knowledge that underlies human performance (Cooke, 1994). One researcher has identified four categories: 1) observations of task performance, 2) various interview techniques, 3) process tracing of sequential behavioral events, and 4) methods to elicit the structure of domain-related concepts (Cooke, 1994; Cooke, 1999). Wei and Salvendy (2004) identify a fifth family – formal models. However, since these typologies are based on processes, analysts may struggle to select an appropriate CTA approach if the desired result is a particular type of knowledge (Yates, 2007).

Pairing Knowledge Elicitation with Knowledge Representation/Analysis

Yates (2007) identified the most frequently used CTA methods and the knowledge types associated with them. The author found it more appropriate to examine CTA as a pairing of knowledge elicitation with an analysis/representation technique, and to classify CTA methods in terms of desired outcome, rather than process (Yates, 2007) .

Effectiveness of CTA

Cognitive task analysis is regarded as a necessary component of research in complex cognitive work, since its use allows for the identification of the explicit and implicit knowledge of experts, which supports effective and efficient training (Hoffman & Militello, 2009). It is seen as an optimal method for capturing knowledge because it emphasizes aspects of tasks that are important to the learner, facilitates understanding of abstract knowledge across domains, and provides a framework for abstract problem solving (Means & Gott, 1988). In educational and work settings, CTA assists researchers in identifying subtle skills, perceptual differences, and procedures (Crandall, Klein, & Hoffman, 2006).

Research has shown that using cognitive task analysis is more cost effective and efficient than other models (Clark, Feldon, van Merrienboer, Yates, & Early, 2008; Clark & Estes, 1996). CTA can reduce total training days by nearly half (Clark, Feldon, van Merrienboer, Yates, & Early, 2008), while producing results comparable to conventional training methods that take longer (Clark & Estes, 1996).

Benefits of CTA for Instruction

Several studies indicate that instruction based on cognitive task analysis is superior to other instructional models (Hoffman & Militello, 2009; Crandall, Klein, & Hoffman, 2006; Clark, Yates, Early, & Moulton, 2010). CTA can allow the data analyst to identify the explicit and implicit knowledge of experts, including domain content, concepts and principles, schemas, reasoning and heuristics, and mental models, all of which can support effective and efficient cognitive training, scenario design, cognitive feedback, and on-the-job training (Crandall, Klein , & Hoffman,2006; Hoffman & Militello, 2009). In addition, CTA leads to guided instruction that is more structured and successful than learning that is based on media, games, or discovery (Clark, Yates, Early, & Moulton, 2010). Overall, CTA has been shown to be effective in capturing expertise and informing instruction in a wide range of professions, including health technicians (Clark, 2014), nursing (Crandall & Gretchell-Reiter, 1993), physicians (Fackler et al., 2009), and education (Crandall, Klein, & Hoffman, 2006).

Conclusion

In summary, educators face inherent difficulties in understanding the conceptual nature of mathematics in general, and algebra and fractions in particular. As a result, providing effective instruction in these concepts poses a problematical endeavor for a significant number of teachers. In response, the standard paradigm for remediating teacher inadequacies in these areas has been predicated on professional development that relies primarily on subject matter experts. However, and as previously outlined, researchers point to an extent of knowledge omissions by such experts during training of novices that is considerable (Clark & Feldon, 2004). Therefore, this study proposes to capture the expertise of teachers proficient in teaching the division of fractions by fractions through the use cognitive task analysis (CTA), a method of eliciting expert knowledge, that research shows to be superior to other models in providing training ((Hoffman & Militello, 2009; Crandall, Klein, & Hoffman, 2006; Clark, Yates, Early, & Moulton, 2010).

CHAPTER THREE: METHODOLOGY

Overview

The purpose of this study was to conduct a Cognitive Task Analysis to determine the knowledge and skills, represented by the action and decision steps, as well as other knowledge, that expert middle school math teachers (subject matter experts, or SMEs) employ when they describe how they teach the division of fractions by fractions. As explicated in Chapter Two of this study, the division of fractions comprises a high-leverage prerequisite for competency in algebra, which itself, and also, as detailed in Chapter Two, is a core prerequisite for achievement in secondary level mathematics overall. Based on the definition of expertise, the researcher assumed that these subject matter experts possessed highly automated declarative and procedural knowledge that was often unconscious. Thus, the researcher assumed it would be difficult for these SMEs to describe accurately and in detail the *what, why, how,* and when of teaching the division of fractions by fractions.

To wit, the research questions that guided the study were:

1. What are the action and decision steps that expert middle school math teachers recall when they describe how they teach the division of fractions by fractions?
2. What percent of action and/or decision steps, when compared to a gold standard, do expert middle school math teachers omit when they describe how they teach the division of fractions by fractions?

Participants

This study identified teachers from three Southern California school districts who are expert in teaching division of fractions by fractions. Each of these teachers had at least five years of recent and continual experience teaching division of fractions by fractions, and this experience was recognized as successful by school and/or district administrators. The researcher explained to these administrators that such successful experience would entail deep content knowledge, best practices pedagogy, assessment-driven practice, and an ability to instill in students a balance of conceptual understanding and procedural fluency. Additionally, each of these subject matter experts (SMEs) have experienced a wide variety of contexts, settings, problems, and applications in their work with teaching division of fractions by fractions, and did not have experience as trainers or instructors in teaching division of fractions by fractions (Yates, 2007). They were also selected on the basis of being verbal, cooperative, and available and willing to participate in audio-recorded, in-person interviews (Yates & Clark, 2011), as determined by the researcher in initial phone conversations.

The researcher made an effort to recruit a fourth SME for the purposes of reviewing the final Gold Standard Protocol. However, due to the great difficulty the researcher encountered in identifying teachers with expertise in division of fractions, the researcher was unable to secure this fourth expert; as a result each of the three SMEs were asked to review the Preliminary Gold Standard Protocol as a complete and accurate aggregation of their individual protocols.

Data Collection for Question 1: *What are the action and decision steps that expert middle school math teachers recall when they describe how they teach the division of fractions by fractions?*

In order to elicit experts' knowledge of teaching division of fractions by fractions, this study adopted the five-stage protocol of cognitive task analysis described by Clark et al. (2008) in which researchers:

1. Collect preliminary knowledge through unstructured interviews, observations, and document analysis.
2. Identify knowledge representations, through the use of flow charts, concept maps or semantic nets.
3. Apply focused knowledge elicitation techniques which can vary based on the type of knowledge required.
4. Analyze and verify the data acquired through coding and review by SMEs.
5. Format the results for the desired application.

In this study, this five-stage process was implemented as outlined in the following sections.

Phase 1: Collect preliminary knowledge. The researcher is an elementary school teacher, with a general knowledge of mathematics instruction. Because the study involved middle school mathematics, the researcher conducted a thorough literature review to collect preliminary information and build a more general understanding of teaching division of fractions by fractions.

Phase 2: Identify knowledge representations. Conducting the literature review on Cognitive Task Analysis allowed the researcher to gain an understanding of the nature of both declarative and procedural knowledge. The researcher also participated, with other researchers and under the guidance of a senior researcher, in practice activities designed to elucidate the differences between these two knowledge types. These practice activities also helped the researcher to identify action steps, decision steps, and conceptual knowledge types, such as concepts, processes, and principles. Familiarity with these knowledge types and action/decision steps was critical to creating the interview protocol.

Phase 3: Apply knowledge elicitation techniques.

***Instrumentation*.** This study employed the semi-structured interview protocol described by Clark, Pugh, Yates, Early and Sullivan (2008) which is based on a series of questions aimed at eliciting (a) conditions/indications; (b) processes; (c) action and decision steps; (d) standards of time and quality; (e) equipment needed; (f) pedagogical reasoning; (g) conceptual understanding, as exemplified by the vocabulary and symbols an expert teacher would need to know. The action and decision steps constitute the critical information a novice would need to perform the target task. Action steps begin with a verb, and state what a person should do. An example would be, "Direct students to compare their answers with a partner." Decision steps take the form of IF/THEN propositions, and usually provide two alternative courses of action. An example would be "IF student provides correct answer, THEN go to step 3.2. IF students does not provide correct answer, THEN provide remediation after class."

This interview protocol was an adaptation of the critical decision method (Hoffman, Crandall, & Shadbolt, 1998), or CDM, that employs cognitive probes to understand how experts assess situations and make decisions during task execution. The protocol, which appears in Appendix A, was also based on the PARI (precursors, actions, results, interpretations) methodology that involves interviewing experts about the aspects of a task that are associated with declarative, procedural and strategic knowledge (Hall, Gott, & Pokorny, 1995).

***Interviews*.** Following IRB approval from the University of Southern California, three SMEs in teaching division of fractions by fractions were interviewed according to the semi-structured protocol described above. Each SME was interviewed and, with SME permission, audio-recorded for approximately 90 minutes. A follow-up interview with each SME also lasted for approximately 90 minutes, and a final phone conversation to review the preliminary protocol lasted approximately 30 minutes. Thus, in aggregate, the researcher spent about three and a half hours in conversation with each SME. The interview protocol was designed to capture the explicit action steps, as well as the implicit, non-observable decision steps, judgments and other cognitive processes that are associated with expert instruction in division of fractions by fractions.

Phase 4: Data analysis. Audio recording of the interviews, coupled with verbatim transcription, provided by a professional transcriber, gave the researcher the ability to acquire, through multiple read-throughs of the transcripts, a deeper, richer understanding of what was revealed in the interview by the subject matter expert, as opposed to not recording and transcribing, which would have required the researcher to rely on memory only.

Coding. Once each interview recording was transcribed, the transcripts were coded according to an *a priori* scheme, based on Clark's (2006) concepts, processes and principles method. Examples of the codes used include "main procedure, "action step", and "decision step." The coding scheme was also used for calculating inter-rater reliability and is included as part of Appendix B.

Inter-rater reliability. Both the study researcher and a fellow researcher independently coded the transcription of one of the SME interviews to determine consistency of coding between researchers. The two coded transcripts were then compared for inter-rater reliability. An inter-rater reliability was calculated as a percentage of agreement between the two coders, and appears in Appendix B. According to Hoffman, Crandall, and Shadbolt (1998), an inter-rater reliability of 85% or higher indicates that the coding process is consistent and reliable among different coders. The results of the inter-rater reliability, expressed as a percentage, was 93%, and are also included in Chapter Four.

SME protocol and verification. Each coded interview transcript was used to generate a step-by-step cognitive task analysis protocol for teaching division of fractions by fractions. Each protocol was then reviewed by the SME from whose interview transcript it was generated.

Phase 5: Formatting the results.

Gold standard protocol. Once each subject matter expert reviewed, verified, and as necessary, corrected their individual protocol, a synthesis of the three individual protocols was used to generate an aggregate Preliminary Gold Standard Protocol (PGSP). This aggregate protocol was developed by first identifying the clearest, most complete, and most articulately worded individual protocol. Each action and decision step from the other two individual protocols was then compared to those of this ideal protocol. If any language, action or decision steps were found to be the same across individual protocols, then they were attributed to both SMEs. If any language, action, or decision steps were found to be more accurate or complete compared to either of the other two protocols, then the action or decision step was modified to reflect that and attributed to both SMEs. If any action or decision step was unique, and not listed in the ideal individual protocol, then it was added to the ideal individual protocol, to facilitate the building of an aggregated Preliminary Gold Standard Protocol. See Appendix C for a description of the steps involved in creating a GSP. The completed Preliminary Gold Standard Protocol was then reviewed by each of the three SMEs for their review and, when necessary, correction.

Summary. The five phase process described above is also referred to as the 3i = 3r method (Flynn, 2012), which stands for three initial interviews and three reviews. A visual representation of this method appears in Figure 4.

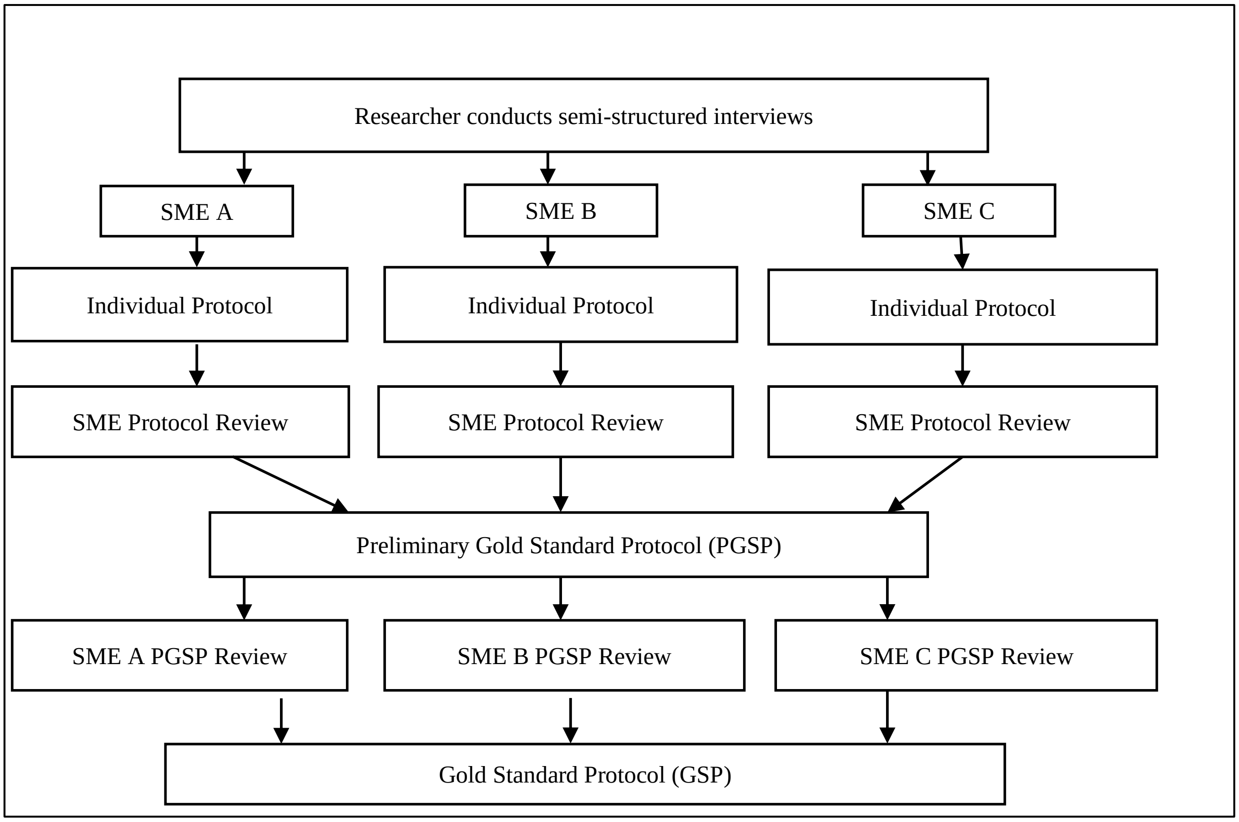

Figure 4. Visual representation of the 3i + 3r CTA method.

Data Analysis for Question 2: ***What percent of action and/or decision steps, when compared to a gold standard, do expert middle school math teachers omit when they describe how they teach the division of fractions by fractions?***

***Spreadsheet analysis*.** Once the Gold Standard Protocol was finalized, each of the action and decision steps of this final version was transferred to an Excel spreadsheet. Each individual SME protocol was analyzed and compared to the GSP action and decision steps listed in the spreadsheet. Four spreadsheet columns, one each for the GSP and the three individual protocols, allowed the researcher to visually represent and assess the degree to which the individual protocols either conformed to the action and decision steps in the GSP or contained omissions. Specifically, each row of the spreadsheet served to represent one of the action or decision steps revealed by the GSP. If an individual SME protocol contained a GSP action or decision step, a "1" was placed in the corresponding cell for that SME; an omission was represented by a "0". Using the spreadsheet, the researcher was able to calculate the number and percentage of total agreements and total omissions of each individual protocol.

CHAPTER FOUR: RESULTS

Overview of Results

This study focuses on the declarative and procedural knowledge of three middle school math teachers, expert in teaching the division of fractions by fractions. This knowledge, which was captured using CTA methodology, takes the form of objectives, standards, cues, conceptual understanding, and action and decision steps. This chapter presents a data analysis of the results of that CTA study, organized by research question.

Research Questions

Question 1

What are the action and decision steps that expert middle school math teachers recall when they describe how they teach the division of fractions by fractions?

Inter-rater reliability. As described in Chapter Three, the researcher and a researcher colleague derived inter-rater reliability through independent coding of one of the SME interview transcripts. Following coding, the two researchers tallied the number of coded items that were in agreement and divided that number by the total number of coded items. This inter-rater reliability was 93%. The tally sheet used to compute this percentage appears in Appendix B. Based on the relatively high inter-rater agreement represented by this value, the researcher then coded the remaining two SME interview transcripts without the assistance of a second coder, and then created an individual protocol for each SME.

Flowchart analysis. The researcher next used SME A’s individual protocol to create a flowchart of the action and decision steps captured in the interview. The goal of the flowcharting process was to determine whether the action and decision steps recalled by the SME in the initial interview represented a logical progression, and additionally, whether there were any decision steps that did not lead to an appropriate action step. A number of questions vis-à-vis the sequence of steps captured in the initial interview revealed themselves during flowcharting. Specifically, several decision steps from the initial interview did not result in a progression to an action step, effectively terminating the progression of the protocol. The flowchart appears in Appendix D. The researcher then addressed these questions during the follow-up interview, which informed the creation of the final SME protocol. This process of initial interview, development of initial protocol, flowcharting, follow-up interview, and creation of final protocol for SME A allowed the researcher insight into how to more effectively conduct both the initial and follow-up interviews for SMEs B and C.

Gold standard protocol. As explained in Chapter Three, the researcher analyzed each of the three SME individual protocols to create an aggregate, preliminary gold standard protocol for teaching the division of fractions by fractions. This analysis revealed a continuum of protocols, from most complete to least complete. SME B's protocol was found to be the most complete, SME C's protocol was identified as slightly less complete, while SME A's protocol was found to be the least complete. As a result, SME B's individual protocol served as the foundational protocol in developing a preliminary gold standard. Each of the action and decision steps for SME B were compared to those for SME C. Where those steps were identical in meaning, attribution was given to both SMEs. In those cases where an action or decision step in SME C's protocol was not present in SME B's, then that step was added to the foundational protocol, and attribution was given to SME C only. Once this process was complete, the individual protocol for SME A was also similarly aggregated into the emerging preliminary gold standard protocol. An example of how each of the SMEs contributed to the preliminary gold standard protocol is illustrated in Figure 5.

SME B – Action Step:

Review that we can convert a fraction division problem into a complex fraction (B)

SME C – Additions (in Bold) to SME B's Action Step:

Review that **converting** a fraction division problem into a complex fraction **results in multiplying the first fraction by the reciprocal of the second fraction, a fact we also discovered when we divided fractions by whole numbers using pictorial representations** (B, C)

SME A – Additions (in italics) to PGSP (As step reads in final GSP)

Review that **converting** a fraction division problem into a complex fraction **results in multiplying the first fraction by the reciprocal of the second fraction, a fact we also discovered when we divided fractions by whole numbers** *and whole numbers by fractions using manipulatives* **and pictorial representations** (A, B, C)

Figure 5. A sequential representation of the process of aggregating action steps from SME C and SME A onto an action step from SME B's foundational individual protocol, resulting in an action step as it appears in the GSP.

Once the preliminary gold standard protocol was developed, the researcher emailed an explanation of the aggregation process along with a copy of the preliminary protocol to SMEs A, B, and C. Once the three SMEs had had time to review the preliminary protocol, the researcher contacted each of them individually by phone, for approximately 30 minutes, to discuss the protocol and then make additions, modifications, and deletions based on their input. The result is the final gold standard protocol.

This final gold standard protocol, which is attached as Appendix E, is a distillation of the action and decision steps that expert middle school math teachers employ in teaching the division of fractions by fractions, and serves, furthermore, as the response to Research Question One. This final protocol consists of a sequence of twelve procedures that the three SMEs who participated in this study identified as necessary for teaching the division of fractions by fractions. These twelve procedures are:

1. Review concepts of multiplication.
2. Review concepts of division.
3. Teach operations with integers.
4. Teach number domains.
5. Review representations of fractions.
6. Review addition and subtraction of fractions and mixed numbers.
7. Review multiplication of fractions and mixed numbers.
8. Teach division of whole numbers by fractions.
9. Teach division of fractions by whole numbers.
10. Teach division of fractions by fractions.
11. Teach division of mixed numbers by mixed numbers.
12. Teach division of fraction word problems.

The following sections contain a description of the disaggregated results for each of the research questions.

Recalled action and decision steps. Actions steps refer to behaviors that are observable. Decision steps take the form of unobservable cognitive processes. These unobservable processes act as cues and/or prompts that allow subject matter experts to decide among alternative courses of action, based on evaluative and interpretive analysis. The sum total of action and decision steps captured from subject matter experts that appears in a final gold standard protocol makes up the information necessary for novice practitioners to replicate expert performance. To answer Research Question One, the researcher analyzed the action and decision steps contributed by the individual SMEs, in order to quantify the number of such steps attributable to each.

To analyze the number of action and decision steps captured by each SME, the researcher entered each step from the final gold standard protocol in individual rows of a Microsoft Excel spreadsheet. The spreadsheet appears in Appendix F. The first column of the spreadsheet provides space for coding of the step number, beginning with “1”. The second column provides space for coding each step as either “A” for action or “D” for decision step. The third column contains the wording of the action or decision step, while columns four, five, and six are labeled with the identifiers “SME A”, “SME B”, and SME C”, reflecting the order in which the subject matter experts were originally interviewed. Each time a SME contributed an action or decision step that appears in the final gold standard protocol, a “1” was entered in the spreadsheet cell where the row containing that step and the column bearing that SMEs identifier intersect. Additionally, if an action or decision step appearing in the final protocol was not attributable to a particular SME, a “0” was placed in that SME’s identifier column. As an example, for the row corresponding to column three, action step 5, “Introduce the example 2 x 3”, a “1” appears in the same row in the column labeled “SME A”, while “0” appears in the columns for SME B and SME C, indicating that SME A was the sole contributor of that action step.

A spreadsheet formula was used to total the number of action and decision steps for each SME. These totals appear at the bottom of each of the SME identifier columns. Action and decision step totals for the individual SMEs appear in Table 1.

Table 1

Cumulative Action and Decision Steps Captured for each SME in the Initial Individual Protocols

	Steps		
	Action Steps	Decision Steps	Total Steps
SME A	110	20	130
SME B	279	106	385
SME C	199	28	227
Total			742

Action and decision steps contributed by each SME. The total numbers of action and decision steps recalled by each SME are summarized in Table 1. As a clarification, there were multiple cases in which two or more SMEs recalled the same action or decision steps. Because the final gold standard protocol represents non-repeating action and decision steps only, the total number of such steps in this final gold standard protocol, 632, is less than the total of 742 steps for all three SMEs.

Figure 6 provides a graphic representation of the action and decision steps reported in Table 1.

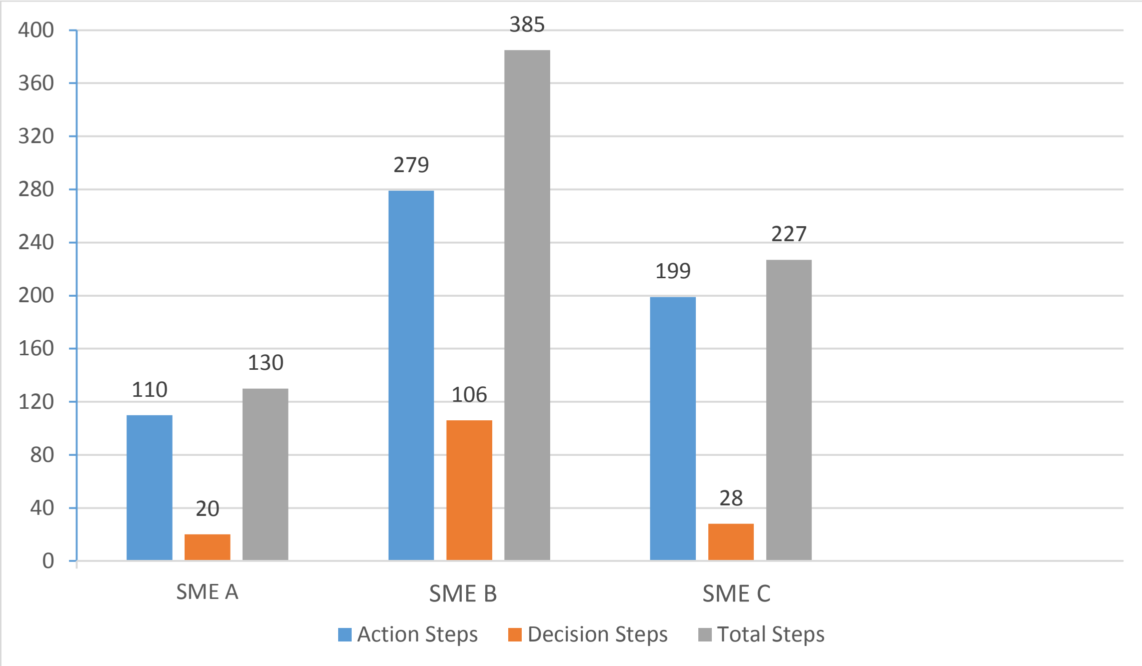

Figure 6. Total action and decision steps from the CTA study captured by individual SMES as they appear in the final gold standard protocol. Compare to gold standard protocol non-repeating action and decisions steps: action steps – 490, decision steps – 142, total action and decision steps – 632.

Collectively, as reported in the final gold standard protocol, the three SMEs described 632 action and decision steps. On an individual basis, however, the range of total action and decision steps reported by the three SMEs varied from a low of 130, or 20.57% of the total steps in the final gold standard protocol, to a high of 385, or 60.92% of the total gold standard protocol steps. For each of the three SMEs, individually, there were more action than decision steps reported, and implications of this are discussed in detail in Chapter 5. SME A recalled 110 action steps compared to 20 decisions steps, SME B recalled 279 action steps and 106 decision steps, while SME C recalled 199 action steps compared to 28 decision steps. In terms of action steps, SME B reported the most, comprising 56.94% of the action steps in the final protocol, while SME A recalled the least, accounting for just 22.45% of the final protocol action steps. A similar disparity characterizes the decision steps. To wit, SME B recalled the most decision steps, at 74.65% of the total decision steps in the final protocol, while SME A reported the least, comprising 14.08% of final protocol decision steps.

Additional action and decision steps captured during follow-up interviews and SME review of preliminary gold standard protocol. To provide a more nuanced answer to Research Question One, the researcher additionally tabulated the number of action and decision steps that were added, deleted or modified during the follow-up interviews and as a result of the review by the three SMEs of the preliminary gold standard protocol. These figures are shown in Table 2.

Table 2

Additional Expert Knowledge Captured, in the Form of Action and Decision Steps, During Follow-up Interviews and SME Review of the Preliminary Gold Standard Protocol

	Additional Expert Knowledge Captured					
	Action Steps			Decision Steps		
	Added	Modified	Deleted	Added	Modified	Deleted
SME A	24	21	12	7	2	3
SME B	80	0	0	34	0	0
SME C	30	13	1	0	1	0

All three SMEs made multiple additions to action steps, while only two, SME A and SME B, made additions to decision steps during the follow-up interviews and preliminary gold standard review. Of the three, SME A provided the widest *variety* of additional expert knowledge, contributing a total of 31 additions, 23 modifications, and 15 deletions, whereas

SME B provided the greatest *quantity* of additional knowledge, with 80 additional action steps and 34 additional decision steps. The implications of this capture of additional knowledge are discussed in Chapter5.

Alignment of SMEs in describing the same action and decision steps. The spreadsheet analysis also allowed the researcher to gauge the degree to which each of the action and decision steps in the final gold standard protocol indicated alignment among the three SMEs. For each step, a column at the far right of the spreadsheet provided a cell where the researcher could enter one of three values, “1”, “2”, or “3”. If an action or decision step was attributed to one SME only, then a “1”, signifying *slight alignment,* was entered. If an action or decision step was attributed to two SMEs, then a “2”, signifying *partial alignment,* was entered, whereas if an action or decision step was attributed to all three SMEs, then a “3” was entered, indicating *high alignment* among the three SMEs. The results of this analysis appear in Table 3.

Table 3

Number and Percentage of Total Action and Decision Steps: Highly Aligned, Partially Aligned, and Slightly aligned.

	Frequency	Percentage
Highly Aligned	21	3.32%
Partially Aligned	68	10.76%
Slightly Aligned	543	85.92%

On a collective basis, there were 21 action or decision steps that were highly aligned among the three SMES, 68 that were partially aligned, and 543 that were slightly aligned.

Percentagewise, 3.32% of the steps were highly aligned, 10.76% were partially aligned, and 85.92% were slightly aligned. The steps in the final gold standard protocol that were highly aligned were steps 254-256 and 258-259, involving review fraction addition and division, steps 422 and 428, involving making meaning of the division of a whole number by a fraction, steps 459, 510, 563, and 609-615, involving the standard fraction division procedure for inverting the divisor and then multiplying, and step 585, involving division of mixed numbers. The implications of these alignment totals also are discussed in Chapter 5.

Question 2

What percent of action and/or decision steps, when compared to a gold standard, do expert middle school math teachers omit when they describe how they teach the division of fractions by fractions?

Total knowledge omissions. The spreadsheet analysis also allowed the researcher to determine the number of action and decision steps individual SMEs omitted when they recalled the expert knowledge necessary to teach the division of fractions by fractions. If an action or decision step was included in the final gold standard protocol, but was not attributed to an individual SME, then a "0" was entered in the same column that was used to determine knowledge *alignment* among the SMEs, as described in the previous section. The spreadsheet included a formula to total the number of such omissions for each SME, and additionally divided that figure by the total number of action and decision steps that comprise the final gold standard protocol, which served to calculate an omission percentage. A summary of the action and decision step omissions for each of the three SMEs appears in Table 4.

Table 4

Total Expert Knowledge Omissions by SME as Compared to the Final Gold Standard Protocol

	Steps Omitted					
	Total Action & Decision Steps Omitted	%	Action Steps Omitted	%	Decision Steps Omitted	%
SME A	502	79.43%	380	77.55%	122	85.92%
SME B	247	39.08%	211	43.06%	36	25.35%
SME C	405	64.08%	291	59.39%	114	80.28%
Mean Omissions	107.33	60.86%	294.0	60.00%	90.66	63.85%
Range	255		169		86	
SD	105.09		69.02		38.79	

Taking all three SMEs into consideration, there were an average of 107.33 total action and decision step omissions (SD ± 105.09) when recalling how to teach the division of fractions by fractions. For action steps alone, the three SMEs, on average omitted 294.0 steps (SD ± 69.02), while for decision steps alone, the three SMEs averaged 49.67 omissions (SD ± 38.79).

On an individual basis, there was significant variance among the three SMEs in terms of knowledge omissions. Total individual action and decisions step omissions ranged from a low of 39.08% to a high of 79.43%. Total individual action step omissions ranged from a low of 43.06% to a high of 77.55%, while total individual decision step omissions ranged from a low of 25.35% to a high of 85.92%. The implications of these knowledge omissions are addressed in Chapter 5.

Analysis of action and decision step omissions. The action and decision step omissions for the three SMEs, as compared to the final gold standard protocol, are presented in Figure 7.

Figure 7

Total SME Knowledge Omissions as Compared to the Final Gold Standard Protocol

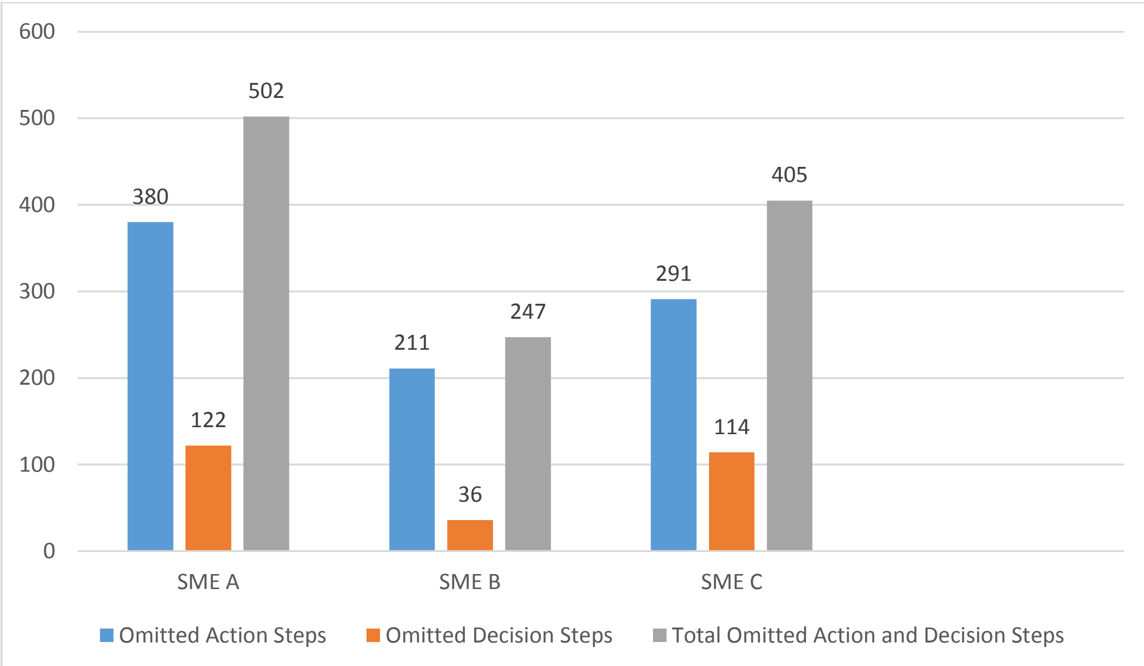

Figure 7. Total individual action and decision step omissions for SME A, SME B, and SME C. Compare to gold standard protocol non-repeating action and decision steps: action steps – 490, decision steps – 142, total action and decision steps – 632.

The next chapter presents an overview of the study, a discussion of findings, as well as limitations of the present study, its implications, and possible avenues for future research.

CHAPTER FIVE: DISCUSSION

Overview of the Study

The purpose of this study was to use cognitive task analysis to capture the knowledge and skills, represented by the action and decision steps, that middle school mathematics teacher experts recall when they teach the division of fractions by fractions. As explicated in Chapter Two of this study, the division of fractions comprises a high-leverage prerequisite for competency in algebra, which itself, and also, as detailed in Chapter Two, is a core prerequisite for achievement in secondary level mathematics overall. Additionally, this study sought to determine the number and percentage of action and decision steps that teacher experts in this instructional process omitted during their recall.

As shown in the literature review, there is a record of underachievement by K-12 students in mathematics in the United States. This record of underachievement has roots in students' understanding of algebra, of which the division of fractions by fractions is a critical component. In recent years, there has been a movement to establish standards-based curricula to address such underachievement in mathematics, as well as in other subject areas. To enable teachers to implement these curricula, school districts are increasingly required to pay greater attention to the manner in which they provide instructional support, both for novice teachers and seasoned veterans (Darling-Hammond, 2004; Polly et al., 2014). Instrumental to this endeavor has been an emphasis on providing ongoing professional development that builds teachers' capacity (Darling-Hammond, 2004). Currently, many school districts rely on subject matter experts in content knowledge and pedagogy to conduct these professional development programs (Darling-Hammond, 2004; Polly et al., 2014).

Unfortunately, the use of such experts poses a number of problems. As experts' new knowledge becomes automated and unconscious, they are often unable to completely and accurately recall the knowledge and skills that comprise their expertise (Chi, 2006; Feldon, 2007). Experts often are overly-confident, overlook details, make inaccurate predictions and offer faulty advice (Chi, 2006). In addition, as their skills improve, experts' self-report errors and omissions tend to increase, while their accuracy of introspection decreases (Feldon, 2007). Furthermore, because experts' schemas are adapted to problem solving, they can fail to articulate relevant cues, and can unintentionally fabricate consciously reasoned explanations for their automated behaviors (Feldon, 2007).

Because automaticity and the accuracy of self-reporting have been found to be negatively correlated, experts in an instructional role may unintentionally leave out information that learners must master when learning procedural skills (Feldon, 2004). In fact, experts may omit up to 70% of the critical information necessary to perform a task (Clark et al., 2011). The automated nature of knowledge causes procedural steps to blend together in experts' minds and makes it difficult for them to share the complex thought processes of technical skill execution (Clark et al., 2011). As a result, teacher professional development that is guided by such experts remains an imperfect model.

Studies indicate that instruction based on cognitive task analysis is superior to other instructional models (Clark, Yates, Early, & Moulton, 2010; Crandall, Klein, & Hoffman, 2006; Hoffman & Militello, 2009). CTA can allow the data analyst to identify the explicit and implicit knowledge of experts, which can support effective and efficient cognitive training, scenario design, cognitive feedback, and on-the-job training (Crandall, Klein, & Hoffman, 2006; Hoffman & Militello, 2009). In addition, CTA leads to guided instruction that is more structured and successful than other types of learning (Clark, Yates, Early, & Moulton, 2010). To that end, this study sought to use CTA to capture the knowledge and skills of math practitioners' expertise in teaching the division of fractions by fractions. The resulting gold standard protocol could serve to inform professional development for pre-service and in-service middle-school mathematics teachers.

This chapter presents a discussion of the process of conducting the task analysis, a discussion of the findings, an analysis of the study's implications, and an exploration of avenues for future research.

Process of Conducting the Cognitive Task Analysis

Selection of Experts

A key feature of a sound qualitative research study is *purposeful* selection of interview subjects. Subjects selected purposefully are best positioned to help the researcher answer the research questions because they have the most knowledge to impart (Creswell, 2009; Merriam, 2009). In the field of cognitive task analysis, researchers have identified several criteria that can assist in such purposeful selection of experts. These are at least 3-5 years of recent, consistent and successful task performance, with 10 or more years being optimal; history of performance in a wide and varied array of settings; and, no experience as trainers or instructors in the task (Clark, 2014; Clark et al., 2008; Feldon, 2007).

Identifying expert teachers can be a more complex endeavor. Beyond simply appraising years of experience, diversity of experience, peer recommendation and student test scores, it is necessary to consider a host of other variables. For example, Smith and Strahan (2004) identified six broad predictors of teaching expertise: (a) a sense of confidence; (b) a view of the classroom as a learning community; (c) the ability to develop nurturing, trusting relationships with students; (d) a student-centered approach; (e) professional contributions; and, (f) content mastery.

Taking all of these criteria into consideration, the researcher approached district administrators and school principals in 11 Southern California school districts, and over 100 individual schools, seeking expert math teachers. It soon became clear that finding math teachers that meet these criteria is extremely difficult, time consuming and produces few suitable candidates. Although the number of middle school math teachers employed in all of the 11 districts and more than 100 schools stands at somewhere between 400 and 500, it took the researcher approximately five months of continual search and outreach to locate just 3 teachers that matched many, but not all of the above criteria. Table 5 summarizes the selection criteria against which each SME was evaluated.

Table 5

Criteria for Selection of Expert Mathematics Teachers

	SME A	SME B	SME
5 years teaching experience	Yes	Yes	Yes
10 or more years' teaching experience	No	Yes	Yes
Diversity of experience (Little, Some, Much)	Little	Much	Much
Recommendation/District administrator	Yes	No	No
Recommendation/School site administrator	No	Yes	No
Recommendation/Peer	No	No	Yes
Student achievement results	No	No	No

As can be seen from Table 5, beyond measuring years of experience, other quantifiable criteria, such as student achievement data, did not come into play. The reason for this is simple: Perhaps most difficult of all was finding teachers with quantifiable measures of student achievement data. Few districts were either willing or able to provide such data, or to identify teachers on such a basis. In the end, because of this challenge, the selection of experts for this study excluded student achievement data. This conforms to the experience of McZeal (2014) who struggled to obtain achievement data in a study of special education teachers, and is in contrast to Mutie (2015) who was able to base selection of expert secondary math teachers partly on student achievement data.

This lack of the use of achievement data in the present study can be put into context. For example, Shanteau (1992) defines professional expertise as enhanced quality of task performance that is a function of additional experience. In certain fields of endeavor, increases in years of experience lead to a *quantifiable* degree of *quality* of task performance. For example, chess masters record more wins than losses or draws; expert test pilots have superior records of test mission successes, with few if any mishaps; expert insurance analysts are directly responsible for minimizing loss claims. Does this mean, then, that expertise in teaching leads to quantifiable gains in student achievement, or conversely, that a superior record of student achievement is an indicator of teaching expertise?

Because of the number of variables that contribute to expertise in teaching, it is a moot point as to whether quantifiable measures, such as student achievement results alone, give an accurate assessment of a teacher’s expertise. In fact, Sternberg and Horvath (1995) found that this criterion, achievement data, as well as many of the other commonly accepted measures of teaching expertise, including years of experience, provide inconclusive evidence of success. Both the number of variables that exist in the profession of teaching, and the difficulty in identifying expertise is akin to the nature of other professions, where research has shown that years of experience do not correlate with quality of task performance. One such field is psychotherapy, where research has shown no measurable correlation between years of experience and accuracy or skill (Tracey, Wampold, Lidhtenberg, & Goodyear, 2014). In addition, much as in teaching, where student learning is partly dependent on the motivation of students to learn, a psychotherapist's ability to effect patient outcomes also rests partly on the desire and motivation of said patient to participate earnestly in treatment (Tracey et al., 2014).

The implication is that present methods for identifying expert teachers are imperfect at best. One problem lies in the ways professional expertise, including that among teachers, is identified. The typical course of action is to identify particular traits, attributes, or metrics, and then to seek individuals that conform to these. Such a *confirmatory* approach often leads to searches that uncover partial evidence only, since the tendency is to ignore evidence that refutes expertise (Tracey et al., 2014). An alternative approach adopts a *disconfirmatory* stance, in which one seeks to identify traits or metrics that would render a professional *inexpert* (Tracey, et al., 2014). Such an approach yields more complete, less biased information, and leads to better decision making (Tracey et al., 2014). The use of such an approach could serve as an appropriate model for the identification of expertise in education. It should be noted that the current study did not adopt such an approach.

Data Collection

The researcher has 15 years' experience as an elementary school teacher, and as such, has no experience in teaching middle school mathematics. As a result, a primary means of conducting the initial two stages in the CTA process, *collect preliminary knowledge* and *identify knowledge representations* (Clark et al., 2008) involved a literature review of secondary mathematics content and pedagogy in general, and algebra and fractions content and pedagogy in particular. In this way, the researcher was able to gain a general appreciation of the types of knowledge possessed by middle school teacher experts in the division of fractions by fractions, and of the major tasks and subtasks they typically perform while delivering instruction.

Data collection took the form of a multi-stage, semi-structured interview technique based on the *Concepts, Processes* and *Principles* (CPP) procedure (Clark et al., 2008), a model designed to capture the automated, unconscious knowledge experts acquire through extended experience. The interview protocol appears in Appendix A.

During the course of the interviews, it became apparent that each SME approached the target task in markedly different ways. For example, SME A's approach to the division of fractions relied heavily on review of the prior knowledge necessary to make sense of this target task. In addition, this SME stressed the importance of students acquiring procedural fluency in the invert and multiply approach, but was less concerned with their acquiring conceptual understanding. SME B also recalled a considerable number of action and decision steps involving review of prior knowledge, but unlike the other two SMEs, also stressed the importance of helping students build connections between different number types, such as integers and rational numbers. This SME was also the only one of the three to employ a complex fraction procedural approach as a bridge to conceptual understanding. SME C recalled the fewest action and decision steps involving review of critical prior knowledge, but was most heavily invested in helping students acquire conceptual understanding, as exemplified by a reliance on realia, pictorial representations, and the use of real life examples.

The researcher found the semi-structured approach, which allows for the use of extemporaneous probes, an ideal format for responding to these differences in approach to the target task. Each time a SME began to describe a feature of instruction that differed from features described in the literature, or features described by the other two SMEs, it was possible to use probes to tease out much of the detail, complexity, and underlying logic inherent in such features. This would have been impossible with a structured approach, in which there is essentially no leeway for deviating from a script of questions. The researcher's experience seems to confirm Merriam (2009) contention that a semi-structured approach best enables an interviewer "…to respond to the situation at hand, to the emerging worldview of the respondent, and to new ideas on the topic" (p.90).

Contrary to the view of Crandall, Klein and Hoffman (2006), who recommend that the interviewer take detailed, hand-written notes during the interview, and then transcribe these to a text file, the researcher audiotaped each interview. In addition, the researcher took general notes, yet only when necessary, to record impressions, underscore a point made by the SME, or to note emerging themes. This primary reliance on audiotaping became an important feature of this study. To wit, eschewing the need to attend to intensive note taking, the researcher was able to lessen cognitive load and free up working memory (Sweller, 1988). In this way, the researcher avoided becoming distracted from the dynamics of the interview (Clark et al., 2008), and with a lightened cognitive load, could respond thoughtfully with the judicious use of extemporaneous probes to the data as it emerged, while also ensuring that everything the SME articulated was preserved on tape for later analysis (Merriam, 2009).

Following each interview, the researcher sent the audio files to a professional transcription service, which provided verbatim transcriptions of the interviews. A discussion of the researcher's use of these transcripts for data analysis appears in the next section.

Data Analysis

After gaining permission from each of the three SMEs, the researcher audiotaped the interviews in their entirety, and then sent the audio files to a professional transcription service, where they were converted to typed, verbatim transcripts. This is contrary to the practice of Zepeda-McZeal (2014), who created edited transcripts, containing only information pertinent to the procedural steps of the target task. This researcher preferred using verbatim transcripts, finding that reading through such scripts, while simultaneously referring to his own interview notes, brought forth vivid mental images of each interview, of the SME's tone and facial expressions, of sensory impressions, and other non-verbal nuances of the interview. This added immeasurably to the researcher's ability to maintain a sense of proximity to the interviews when later reconstructing the transcripts into the individual protocols by adding the nuances of the interview recalled in memory and the resulting details of the actions and decisions recalled by the SME. The researcher's experience also runs counter to the position taken by Crandall, Klein and Hoffman (2006) that a verbatim transcript, "…gives you the words, but leaves everything else out." (p. 280). The verbatim transcripts were then analyzed, coded, and converted by the researcher into initial protocols of action and decision steps.

There are various ways to verify an initial SME protocol. This researcher met in person with each SME and presented his or her protocol for review. This runs contrary to the work of Clark et al. (2008), who recommend giving the protocols of each SME to one of the other SMEs for review, and conforms to the work of Crispin (2010), Embrey (2012), Flynn (2012) and

Zepeda-McZeal, who also asked each SME to review his or her own protocol. The researcher's rationale for adopting this approach is twofold. In the first place, giving a protocol of one SME to another SME for review would have been too time consuming. Secondly, and more importantly, looking at an initial protocol, which is essentially a detailed list of action and decision steps, can be a bit disorienting for SMEs. Such a representation differs markedly from the structure of the interview conversation, and this researcher felt it best to allow each SME to reconcile his or her recollection of the interview with the completed protocol, and accordingly to verity it for accuracy and integrity.

As mentioned earlier in this chapter, the researcher encountered great difficulty in identifying, contacting, and securing the cooperation of subject matter experts. Because it took anywhere from one month to six weeks to identify each additional expert, the researcher ended up adopting a specific pattern of data analysis. Following each interview, the researcher arranged for transcription of the audio file, received a transcript, coded and then converted the transcript into an individual task protocol. Then the researcher met with the SME for a follow-up interview during which the SME suggested revisions. Based on these recommendations, the researcher then produced a revised individual protocol. All this time, the researcher was also engaged in the process of locating additional SMEs. As it happened, for each protocol, no additional SMEs were located before the researcher had had time to complete this entire process of interviewing, coding, developing a protocol, conducting a follow-up interview, and then producing a revised individual protocol. As a result, the researcher was able to concentrate exclusively on each SME's version of the target task. Interview transcripts did not sit around "gathering dust" while additional interviews were being arranged and conducted, the researcher did not have to juggle in his mind alternate SME versions of the task procedures, nor did the researcher ever experience a sense of “distance” from the immediacy of the interview experience. The researcher accordingly was able, as closely as possible, to experience an *emergent* qualitative research study (Merriam 2009). Data collection and analysis evolved rhythmically, unbrokenly, and, at times, approached *simultaneity* (Merriam, 2009). It would be interesting to compare the results of this study to other CTA studies where data collection and analysis took place in a more fragmented, less continuous manner.

Discussion of Findings

No formal hypotheses were developed for this research study. Instead, the study was guided by two main research questions.

Research Question 1

What are the action and decision steps that expert middle school math teachers recall when they describe how they teach the division of fractions by fractions?

Action steps versus decision steps. Each of the three SMEs recalled more action steps than decision steps. As reported in *Chapter Four: Results*, SME A recalled 110 action steps compared to 20 decision steps, SME B recalled 279 action steps versus 106 decision steps, and SME C recalled 199 action steps as compared to 28 decision steps. On a collective basis, the three SMEs recalled an average of 196 action steps versus an average of 51.34 decision steps.

Decision steps, by their very nature, involve unobservable, cognitive processes. Through repeated task performance, experts’ execution of these processes becomes automated (Anderson, 1996; Feldon, 2007), which makes it difficult for experts to consciously explain how they go about deciding on a course of action (Clark & Elen, 2006; Clark & Estes, 1996). Therefore, the preponderance of reported action steps over decision steps in the present study appears to conform to this research on automaticity (Clark, 2014). In addition, the results from the three

SMEs in this study are in line with other CTA studies, which have also reported a greater number of action steps than decision steps, such as Canillas' (2010) study of a central venous catheter placement, Crispen's (2010) study of an open cricothyrotomy procedure, Embry's (2012) study of tracheal extubation, and Zepeda-McZeal's (2014) study of reading instruction in a special education environment.

Of additional interest is the degree to which the *differences* between the numbers of action and decision steps, both individually and collectively, compare to such difference in other CTA studies. One analytical tool for making such a comparison, percentage difference, calculates a value that is a function of two compared values, in this case, the number of action steps and the number of decision steps. This value is derived by subtracting a relative value (the number of decision steps) from a reference number (the number of action steps). The resulting value is divided by the average of the relative and reference numbers, and then multiplied by 100 to convert it to a percentage (Zepeda-McZeal, 2014). In the current study, the three SMEs, on average, collectively recalled 116.97% more action steps than decision steps. Individually, SME A recalled 138.46% more action steps, SME B recalled 89.87% more action steps, and SME C recalled 150.66% more action than decision steps.

Compared to other CTA studies, these figures appear, at first, to be excessively large. In a study of instruction among special education teachers, Zepeda-McZeal (2014) reported a collective average difference of 20.56%, and individual differences that ranged from 15.73% to 27.78%. In a study involving a medical procedure, open cricothyrotomy, Tolano-Leveque (2010) reported a collective average difference between action and decision steps of 37.95%, while in a study of K-12 principals, Hammitt (2015) reported a collective average difference of

13.53%. Clearly, the percentage differences reported in the current study are significantly greater, and do not seem to conform to the results of other studies.

However, taking into consideration the nature of the tasks under study, and of the number of total reported action and decision steps, the large percentage differences reported in the present study can be put into context. In the present study, the final gold standard protocol contains 632 total action and decision steps, and 348 more action steps than decision steps. In the other studies cited, the total number of action and decision steps is significantly smaller, and the overall difference between action and decision steps is as well. For example, in the study by Zepeda-McZeal (2014), the final gold standard protocol contains 179 total action and decision steps, with just 21 more action than decision steps. In Hammitt's (2015) study, the final gold standard protocol contains 196 total action and decision steps, with a mere 14 more action steps than decision steps. It may be that the tasks in the two studies referenced, in one case instruction in informational text in a special education environment (Zepeda-McZeal, 2014), and in the other, observations by K-12 principals of instructional practice (Hammitt, 2015), do not lend themselves to comparison to the task analyzed in the current study, instruction in fraction division, either because of differences in the nature of the tasks, and/or because of the disparities in the number of action and decision steps in the tasks. This possibility is also suggested by Hoffman (1987), where the extent of knowledge extraction with experts was found to vary according to task complexity.

One CTA study that also involved mathematics instruction sheds some light on this question. Mutie (2015) studied four instructors expert in teaching quadratic equations to middle school eighth-graders. Much like the task in the present study, the task in that study also involved a large number of action and decision steps, with a concomitantly large disparity between types of steps. In the final gold standard protocol, Mutie (2015) captured 404 total action and decision steps, while there were 234 more action steps than decision steps. In terms of percentage difference, the four SMEs in that study collectively recalled, on average, 118.52% more action than decision steps. Individually, SME A recalled 134.02% more action steps, SME B recalled 113.51% more actions steps, SME C recalled 110.06% more action steps, and SME D recalled 130.94% more action steps than decision steps. In light of the similarities in the percentage differences between the present study and that of Mutie (2015), and in the similarities in the numbers of steps and in the disparities of types of steps, the results of the present study seem to conform to, in at least one case, studies that analyze tasks with similar characteristics.

Differences in recall among SMEs. SME A recalled 130 action and decision steps, the fewest among the three SMEs. This SME was recommended, on the basis of instructional performance, by a district director of secondary mathematics, possessed five years of experience teaching middle school mathematics, and entered teaching mid-career, following several years working in an unrelated field. The relatively low rate of recall of this SME might be attributable to a couple of factors. First of all, five years is seen as the low end of the range for the development of expertise, with ten years typically regarded as a more predictive time span (Ericsson, 2004; Ericsson, Krampe, & Tesch-Romer, 1993). Thus, this SME may have lacked the same degree of expertise displayed by the other two SMEs, who possessed, respectively, twenty-one years' experience, and twenty-two years' experience. Furthermore, although five years of task experience is viewed as sufficient for practitioners to attain task automaticity, this time frame also represents a point at which performance can, under certain conditions, reach a stable plateau (Ericsson, 2004). Without any additional time, beyond five years of practice, in which to engage in a conscious, orderly and deliberate approach to improvement, this SME may have been rooted in this transitional stage of expertise, one that is typified by arrested development of further skill acquisition (Ericsson, 2004).

SME B recalled the most action and decision steps among the three SMEs, with a total of 385. This SME was recommended on the basis of instructional performance by the school principal, and had twenty-one years' experience teaching middle school mathematics. The fact that this SME provided the most recall of combined action and decision steps can be viewed as running counter to research findings. In general, because of the length of time this expert had been developing conscious declarative knowledge of task execution, the more automated in nature this knowledge should have become (Feldon, 2007). Because such knowledge is often difficult for experts to articulate, due to its non-conscious, automated nature (Kirschner, Sweller, & Clark, 2006) it seems counterintuitive that this SME's recall of expertise was so extensive. Additionally, and more specifically, this SME reported far more decision steps than either of the other two SMEs: 106 decision steps versus 20 decision steps for SME A and 28 decision steps for SME C. As delineated earlier in this chapter, decision steps, by their very nature, involve unobservable, cognitive processes. Through repeated task performance, experts' execution of such cognitive processes also is subject to becoming automated (Anderson, 1996; Feldon, 2007), making it difficult for such experts to consciously explain how they go about making decisions pursuant to a course of action (Clark & Elen, 2006; Clark & Estes, 1996). Thus, this SME's superior recall of this variety of expertise also presents a contradiction. Yet it is also possible that individual SMEs have different understandings of how much review of prior knowledge is necessary when providing instruction, and thus, could have different conceptions of when a task actually starts or ends.

One explanation for the degree to which SME B was able to provide recall of expertise, compared to the other two SMEs, may lie in the fact that this SME was, during the course of this study, involved with a group of educators in the collaborative design of a forthcoming middle school mathematics textbook. Involvement in such a process, which involves working with other experts to delineate the knowledge inherent in a specific curriculum, including the division of fractions, would seem to be representative of the deliberate and orderly nature of practice that research has identified as being essential to the improvement of performance (Ericsson, 2004). Such a motivation to improve performance (Ericsson, Krampe, & Tesch-Romer, 1993) would no doubt immerse an educator in deep and thoughtful consideration of the nature of each instructional task in that curriculum, and of each and every element necessary to execute those tasks, thus allowing greater recall of attendant expertise as reported in this study. Such mental effort is reminiscent of the *proficiency* stage of Alexander's (2003) Model of Domain Learning, in which practitioners make concerted effort to contribute new knowledge to a field of endeavor. This deep, thoughtful, and conscious consideration brought to the task by this SME can also be compared to Anderson's (1996) associative stage of automaticity, in which a learner must still consciously work through the steps of a procedure during task execution, as not all declarative and procedural task knowledge has been fully automated. Perhaps this SME, as a result of participation in a textbook committee, had actually regressed from full to partial automaticity.

SME C possessed twenty-two years' experience teaching middle school mathematics, was recommended based on her instructional performance by a district administrator, and recalled 227 total action and decision steps. Although this SME recalled more action steps than SME A, this expert recalled significantly fewer steps than SME B, a fact which seems to conform more readily to the research literature on expertise. To wit, with such extensive task experience, this SME may have possessed deeply ingrained, automated procedural knowledge, resulting in a concomitant decrease in the accuracy and completeness of recall, coupled with an increase in self-report errors and omissions (Clark et al., 2011; Feldon, 2007).

Additions, deletions, and revisions captured during review of initial individual protocols and of the preliminary gold standard protocol. As described earlier, following the drafting of an initial, individual protocol, the researcher met with each SME for a review. The researcher described the coding and protocol development process, and then presented the SME his or her protocol. In each case, the SME read over the protocol and then made revisions, a process that was slightly different for each SME. Revisions took three forms: additions, modifications, and deletions. Additions involved an entirely new step, modifications involved changes in the language of pre-existing steps, and deletions involved the complete removal of a pre-existing step. Following this review process, the revised individual protocols were aggregated into a preliminary gold standard protocol, which was also presented to each SME for review.

Review of initial individual protocols. SME A conducted the most thorough, systematic review of the initial protocol, compared to the other two experts. This expert carefully read over each page, line by line, stopping often to pencil in notes in the margins. Following this approximately 45 minute read-through, SME A carefully went over each of these marginalia with the researcher. One result was the addition of 24 action and 7 decision steps. In almost all cases, these additions involved enhancements to checks for student understanding, usually in the form of polling individual students and having the teacher orally reinforce concepts. Most of these were action steps, with the decision steps comprising either moving on if students understood, or conducting a review if comprehension was weak.

SME A made the most modifications, 23, most of which were action steps. The majority of these involved minor revisions to the manner in which the researcher had worded these steps, with little change in semantics. Of interest was this SME's choice to move an eight step sub-procedure, involving conceptual understanding of division of a fraction by a whole number, to an earlier juncture in the larger procedure, commenting, "We've already created understanding of this." This attests to the care and degree of mental effort this SME brought to the review process.

SME A also made 15 deletions. Most of these were action and decision steps that the researcher had inferred from the interview transcript. They mostly involved reinforcing a concept with additional independent practice, a course of action the SME found superfluous. This is of interest, in that it lies in contrast to the numerous steps involving checks for understanding this expert added, as described earlier.

SME B's approach to this process was markedly different. This expert listened to the researcher's description of the protocol development process, was given the protocol, read over each of its 16 pages methodically, yet rapidly, taking just over 15 minutes, and made no written annotations or asked any questions. However, following this read-through, SME B commented that in the two weeks since the initial interview, "…in my work with a committee to write a textbook, I realized that I left out a big piece where we need to relate fractions to decimals." This SME then laid out an additional 80 action steps and 30 decision steps designed to convey that fractions and decimals both are representations of parts of a whole. In most cases, these additional steps involved instruction aimed at helping students perform fraction operations, and then replicate those operations by converting the fractions to decimals and repeating the operations.

While SME B added the most additional expertise of the three SMEs, this expert made no modifications and no deletions. This SME seemed preoccupied at the beginning of this second interview. To wit, the SME answered some preliminary questions perfunctorily, appeared mildly disinterested during the researcher's description of the process of developing the protocol, and, when prompted, indicated that she had no questions. The expert's somewhat cursory read-through of the protocol seemed to confirm this sense of preoccupation. It is possible that SME B was concerned mainly with imparting the additional expertise that work with a textbook committee had uncovered, and that this concern distracted the expert from a thorough review of the initial protocol. There might have been better capture of additional expertise pursuant to the *initial protocol*, had the researcher contacted this SME beforehand by phone, explained the purpose of the second interview, and then emailed the initial protocol for the SME to review at leisure. Although this course of action runs counter to the findings of other such studies (Hammitt, 2015; Zepeda-McZeal, 2014), this SME may have been an atypical case requiring atypical measures.

Among the three experts, SME C contributed the least additional decision-step expertise. This involved a single modification to a decision step involving how to provide remediation to struggling students. The majority of this expert's additions of expertise involved 30 action steps, and of these, the majority involved two categories: directing the teacher to make use of real-world objects (realia), such as apples, pizzas, bagels and sheets of paper to reinforce conceptual understanding, and, asking students to formulate real-life examples of the concepts under consideration. This SME also made 13 modifications to action steps, and the majority of these involved changes to the types of realia employed.

This expert's focus on conceptual understanding, either through the use of realia or of real-life examples is consistent with the fact that this SME was the only one of the three to include division of fraction word problems in the initial protocol. Such word problems are regarded as essential in helping students make sense of mathematical concepts (NMAP, 2008). This SME remarked that too few teachers understand the importance of such conceptual understanding and the result is that numerous teachers, "… complain to me that they taught this concept two weeks ago, and the students have already forgotten it." This SME, with the most years of teaching experience of the three, may exemplify some of the more advanced characteristics of expertise: the possession of a profoundly organized body of knowledge, with well-developed schemas, and the ability to create and present to students mental models, which can be quickly and efficiently retrieved from long term memory (Bedard & Chi, 1992).

Review of the preliminary gold standard protocol. Subsequent to these follow-up interviews, the researcher revised each individual protocol, and then aggregated these three into a single preliminary gold standard protocol. Each SME was contacted by phone, and agreed to review an emailed copy of this preliminary gold standard. Once each SME had indicated by return email that he or she had had time to review it, a second phone call was made to discuss any possible revisions.

At this point, there was essentially, no further revision. Aside from a few questions about how the other SMEs had approached the instructional task, or for an explication of the rationales involved in some of the procedures that had been suggested by other SMEs, each expert indicated that the protocol, as it stood, was acceptable to him or her.

This runs counter to the experience of the researcher during the initial follow-up interviews, during which, as has been described, each SME read over the protocol and made not insubstantial revisions. As a result, perhaps, and as also suggested by Zepeda-McZeal (2014), the researcher may have been able to capture additional expertise for the preliminary gold standard protocol through the use of an in-person review. However, choosing to accept the avowal by each of the three SMEs that the preliminary protocol was complete and could be used to inform instruction, the researcher chose not to pursue an in-person interview format, and thus, the preliminary gold standard protocol basically transmogrified into the final gold standard protocol, with no further additions, revisions, or deletions. As a result, it is impossible to say that the final gold standard protocol represents a complete listing of the action and decision steps necessary to teach the division of fractions by fractions.

Alignment of SMEs vis-à-vis total action and decision steps. As described in Chapter 4, this study included an analysis of the degree to which there was alignment among the three SMEs in their recall of total action and decision steps. As reported, just 21, or 3.32% of the total steps in the final gold standard protocol were *highly aligned*, meaning that they were recalled by each of the three SMEs. Additionally, a mere 68, or 10.76% of the total action and decision steps in the final protocol were recalled by two of the SMEs in the study, and were thus classified as *partially aligned.* The majority of recalled action and decision steps, 543 steps, comprising 85.92% of the total, were *slightly aligned,* meaning they were attributable to a single SME only.

As research shows, the evidence-based, highly automated and unconscious knowledge of experts is difficult to articulate, and experts' self-report errors and omissions tend to increase as skills improve (Feldon, 2007; Kirschner, Sweller, & Clark, 2006). Because the highly developed and adaptive schemas of such experts can interfere with the accurate recall of problem situations (Feldon, 2007), CTA methods that rely on multiple experts to elicit expertise have been found to be an effective means of informing instruction (Clark & Elen, 2006). As this study's analysis of action and decision step alignment shows, reliance on a single subject matter expert to inform instruction could result in an imperfect approach in teaching teachers how to provide instruction in the division of fractions by fractions. This is borne out by a number of recent studies in the fields of medical instruction and K-12 instruction (Bartholio, 2010; Canillas, 2010; Mutie, 2015; Zepeda-McZeal, 2014), in which the use of multiple experts led to significant increases in knowledge capture.

Research Question 2

What percent of action and/or decision steps, when compared to a gold standard, do expert middle school math teachers omit when they describe how they teach the division of fractions by fractions?

Expert knowledge omissions. Research indicates that experts' unconscious, automated knowledge is difficult to articulate, and this leads to omissions by such experts during recall (Kirschner, Sweller, & Clark, 2006). Therefore, the researcher conducted an omission analysis for this study by comparing the action and decision steps in the individual protocol for each SME to the action and decision steps in the final gold standard protocol. On average, the three SMEs collectively omitted 60.86% of total action and decision steps, 60.0% of action steps, and 63.85% of decision steps, with a difference between action and decision step omissions of 3.85%. The small difference between these average action step and decision step omissions conforms to the findings of other recent CTA studies. For example, Mutie (2015) found a difference between average action and decision steps of 3.10%, Hammitt (2015) found a difference of 0.34%, while Zepeda-McZeal derived a difference between average action and decision step omissions of 1.15%.

The researcher further analyzed the extent to which each SME individually omitted combined action and decision steps. SME A omitted 79.43% of the action and decision steps in the final gold standard protocol, SME B omitted 39.08% of the action and decision steps in the final protocol, while SME C omitted 64.08% of the combined action and decision steps in the final gold standard protocol. The mean of these omissions is 60.86%, with a standard deviation of 20.37%. Of interest to the researcher was the degree to which these omissions either did or did not conform to the research finding that experts may omit up to 70% of action and decision steps when prompted to describe task execution (Clark, et al., 2011; Feldon, 2004).

The researcher performed a one-sample, two-tailed *t*-test to determine whether the omission percentages found in the study conform to the hypothesized omission value of 70%, or whether the findings were due to chance. The *t*-test computation used the following values: $n = 3$ participants, sample mean of omissions = .6086, with a sample standard deviation (SD) = .2037, and a population mean = .70. The resulting t value was -0.7773, while the two-tailed p value based on an alpha level of $\alpha = 0.05$ was .5183. The magnitude of the p value indicates that the findings of combined action and decision steps for the three SMEs in this study conform to the research finding that experts may omit up to 70% of critical action and decision steps during recall, and were not the result of chance (Clark et al., 2011; Feldon, 2004).

Limitations

The results of the current study were consistent with those of other CTA studies seeking to capture and analyze expert knowledge recall and omissions. Following is a discussion of the limitations inherent in this study.

Confirmation Bias

Confirmation bias is the tendency among researchers to give greater credit to information that aligns with their own preconceptions, whether or not the information is actually true (Corbin & Strauss, 2008). As Clark (2014) further notes, in the realm of cognitive task analysis, this bias is seen when knowledge analysts are experienced in the domain of the task under study, and can lead to the analyst unconsciously editing knowledge captured from SMEs. The researcher, at the time of the present study, had 15 years of instructional experience in K-12 settings, specifically at the elementary level. Although those 15 years of instructional experience did include mathematics instruction, the researcher was not overly familiar with the middle school mathematics that were the focus of the study. Thus the researcher sought to bootstrap the information necessary to become familiar with the domain of middle school mathematics, primarily through a search of the literature (Schraagen et al., 2000). The researcher's intent was to acquire domain familiarity commensurate with of that of an accomplished novice (Schraagen, et al., 2000), with the aim of minimizing the tendency that would be characteristic of a more accomplished expert to filter or edit the knowledge captured.

It should additionally be noted that the researcher does have 15 years' experience in teaching *general* fraction concepts. As a result, during interviews, when SMEs were describing activities involving review in such general concepts, as a precursor to describing more specialized instruction in the *target* task, the researcher strove to avoid making judgments, based on those 15 years of experience that might have resulted in filtering, editing, or otherwise altering the data.

Internal Validity

Maxwell (2005) cautions that the aim of research is to capture reality, and that this can be a daunting, elusive goal. In the present study, internal validity represents the degree to which reality, namely the actual enactment of the target task by the SMEs on a daily basis in their classrooms, matches the descriptions of such practice as delineated in the gold standard protocol (Merriam, 2009). To that end, internal validity for this study would involve triangulation through the use of multiple data sources (Merriam, 2009): one source involving observation of the three SMEs teaching the target task, and a second source consisting of an analysis of the number of action and decision steps found in the gold standard manifested in their classroom practice. For the purposes of the present study, such an internal validity analysis by observation was not performed, and thus presents a limitation of the study.

External Validity

External validity is a measure of the degree to which the results of a study are generalizable to other settings (Merriam, 2009). Threats to external validity in the present study include the small sample size (n = 3), and the fact that the sampling of experts was non-random and included only teachers from three somewhat adjacent Southern California school districts. It remains moot, therefore, whether the gold standard protocol produced in this study would be applicable to other teachers in school districts in other regions and/or states.

Yet Merriam (2009) takes a somewhat contrarian, yet intriguing, position. Because case studies such as this are generally rich in qualitative description, they provide for the reader opportunities to apply what is gleaned from such description to similar cases, possibly serving as the basis for teacher education or evaluation (Eisner, 1991; Merriam, 2009). Additionally, it is always possible to apply lessons learned from particular cases, such as this, to others that are similar, although it should also be noted that generalizability is not the immediate aim of case studies (Erickson, 1986; Merriam, 2009). Nevertheless, future CTA studies that aim to elicit the declarative and procedural knowledge of middle school teachers expert in the division of fractions by fractions couldn't help but serve to increase the present study's external validity by employing a larger, more diverse, and ideally, randomly selected sample of subject matter experts.

Implications

The current movement to establish standards-based curricula to address underachievement among K-12 students has led school districts to pay greater attention to the manner in which they provide ongoing professional development to build teachers' capacity. The current model, in which districts rely on subject matter experts in content knowledge and pedagogy to conduct professional development programs (Darling-Hammond, 2004; Polly et al., 2014), has been found to be problematic, as research shows that such experts may unintentionally leave out up to 70% of the information that learners must master when learning new tasks (Feldon, 2004). Several studies indicate that instruction based on cognitive task analysis is superior to other instructional models (Canillas, 2010; Clark, Yates, Early, & Moulton, 2010; Crandall, Klein, & Hoffman, 2006; Hoffman & Militello, 2009; Zepeda-McZeal, 2014). CTA can allow data analysts to identify the explicit and implicit knowledge of experts, which can support effective and efficient cognitive training, scenario design, cognitive feedback, and on-the-job training (Crandall, Klein, & Hoffman, 2006; Hoffman & Militello, 2009). Similarly, the findings of the current study provide support for the use of CTA to inform teacher professional development in K-12 instructional tasks, such as the division of fractions by fractions, an area of the mathematics curriculum that has been identified as critical to overall mathematics achievement (NMAP, 2008).

There is, however, a major caveat to the use of this study's final gold standard protocol as the basis for teacher professional development in the division of fractions by fractions. It must be stressed that developing a gold standard protocol through the use of CTA methods is predicated on the educational context within which the subject matter experts chosen operate. Thus, attempting to generalize such a gold standard protocol to other educational contexts must be done with caution. Differences in how individual students learn and how individual teachers deliver instruction would preclude the possibility of such a gold standard protocol representing *the* definitive method of best practice in division of fractions by fractions. In addition, the concerns discussed earlier in this chapter regarding the paucity of further input by the three SMEs when asked to review the preliminary gold standard protocol further leads to the conclusion that the gold standard protocol developed in the current study cannot be regarded as representing 100% of the action and decision steps comprising best practices instruction in the division of fractions by fractions.

Future Research

There appear to be no other CTA studies that have investigated the focus task of this study: instruction in the division of fractions by fractions. A search of the literature on cognitive task analysis and this branch of mathematics was inconclusive. This tends to suggest, therefore, that an avenue for future research might be to use the gold standard protocol developed in the current study as the basis for a randomized experimental study with middle school teachers tasked with instruction in the division of fractions by fractions. The intent would be to compare the learning gains realized among students receiving instruction in traditional methods as compared to students receiving instruction in accordance with the gold standard protocol.

Of further interest, and as mentioned earlier, this study employed a specific method for expert review of the preliminary gold standard protocol. To wit, once this protocol had been aggregated, the three SMEs responsible for its creation were tasked with its final review. As reported, this review resulted in no additional capture of expertise. Two avenues for future research present themselves as a result of this outcome. First of all, the review process did *not* involve an in-person discussion between the researcher and each individual SME. A future study in which such an in-person review was incorporated might shed light on whether or not such practice led to increased capture of expertise. Additionally, future research might employ an alternative review method, one in which review of the preliminary protocol were conducted not by the SMEs responsible for the preliminary gold standard, but rather, by an independent expert. It might be instructive to determine whether the use of such a fourth expert also could lead to an increase in capture of expertise.

Also, this study drew attention to the large disparities in percentage differences between recalled action steps versus decision steps among the three SMEs, as compared to other CTA studies (Hammitt, 2015; Tolano-Leveque, 2010; Zepeda-McZeal, 2014). As discussed, it is possible that these differences were a result of differences in the nature of the task in this study versus the nature of the tasks in the comparison studies. Although, upon further analysis, these percentage differences were found to be similar to the percentage differences in another CTA study that also involved a mathematics instructional task (Mutie, 2015), future research might investigate whether other CTA studies have investigated K-12 mathematics instruction, and to what extent such percentage differences between recalled action and decision steps in those studies compare to those in the present study.

Finally, for the purposes of assessing internal validity, there are two avenues for future research. In the short term, either the current researcher, or another researcher, might conduct an analysis of the actual, classroom practice of the three SMEs that were the focus of this study. The aim would be to determine the degree to which that practice conforms to the action and decision steps reported in the final gold standard protocol. In the long term, future CTA studies focused on the division of fractions might also include observation of SME instructional practice to validate the credibility of any resulting final protocols.

Conclusion

This study builds on the current body of knowledge concerning the use of cognitive task analysis to capture the knowledge and skills of experts involved in complex tasks, and additionally, to analyze the omissions such practitioners make when recalling their expertise. As mentioned, this is the first known CTA study involving instruction in the division of fractions by fractions in a middle school setting. With respect to knowledge omissions among experts, this study found an average of omissions of action and decision steps among three subject matter experts of just over 60%, which conforms statistically to established research findings that experts may omit up to 70% of critical knowledge and skills when asked to recall their expertise (Feldon, 2004). Compared to the amount of knowledge and skills that were captured from the three SMEs *individually*, the extent of such knowledge and skills that was captured from these three experts *in aggregate* confirms the superiority of the use of multiple experts for knowledge capture, and of the use of CTA methods. The resulting final gold standard protocol from this study could serve as the basis for a professional development program for novice and veteran middle school teachers alike, with the aim of providing a model of teacher capacity building that more successfully improves student mathematics achievement, as compared to current models, which rely on individual subject matter experts.

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Appendix A
Cognitive Task Analysis Interview Protocol

Begin the Interview: Meet the Subject Matter Expert (SME) and explain the purpose of the interview. Ask the SME for permission to record the interview. Explain to the SME the recording will be only used to ensure that you do not miss any of the information the SME provides.

Name of task(s): How to teach division of fractions by fractions

Performance Objective:
Ask*: "What is the objective of teaching how to divide fractions by fractions?" "What action verb should be used?"*

Step 1:
Objective: Capture a complete list of student learning outcomes for teaching division of fraction by fractions.

A. Ask the Subject Matter Expert (SME) to list student outcomes when these tasks are complete. Ask them to make the list as complete as possible
B. How is the student assessed on these outcomes?

Step 2:
Objective: Provide practice exercises that are authentic to the context in which the tasks are performed

A. Ask the SME to list all the contexts in which these tasks are performed (e.g. classroom; small group; whole group; addition vs. subtraction, etc.)
B. Ask the SME how the tasks would change for context/setting

Step 3:
Objective: Identify main steps or stages to accomplish the task

C. Ask SME the key steps or stages required to accomplish the task.
D. Ask SME to arrange the list of main steps in the order they are performed, or if there is no order, from easiest to difficult.

Step 4:
Objective: Capture a list of "step by step" actions and decisions for each task

A. Ask the SME to list the sequence of actions and decisions necessary to complete the task and/or solve the problem

Ask*: "Please describe how you accomplish this task step-by-step,* *so a novice trainee could perform it."*

For each step the SME gives you, ask yourself, "Is there a decision being made by the SME here?" If there is a possible decision, ask the SME.

If SME indicates that a decision must be made…

Ask*: "Please describe the most common alternatives (up to a maximum of three) that must be considered to make the decision and the criteria trainees should use to decide between the alternatives".*

Step 5:

Objective: Identify prior knowledge and information required to perform the task.

A. Ask SME about the prerequisite knowledge and other information required to perform the task.

1. Ask the SME about Cues and Conditions

Ask: "*For this task, what must happen before someone starts the task? What prior task, permission, order, or other initiating event must happen? Who decides?"*

2. Ask the SME about New Concepts and Processes

Ask*: "Are there any concepts or terms required of this task that may be new to the trainee?"*

Concepts – terms mentioned by the SME that may be new to the novice teacher

Ask for a definition and at least one example

Processes - How something works

If the trainee is operating equipment, or working on a team that may or may not be using equipment, ask the SME to *"Please describe how the team and/or the equipment work - in words that novices will understand. Processes usually consist of different phases and within each phase, there are different activities – think of it as a flow chart"*

Ask*: "Must trainees know this process to do the task?" "Will they have to use it to change the task in unexpected ways?"*

IF the answer is NO, do NOT collect information about the process.

3. Ask the SME about Equipment and Materials

Ask*: "What equipment and materials are required to succeed at this task in routine situations? Where are they located? How are they accessed?*

4. Performance Standard

Ask*: "How do we know the objective has been met? What are the criteria, such as time, efficiency, quality indicators (if any)?"*

5. Sensory experiences required for task

Ask*: "Must trainees see, hear, smell, feel, or taste something in order to learn any part of the task? For example, are there any parts of this task they could not perform unless they could smell something?"*

<u>Step 6:</u>

Objective: Identify routine or simple problems that can be solved by using the procedure.

A. Ask the SME to describe at least one routine or simple problem and two to three complex problems that the trainee should be able to solve if they can perform each of the tasks on the list you just made.

Ask*: "Of the task we just discussed, describe at least one routine problem and two to three complex problems that the trainee should be able to solve IF they learn to perform the task".*

Appendix B
Inter-rater Reliability Code Sheet for SME A

Inter-rater Reliability Code Sheet

Doug Wieland — Main Coder

Acquiclahs Mutie — Secondary Coder

Code	Tally Total	Agree	Disagree	% Agreement (IRR)
O (Objective)	2	2	0	100%
C (Conditions/cues)	13	12	1	92%
M (Main Procs.)	10	9	1	90%
A (Action Step)	182	172	10	95%
D (Decision Step)	14	13	1	93%
S (Standards)	10	10	0	100%
E (Equipment)	7	6	1	86%
P (Prerequisite)	7	6	1	86%
NCONC (New Concept)	1	1	0	100%
NPROS (New Process)				
NPRIN (New Principle)				
SENSE (Sensory Info)				
REASON (Reasons)	26	22	4	85%
PROB (Problems)				
SAFE (Safety Factors)				
REF (References)				
TOTAL	272	253	19	93%

Appendix C
Job Aid for Developing a Gold Standard Protocol
Richard Clark and Kenneth Yates (2010,
Proprietary)

The **goals** of this task are to 1) aggregate CTA protocols from multiple experts to create a "gold standard protocol" and 2) create a "best sequence" for each of the tasks and steps you have collected and the best description of each step for the design of training.

Trigger: After having completed interviews with all experts and capturing all goals, settings, triggers, and all action and decision steps from each expert – and after all experts have edited their own protocol.

Create a gold standard protocol

STEPS: Actions and Decisions

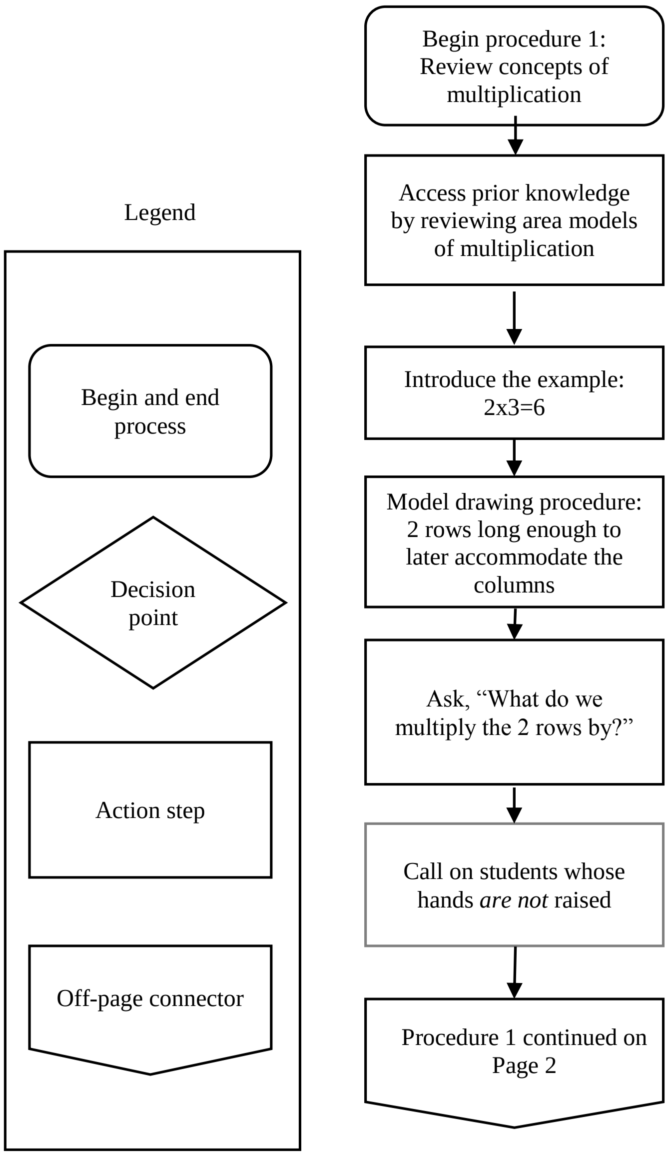
1. For each CTA protocol you are aggregating, ensure that the transcript line number is present for each action and decision step.
    a. If the number is not present, add it before going to Step 2.
2. Compare all the SME's corrected CTA protocols side-by-side and select one protocol (marked as P1) that meets all the following criteria:
    a. The protocol represents the most complete list of action and decision steps.
    b. The action and decision steps are written clearly and succinctly.
    c. The action and decision steps are the most accurate language and terminology.
3. Rank and mark the remaining CTA protocols as P2, P3, and so forth, according to the same criteria.
4. Starting with the first step, compare the action and decision steps of P2 with P1 and revise P1 as follows:
    a. IF the step in P2 has the same meaning as the step in P1, THEN add "(P2)" at the end of the step.
    b. IF the step in P2 is a more accurate or complete statement of the step in P1, THEN revise the step in P1 and add "(P1, P2)" at the end of the step.
    c. IF the step in P2 is missing from P1, THEN review the list of steps by adding the step to P1 and add "(P2N)"* at the end of the step.
5. Repeat Step 4 by comparing P3 with P1, and so forth for each protocol.
6. Repeat Steps 4 and 5 for the remaining components of the CTA report such as triggers, main procedures, equipment, standards, and concepts to create a "preliminary gold standard protocol" (PGSP).
7. Verify the PGSP by either:
    a. Asking a senior SME, who has not been interviewed for a CTA, to review

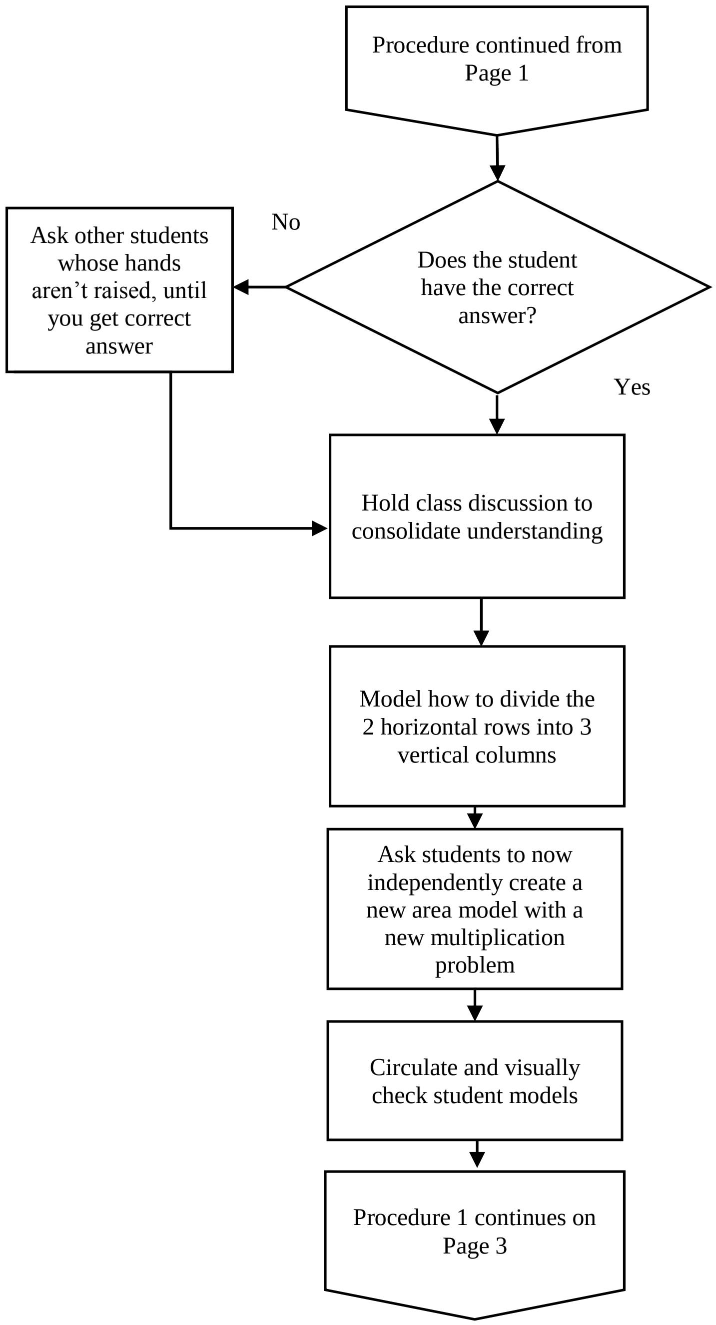
the PGSP and note any additions, deletions, revisions, and comments.

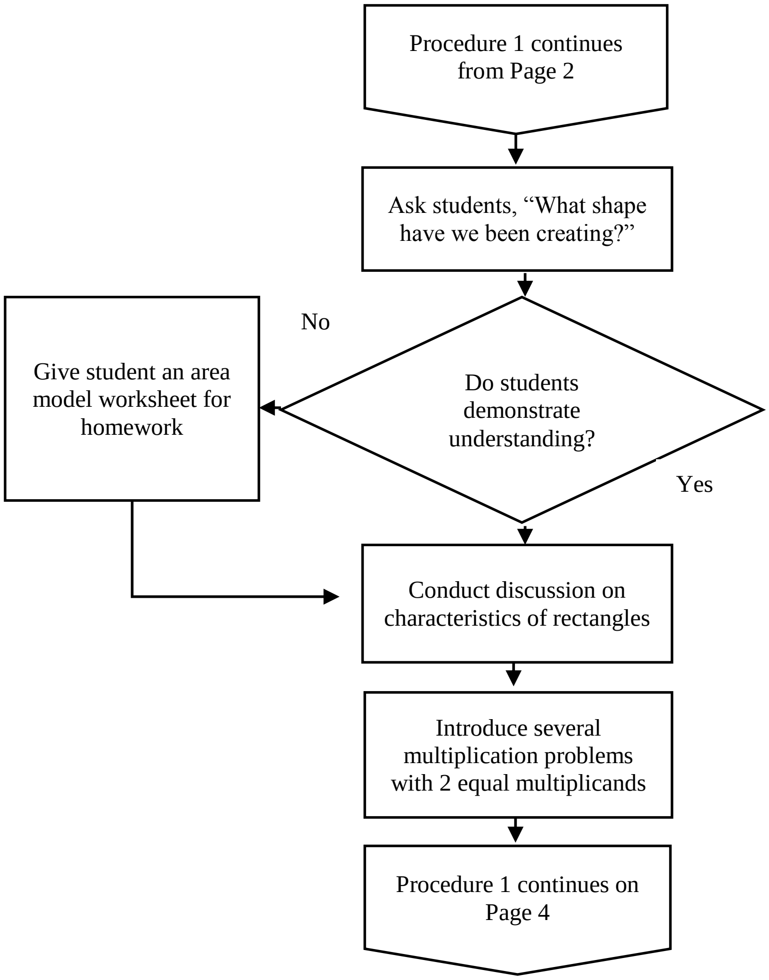
b. Asking each participating SME to review the PGSP, and either by hand or using MS Word Track Changes, note any additions, deletions, revisions, or comments.
    i. IF there is disagreement among the SMEs, THEN either
        1. Attempt to resolve the differences by communicating with the SMEs, OR
        2. Ask a senior SME, who has not been interviewed for a CTA, to review and resolve the differences.

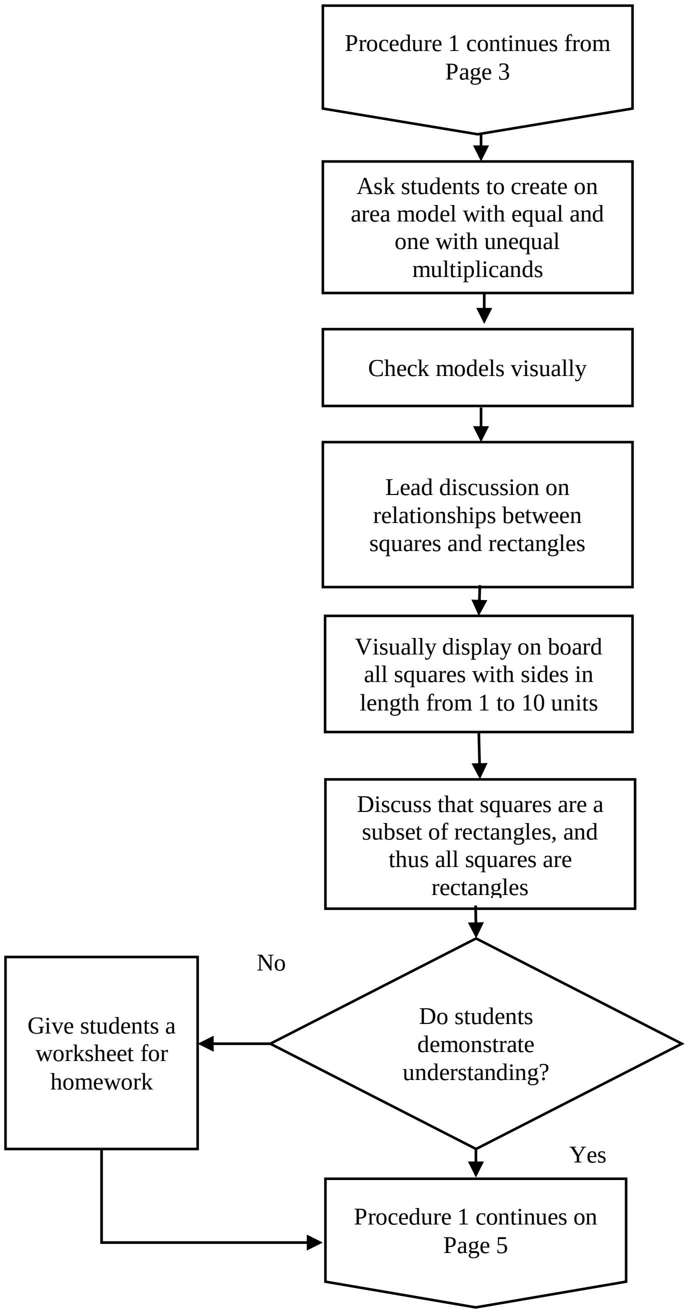
8. Incorporate the final revisions in the previous Step to create the “gold standard protocol” (GSP).

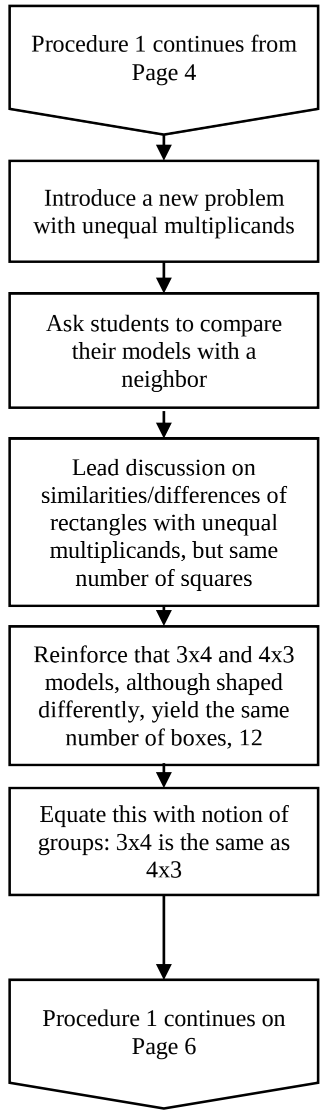
Appendix D
Flowchart for SME A Individual Protocol

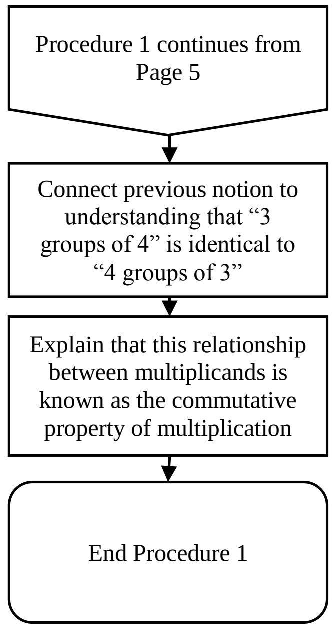


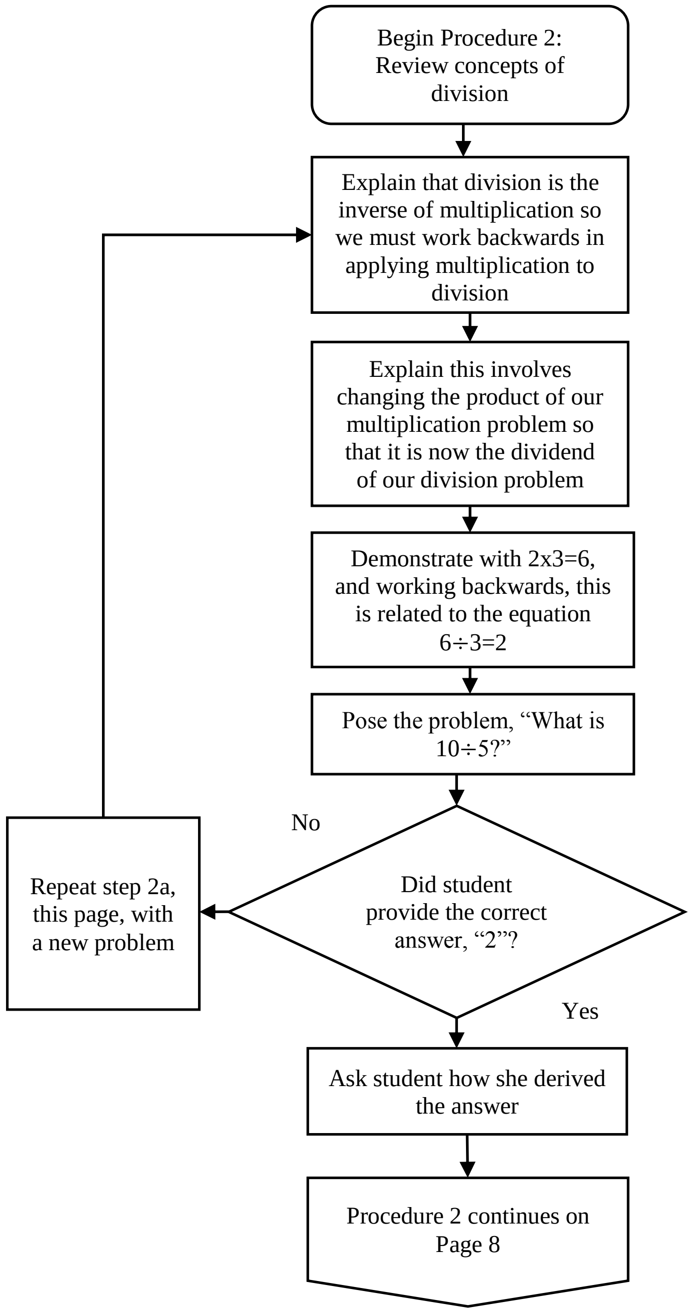

Procedure continued from Page 1
Does the student have the correct answer?
No
Ask other students whose hands aren't raised, until you get correct answer
Yes
Hold class discussion to consolidate understanding
Model how to divide the 2 horizontal rows into 3 vertical columns
Ask students to now independently create a new area model with a new multiplication problem
Circulate and visually check student models
Procedure 1 continues on Page 3

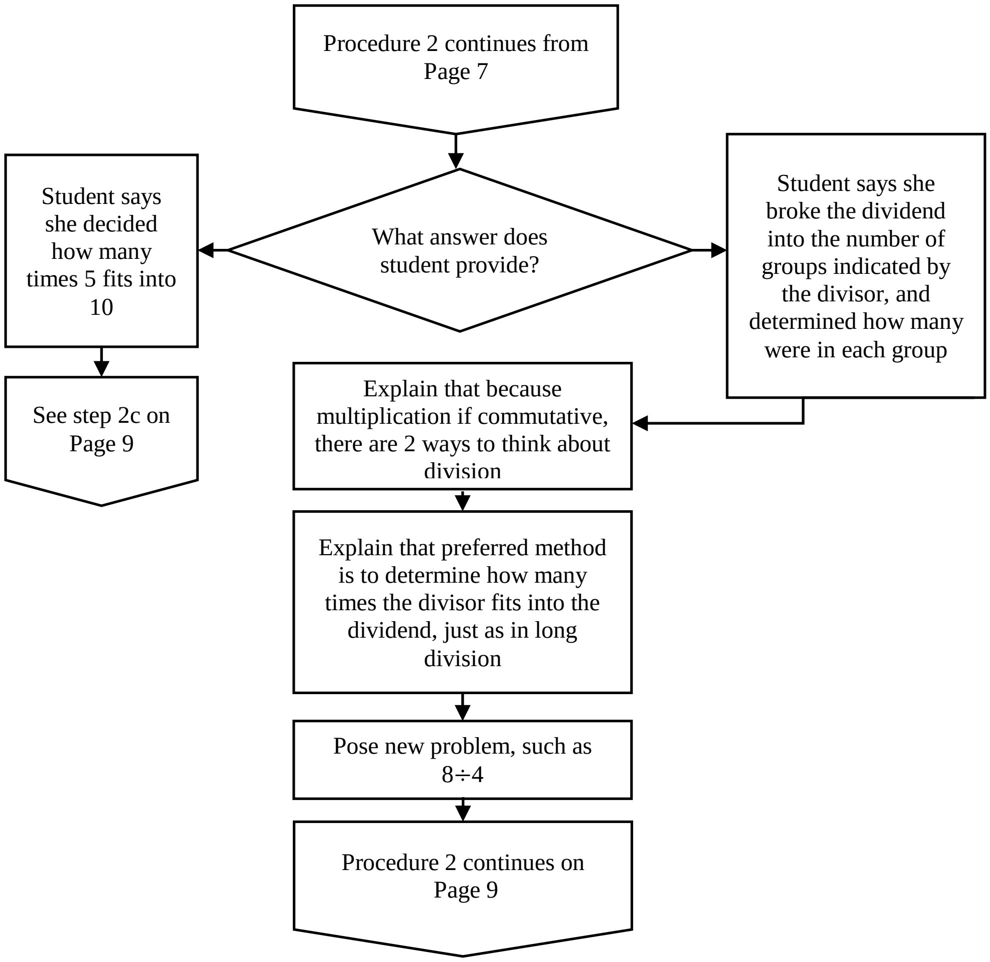

Procedure 1 continues from Page 2
Ask students, “What shape have we been creating?”
Do students demonstrate understanding?
No
Give student an area model worksheet for homework
Yes
Conduct discussion on characteristics of rectangles
Introduce several multiplication problems with 2 equal multiplicands
Procedure 1 continues on Page 4

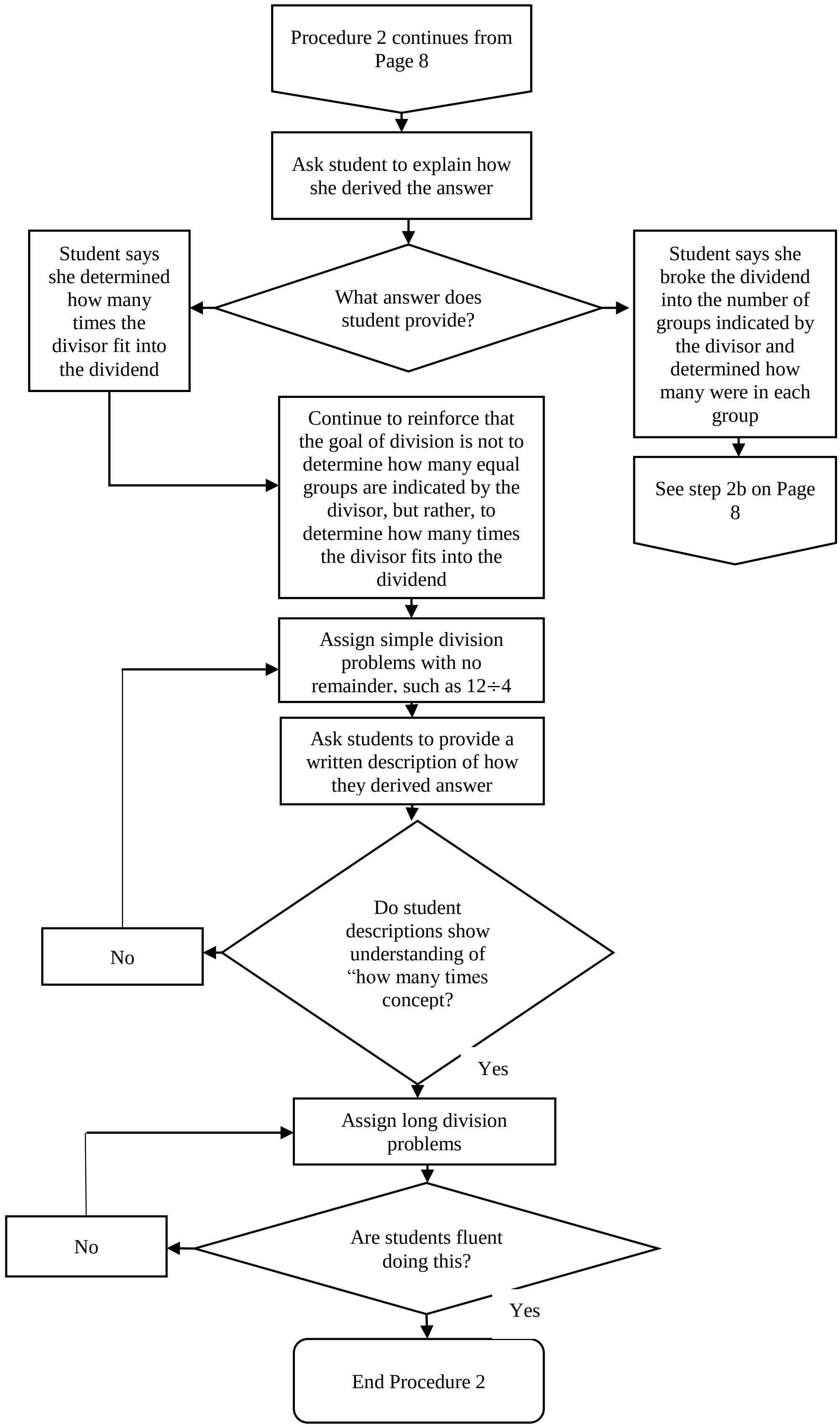

Procedure 1 continues from Page 3
Ask students to create on area model with equal and one with unequal multiplicands
Check models visually
Lead discussion on relationships between squares and rectangles
Visually display on board all squares with sides in length from 1 to 10 units
Discuss that squares are a subset of rectangles, and thus all squares are rectangles
Do students demonstrate understanding?
No
Give students a worksheet for homework
Yes
Procedure 1 continues on Page 5

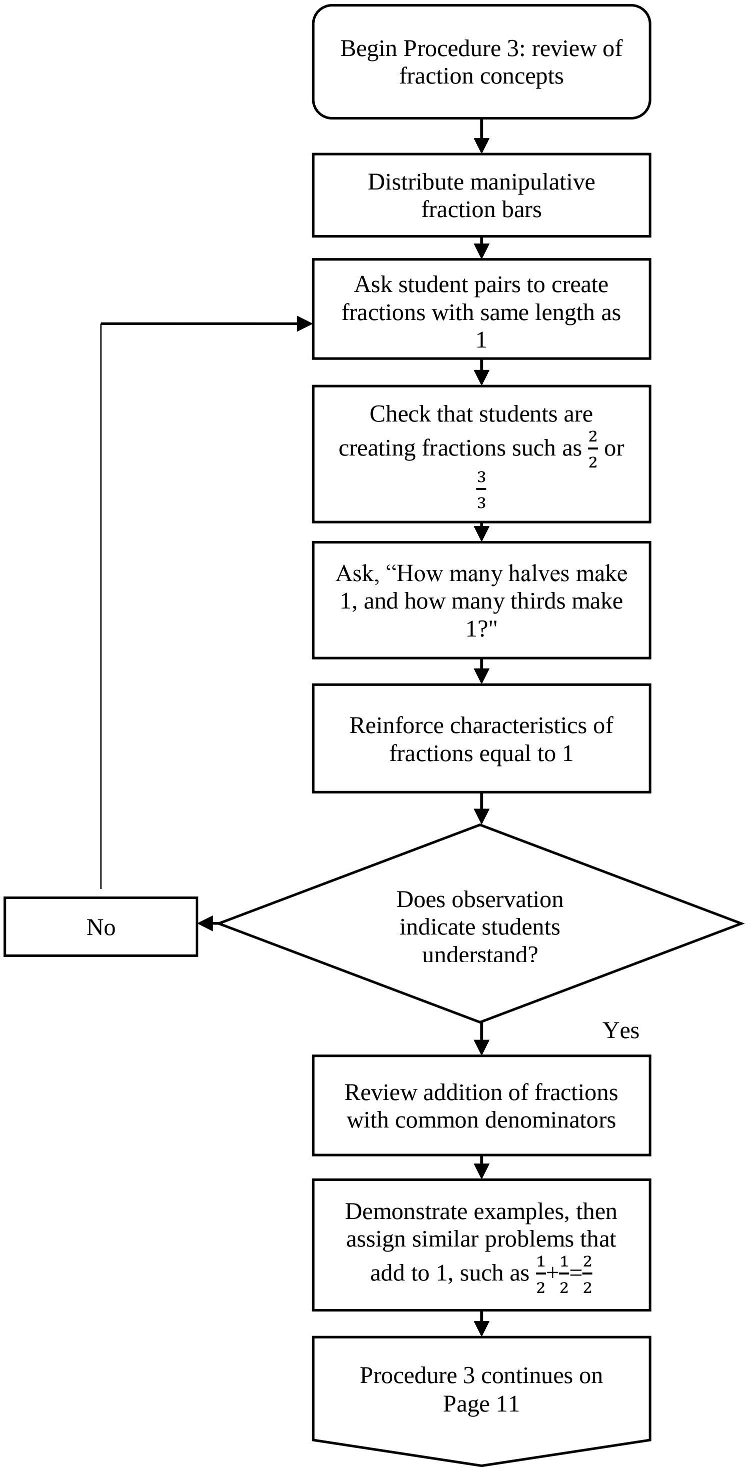

Procedure 1 continues from Page 4
Introduce a new problem with unequal multiplicands
Ask students to compare their models with a neighbor
Lead discussion on similarities/differences of rectangles with unequal multiplicands, but same number of squares
Reinforce that 3x4 and 4x3 models, although shaped differently, yield the same number of boxes, 12
Equate this with notion of groups: 3x4 is the same as 4x3
Procedure 1 continues on Page 6

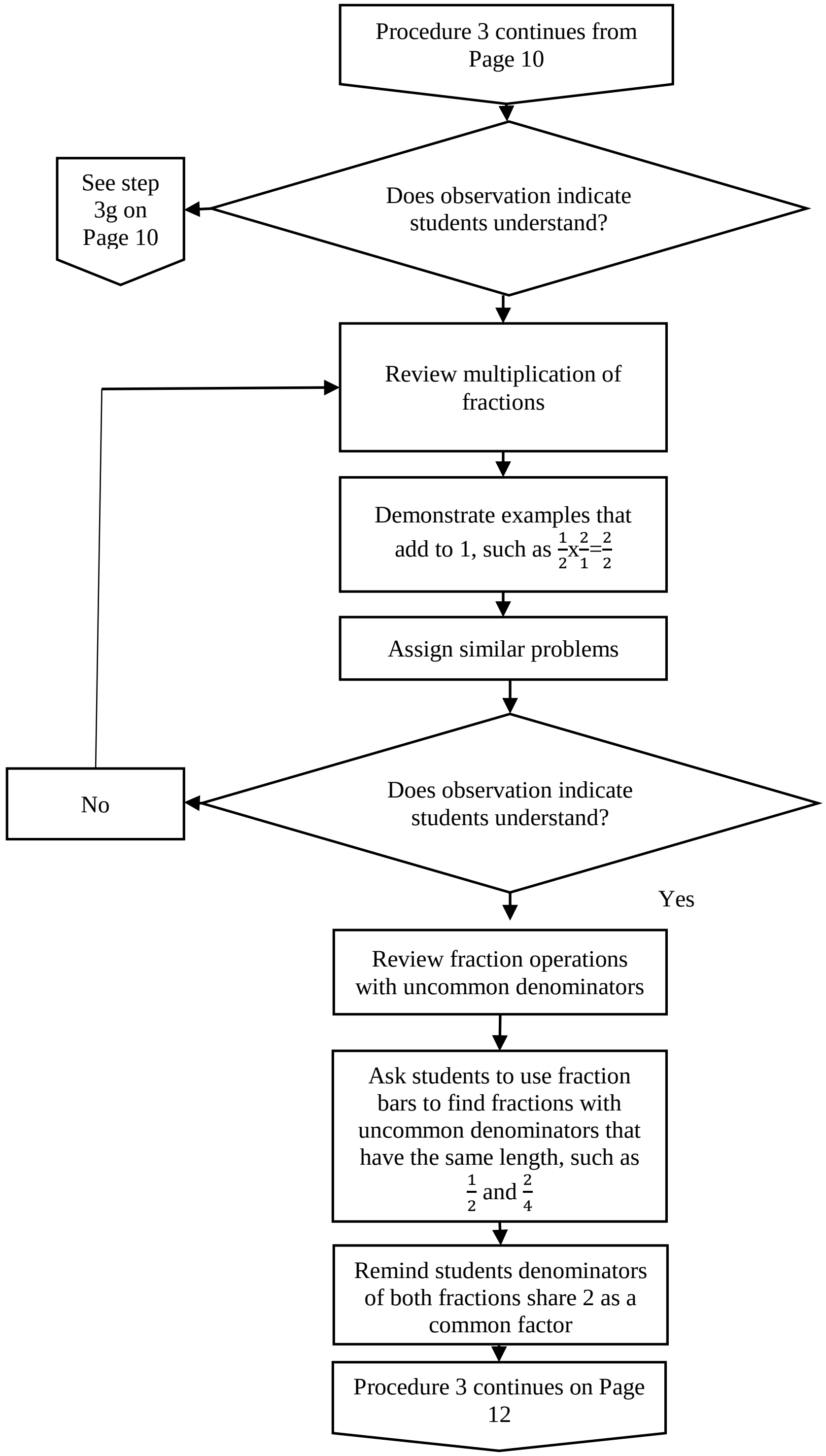

Procedure 1 continues from Page 5
Connect previous notion to understanding that "3 groups of 4" is identical to "4 groups of 3"
Explain that this relationship between multiplicands is known as the commutative property of multiplication
End Procedure 1

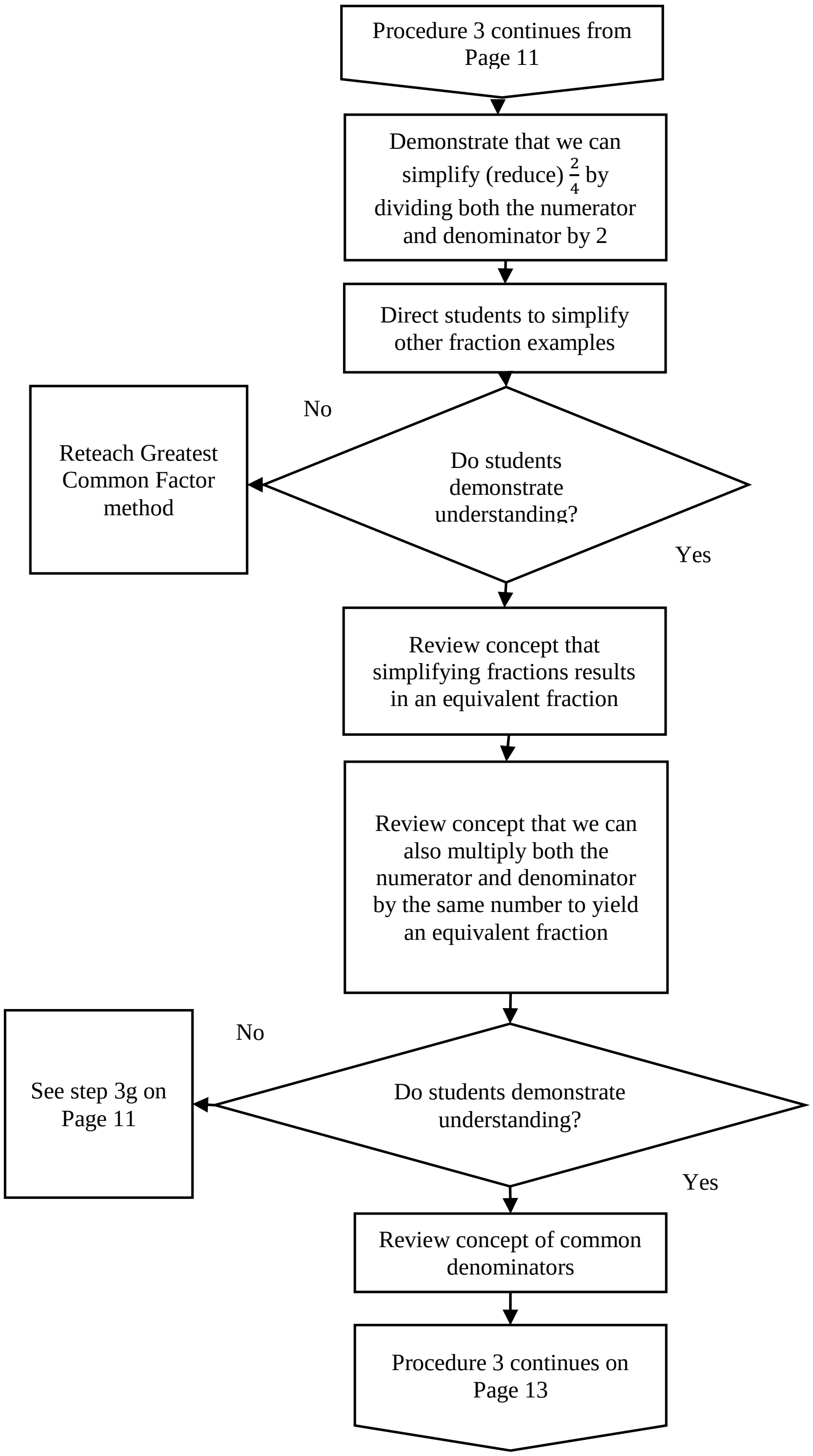

Begin Procedure 2: Review concepts of division
Explain that division is the inverse of multiplication so we must work backwards in applying multiplication to division
Explain this involves changing the product of our multiplication problem so that it is now the dividend of our division problem
Demonstrate with 2x3=6, and working backwards, this is related to the equation 6÷3=2
Pose the problem, "What is 10÷5?"
Did student provide the correct answer, "2"?
No
Repeat step 2a, this page, with a new problem
Yes
Ask student how she derived the answer
Procedure 2 continues on Page 8

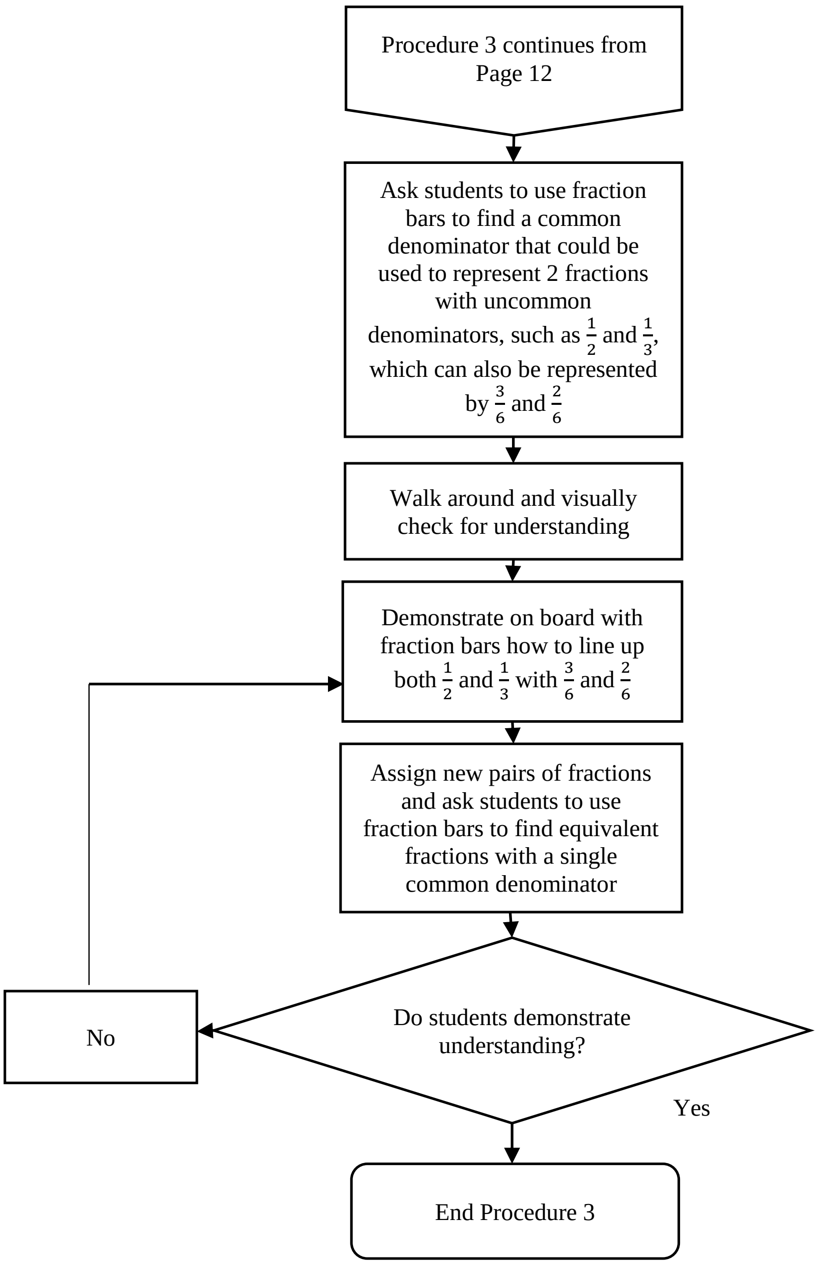

Procedure 2 continues from Page 7
What answer does student provide?
Student says she decided how many times 5 fits into 10
See step 2c on Page 9
Student says she broke the dividend into the number of groups indicated by the divisor, and determined how many were in each group
Explain that because multiplication if commutative, there are 2 ways to think about division
Explain that preferred method is to determine how many times the divisor fits into the dividend, just as in long division
Pose new problem, such as 8÷4
Procedure 2 continues on Page 9

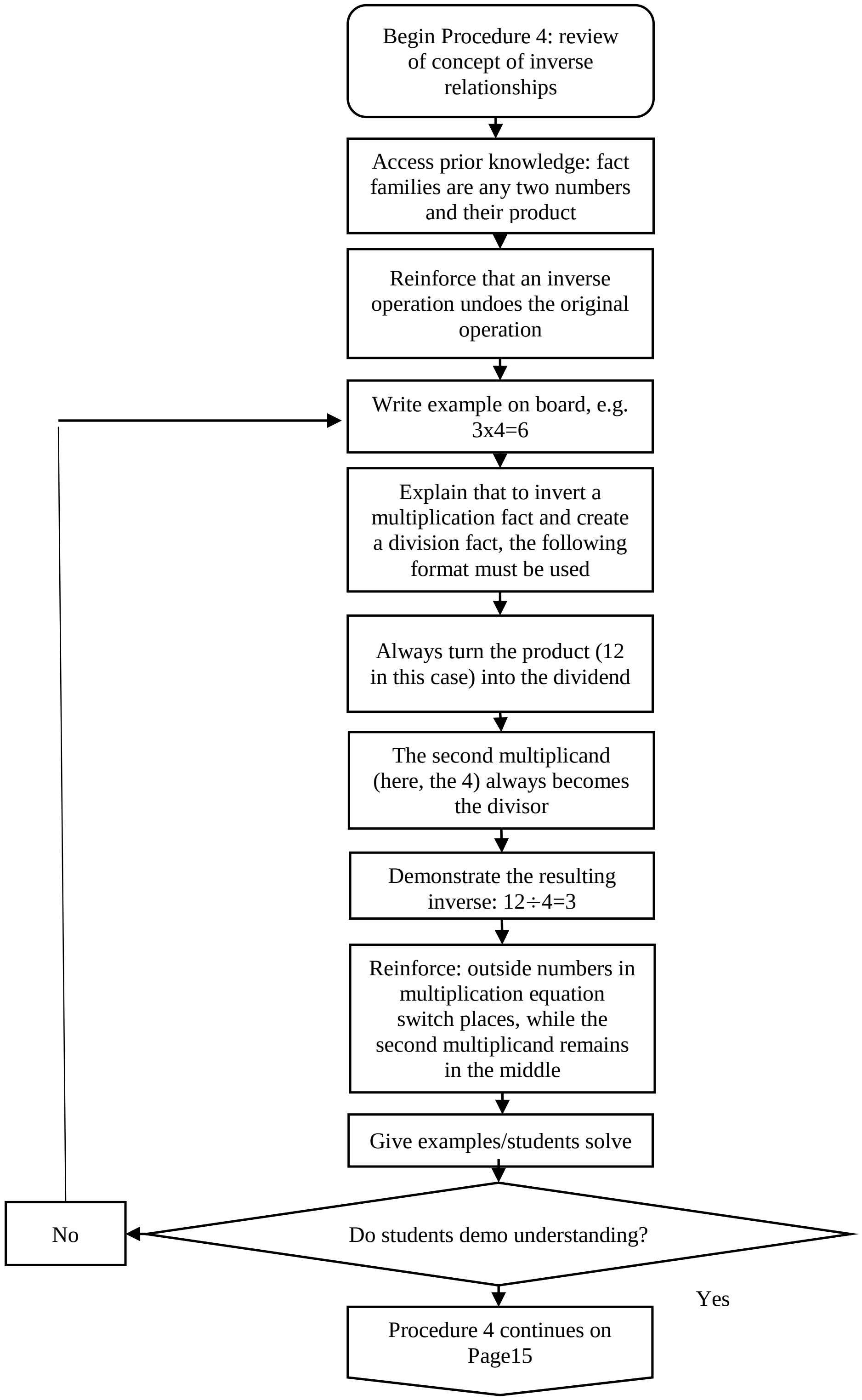

Procedure 2 continues from Page 8
Ask student to explain how she derived the answer
What answer does student provide?
Student says she determined how many times the divisor fit into the dividend
Student says she broke the dividend into the number of groups indicated by the divisor and determined how many were in each group
See step 2b on Page 8
Continue to reinforce that the goal of division is not to determine how many equal groups are indicated by the divisor, but rather, to determine how many times the divisor fits into the dividend
Assign simple division problems with no remainder, such as 12÷4
Ask students to provide a written description of how they derived answer
Do student descriptions show understanding of “how many times concept?
No
Yes
Assign long division problems
Are students fluent doing this?
No
Yes
End Procedure 2

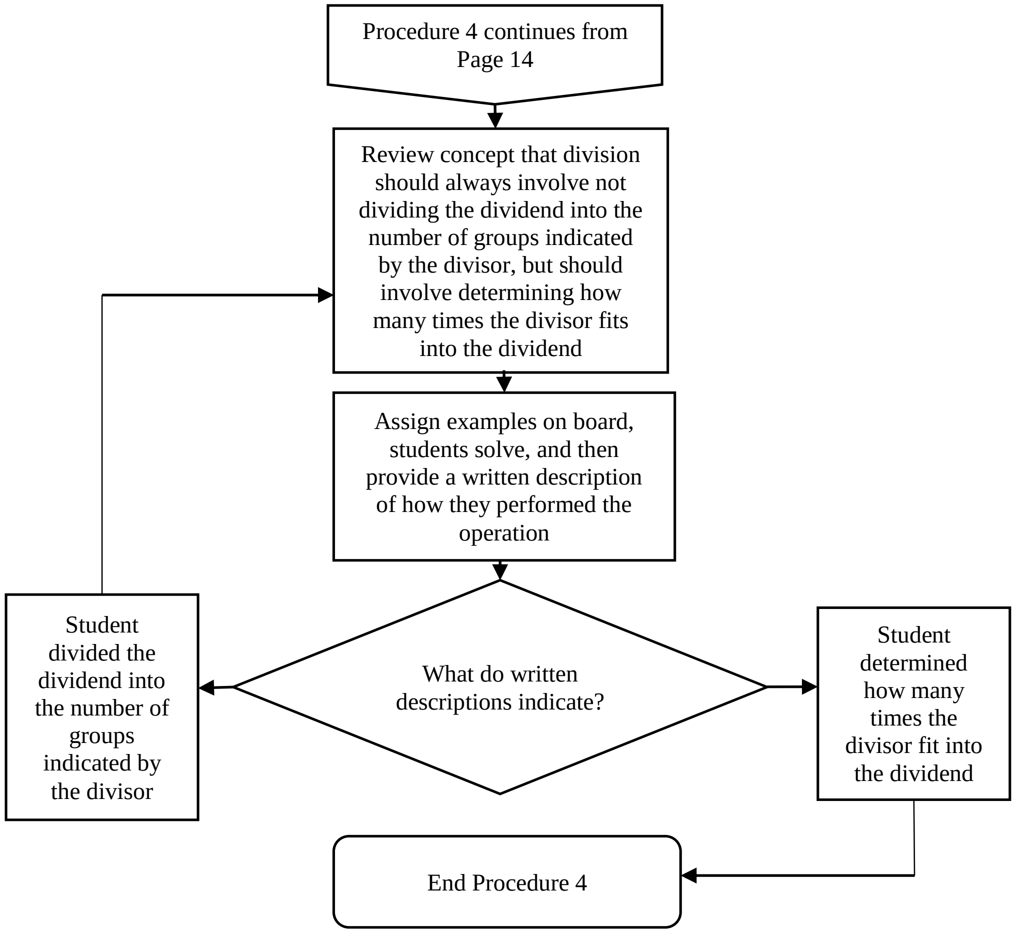

Begin Procedure 3: review of fraction concepts
Distribute manipulative fraction bars
Ask student pairs to create fractions with same length as 1
Check that students are creating fractions such as $\frac{2}{2}$ or $\frac{3}{3}$
Ask, "How many halves make 1, and how many thirds make 1?"
Reinforce characteristics of fractions equal to 1
Does observation indicate students understand?
No
Yes
Review addition of fractions with common denominators
Demonstrate examples, then assign similar problems that add to 1, such as $\frac{1}{2}+\frac{1}{2}=\frac{2}{2}$
Procedure 3 continues on Page 11

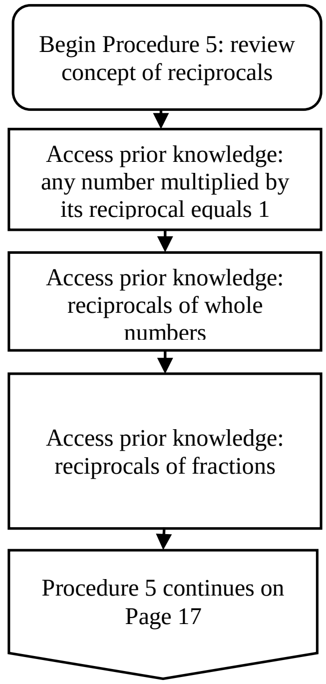

Procedure 3 continues from Page 10
See step 3g on Page 10
Does observation indicate students understand?
Review multiplication of fractions
Demonstrate examples that add to 1, such as $\frac{1}{2}$x$\frac{2}{1}$=$\frac{2}{2}$
Assign similar problems
No
Does observation indicate students understand?
Yes
Review fraction operations with uncommon denominators
Ask students to use fraction bars to find fractions with uncommon denominators that have the same length, such as $\frac{1}{2}$ and $\frac{2}{4}$
Remind students denominators of both fractions share 2 as a common factor
Procedure 3 continues on Page 12

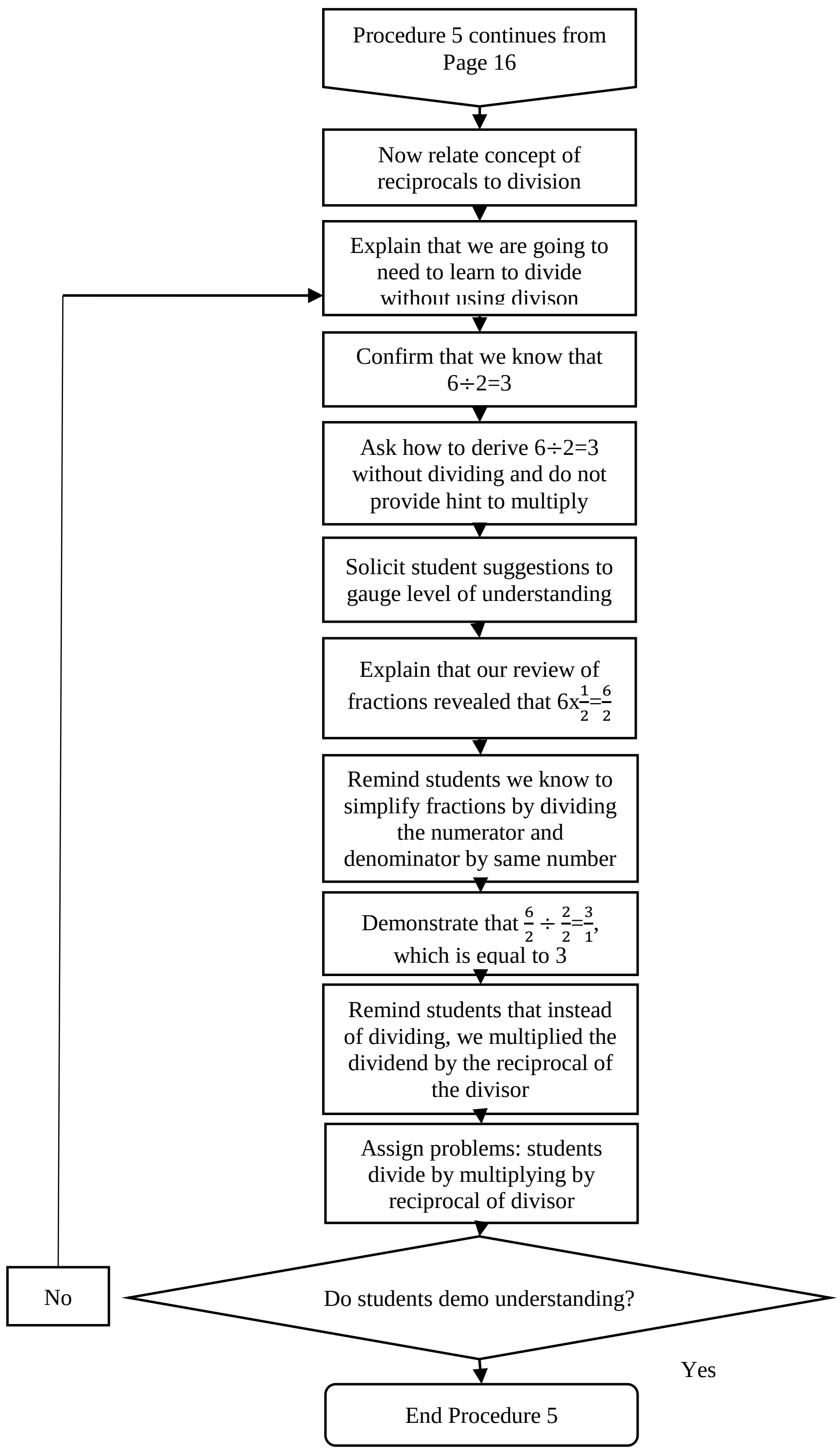

Procedure 3 continues from Page 11
Demonstrate that we can simplify (reduce) $\frac{2}{4}$ by dividing both the numerator and denominator by 2
Direct students to simplify other fraction examples
Do students demonstrate understanding?
No
Reteach Greatest Common Factor method
Yes
Review concept that simplifying fractions results in an equivalent fraction
Review concept that we can also multiply both the numerator and denominator by the same number to yield an equivalent fraction
Do students demonstrate understanding?
No
See step 3g on Page 11
Yes
Review concept of common denominators
Procedure 3 continues on Page 13

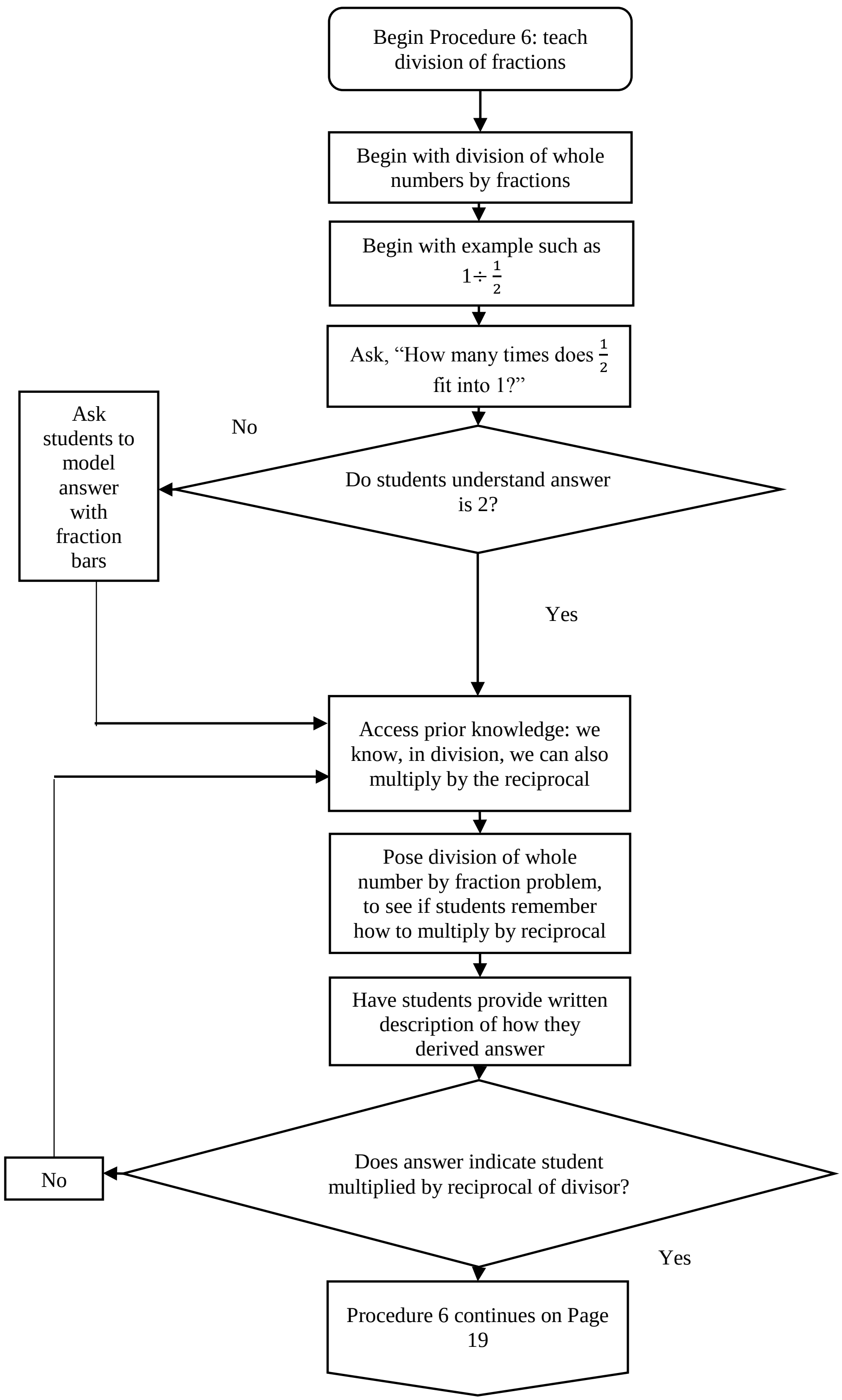

Procedure 3 continues from Page 12
Ask students to use fraction bars to find a common denominator that could be used to represent 2 fractions with uncommon denominators, such as $\frac{1}{2}$ and $\frac{1}{3}$, which can also be represented by $\frac{3}{6}$ and $\frac{2}{6}$
Walk around and visually check for understanding
Demonstrate on board with fraction bars how to line up both $\frac{1}{2}$ and $\frac{1}{3}$ with $\frac{3}{6}$ and $\frac{2}{6}$
Assign new pairs of fractions and ask students to use fraction bars to find equivalent fractions with a single common denominator
Do students demonstrate understanding?
No
Yes
End Procedure 3

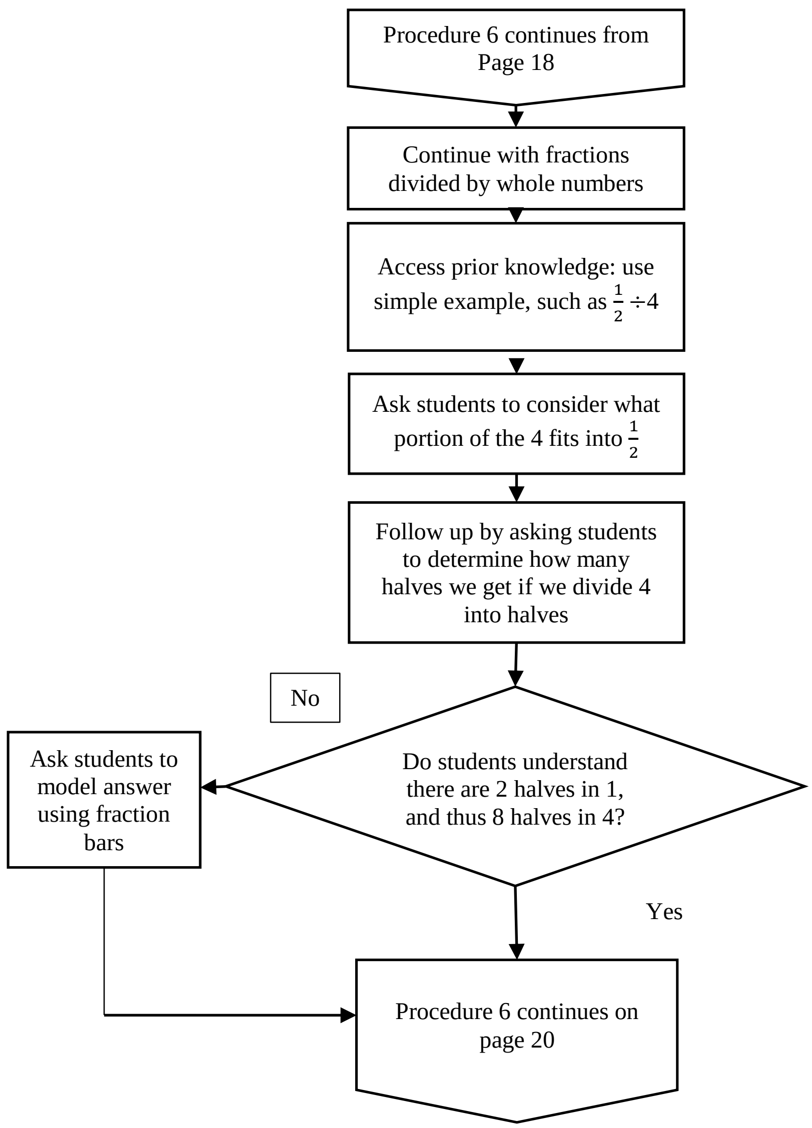

Begin Procedure 4: review of concept of inverse relationships
Access prior knowledge: fact families are any two numbers and their product
Reinforce that an inverse operation undoes the original operation
Write example on board, e.g. 3x4=6
Explain that to invert a multiplication fact and create a division fact, the following format must be used
Always turn the product (12 in this case) into the dividend
The second multiplicand (here, the 4) always becomes the divisor
Demonstrate the resulting inverse: 12÷4=3
Reinforce: outside numbers in multiplication equation switch places, while the second multiplicand remains in the middle
Give examples/students solve
Do students demo understanding?
No
Yes
Procedure 4 continues on Page15

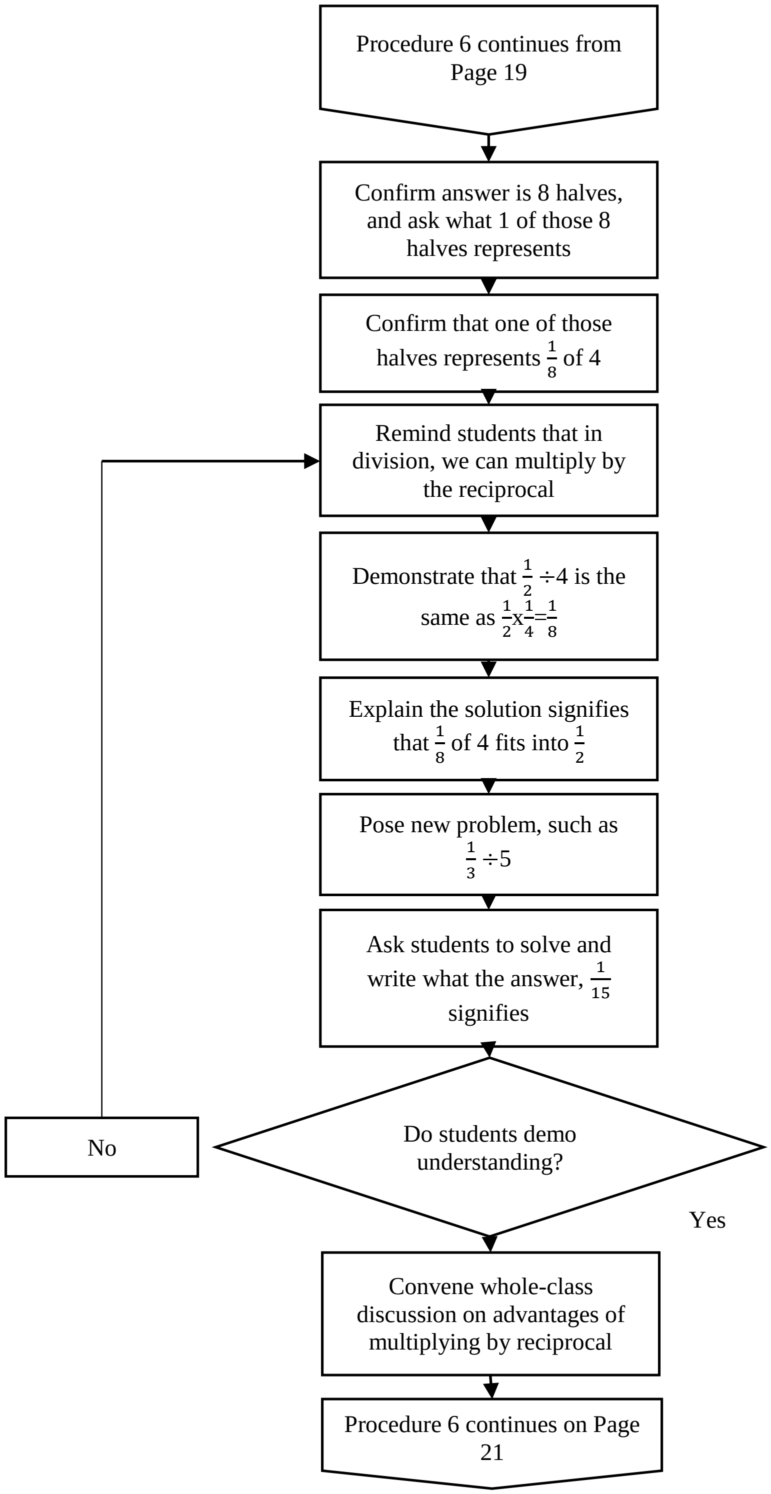

Procedure 4 continues from Page 14
Review concept that division should always involve not dividing the dividend into the number of groups indicated by the divisor, but should involve determining how many times the divisor fits into the dividend
Assign examples on board, students solve, and then provide a written description of how they performed the operation
What do written descriptions indicate?
Student divided the dividend into the number of groups indicated by the divisor
Student determined how many times the divisor fit into the dividend
End Procedure 4

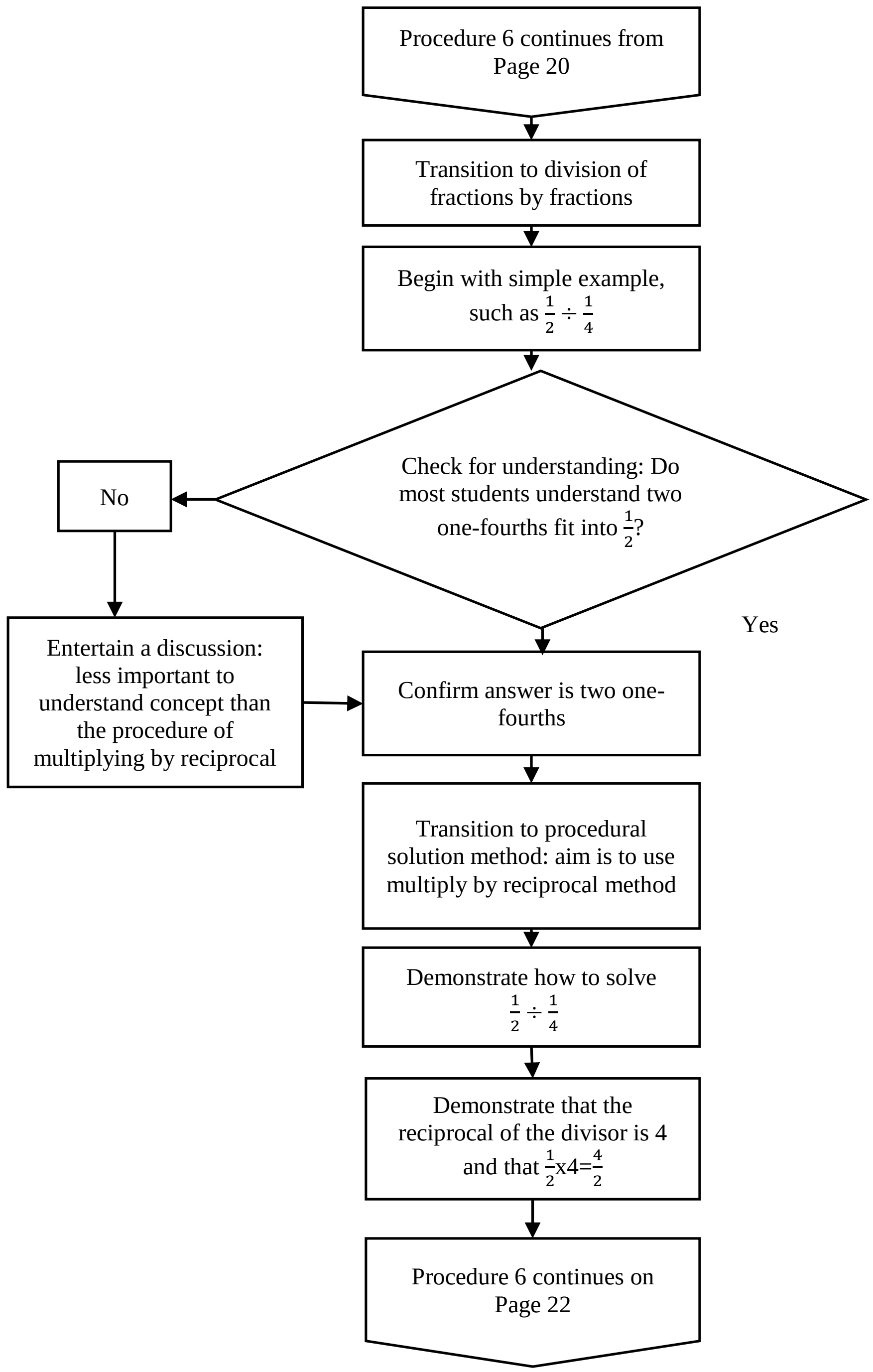

Begin Procedure 5: review concept of reciprocals
Access prior knowledge: any number multiplied by its reciprocal equals 1
Access prior knowledge: reciprocals of whole numbers
Access prior knowledge: reciprocals of fractions
Procedure 5 continues on Page 17

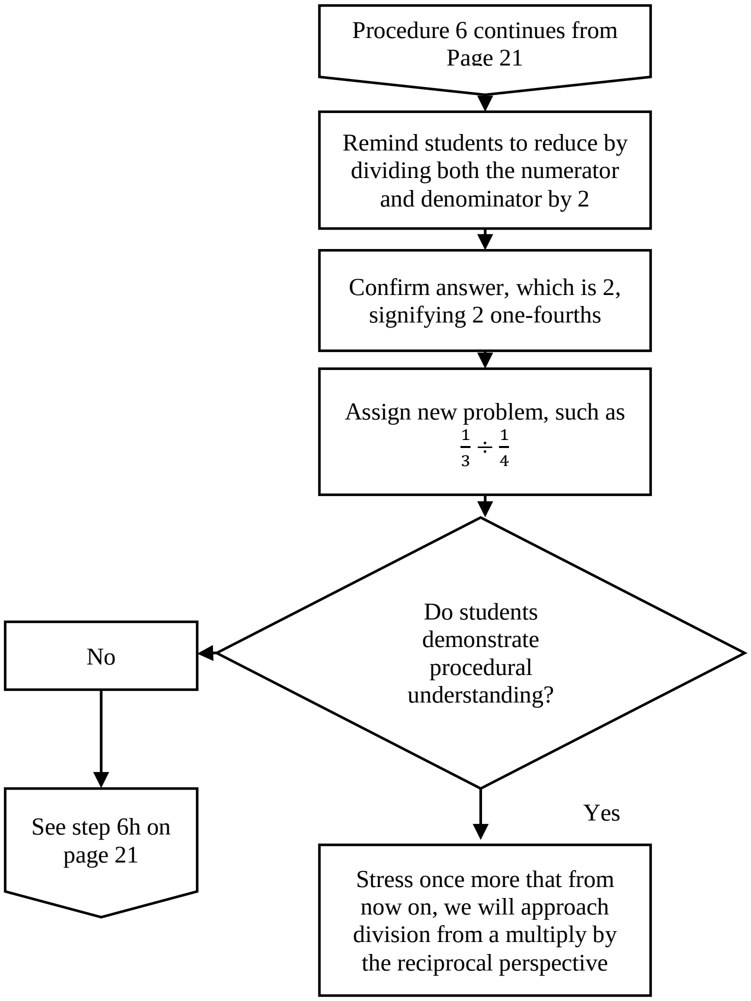

Procedure 5 continues from Page 16
Now relate concept of reciprocals to division
Explain that we are going to need to learn to divide without using divison
Confirm that we know that 6÷2=3
Ask how to derive 6÷2=3 without dividing and do not provide hint to multiply
Solicit student suggestions to gauge level of understanding
Explain that our review of fractions revealed that 6x$\frac{1}{2}$=$\frac{6}{2}$
Remind students we know to simplify fractions by dividing the numerator and denominator by same number
Demonstrate that $\frac{6}{2} \div \frac{2}{2} = \frac{3}{1}$, which is equal to 3
Remind students that instead of dividing, we multiplied the dividend by the reciprocal of the divisor
Assign problems: students divide by multiplying by reciprocal of divisor
No
Do students demo understanding?
Yes
End Procedure 5


Begin Procedure 6: teach division of fractions
Begin with division of whole numbers by fractions
Begin with example such as $1 \div \frac{1}{2}$
Ask, "How many times does $\frac{1}{2}$ fit into 1?"
Do students understand answer is 2?
No
Ask students to model answer with fraction bars
Yes
Access prior knowledge: we know, in division, we can also multiply by the reciprocal
Pose division of whole number by fraction problem, to see if students remember how to multiply by reciprocal
Have students provide written description of how they derived answer
Does answer indicate student multiplied by reciprocal of divisor?
No
Yes
Procedure 6 continues on Page 19


Procedure 6 continues from Page 18
Continue with fractions divided by whole numbers
Access prior knowledge: use simple example, such as $\frac{1}{2} \div 4$
Ask students to consider what portion of the 4 fits into $\frac{1}{2}$
Follow up by asking students to determine how many halves we get if we divide 4 into halves
No
Do students understand there are 2 halves in 1, and thus 8 halves in 4?
Ask students to model answer using fraction bars
Yes
Procedure 6 continues on page 20


Procedure 6 continues from Page 19
Confirm answer is 8 halves, and ask what 1 of those 8 halves represents
Confirm that one of those halves represents $\frac{1}{8}$ of 4
Remind students that in division, we can multiply by the reciprocal
Demonstrate that $\frac{1}{2} \div 4$ is the same as $\frac{1}{2} x \frac{1}{4} = \frac{1}{8}$
Explain the solution signifies that $\frac{1}{8}$ of 4 fits into $\frac{1}{2}$
Pose new problem, such as $\frac{1}{3} \div 5$
Ask students to solve and write what the answer, $\frac{1}{15}$ signifies
Do students demo understanding?
No
Yes
Convene whole-class discussion on advantages of multiplying by reciprocal
Procedure 6 continues on Page 21


Procedure 6 continues from Page 20
Transition to division of fractions by fractions
Begin with simple example, such as $\frac{1}{2} \div \frac{1}{4}$
Check for understanding: Do most students understand two one-fourths fit into $\frac{1}{2}$?
No
Yes
Entertain a discussion: less important to understand concept than the procedure of multiplying by reciprocal
Confirm answer is two one-fourths
Transition to procedural solution method: aim is to use multiply by reciprocal method
Demonstrate how to solve $\frac{1}{2} \div \frac{1}{4}$
Demonstrate that the reciprocal of the divisor is 4 and that $\frac{1}{2}$x4=$\frac{4}{2}$
Procedure 6 continues on Page 22


Procedure 6 continues from Page 21
Remind students to reduce by dividing both the numerator and denominator by 2
Confirm answer, which is 2, signifying 2 one-fourths
Assign new problem, such as $\frac{1}{3} \div \frac{1}{4}$
Do students demonstrate procedural understanding?
No
See step 6h on page 21
Yes
Stress once more that from now on, we will approach division from a multiply by the reciprocal perspective

Appendix E
Gold Standard Protocol

Final Gold Standard Protocol

Task: To teach the division of fractions by fractions in 6th, 7th, and 8th grades.

Definitions

Muir Time: the final period of the day during which students at "JM" Middle School either begin homework in their homerooms, or are directed to come to the class of a teacher who is providing remediation in a specific subject.

Example of Muir Time: a student spends Muir Time in the classroom of his/her social studies teacher, reviewing material that was incorrect on a recent test.

Non-example of Muir Time: a student neither works on homework in his/her homeroom nor reports to another teacher's class for remediation, but rather, attends a Student Council meeting.

Procedures

1. Review concept of multiplication
2. Review concepts of division
3. Teach operations with integers
4. Teach number domains
5. Review representations of fractions
6. Review addition and subtraction of fractions and mixed numbers
7. Review multiplication of fractions and mixed numbers
8. Teach division of whole numbers by fractions
9. Teach division of fractions by whole numbers
10. Teach division of fractions by fractions
11. Teach division of mixed numbers by mixed numbers
12. Teach division of fraction word problems

1. Review concept of multiplication
    - 1.1. Explain that to model multiplication, we will construct rectangles (A)
        - 1.1.1. Introduce notion of rectangles as area models for multiplication (A)
        - 1.1.2. Review concept that area models are arrays with rows and columns (A)
            - 1.1.2.1. Explain that the rows and columns correspond to the two multiplicands
        - 1.1.3. Introduce the example of 2 x 3 (A)
            - 1.1.3.1. Model how to draw the rows: 2 (first multiplicand) rectangles on top of each other (234-5) and that the 2 rectangles need to be long enough to accommodate the columns that correspond to the other multiplicand, 3 (A)
            - 1.1.3.2. Ask, "What do we multiply those 2 rows by?" (A)
            - 1.1.3.3. Call on students whose hands are not raised (A)
                - 1.1.3.3.1. IF student has correct answer, THEN go to step 1.1.3.4 (A)
                - 1.1.3.3.2. IF student has incorrect answer, THEN call on another student whose hand isn't raised, until a student gives the correct answer (A)
            - 1.1.3.4. Check for understanding with discussion (A)
        - 1.1.4. Model how to divide those two boxes by 3, the other multiplicand, so that there are 3 columns (A)

1.1.4.1. Demonstrate how to count the number of boxes created by the 2 rows and 3 columns, which is 6 (A)

1.1.4.2. Reinforce the point of the exercise that 2 rows by 3 columns is a graphical representation of 2x3=6 (A)

1.2. Repeat above sequence with a new multiplication problem with unequal multiplicands; students independently create area models corresponding to that problem (A)

1.3. Wander around and check visually (A)

1.4. Ask students, following several additional problems area models with unequal multiplicands, “What shape have we been creating?” (A)

1.4.1. IF students demonstrate understanding, THEN step 1.5 (A)

1.4.2. IF students are unclear, THEN give them a worksheet for homework (A)

1.5. Reinforce that all examples have been rectangles (A)

1.6. Ask, “What does it mean to be a rectangle?” (A)

1.7. Ask, “How do you know a rectangle when you see one?” (A)

1.8. Challenge students to create a definition of a rectangle (2 pairs of parallel sides, the opposites being equal in length, and 4 right angles) (A)

1.9. Reinforce that what you have built is a rectangle (A)

1.10. Introduce several multiplication problems with 2 equal multiplicands (A)

1.11. Ask students to create one area model with unequal multiplicands and then one area model with equal multiplicands (A)

1.12. Check visually (A)

1.12.1. Build a connection by asking, “What plane figure is created when all sides are equal?” (A)

1.12.2. Ask, after student response of “square” is elicited, “Is a square not also a rectangle?” (A)

1.12.3. Reinforce concept that a square is a rectangle with 4 equal sides (A)

1.12.4. Display visual on board during discussion. (A)

1.12.5. Display all squares whose sides range in length from 1 to 10 units. (A)

1.12.6. Discuss that squares are a subset of rectangles (A)

1.12.7. Discuss here that all squares are rectangles (A)

1.12.7.1. IF students are struggling to conceptualize these understandings, problems. THEN give students a worksheet (A)

1.12.8. IF students understand concepts, THEN step 1.13 (A)

1.13. Introduce a new problem with unequal multiplicands (e.g. 3x4) (A)

1.13.1. Ask students to compare their models with a neighbor (A)

1.13.2. Ask students to decide if their area model looks like their neighbor’s (A)

1.13.2.1. Showcase students whose models differ (e.g. 3x4 area models versus 4x3 area models) (A)

1.13.2.2. Circulate and ask students with different models to draw their models on board and then have a discussion (A)

1.13.2.3. Ask, “If our multiplication problem is 3x4, how many boxes do we expect in our area model,” the answer, of course to which is 12 (A)

1.13.2.4. Challenge students to pair with someone else with a differently shaped area model and establish that both have the same number of boxes (A)

1.13.2.5. Explain that 3x4 and 4x3 area models, although they are shaped differently, yield the same number of boxes (A)

1.13.2.5.1. Equate this with the notion of groups: a 3x4 area model is the same multiplication problem as a 4x3 area model (A)

1.13.2.5.2. Establish a connection to the understanding that “3 groups of 4 is the same idea as 4 groups of 3” and explain that this relationship between multiplicands is known as the commutative property of multiplication (A)

2. Review concepts of division

2.1. Explain that division is the inverse of multiplication so we have to work backwards in applying multiplication to division (A)

2.1.1. Explain this involves changing the product of our multiplication problem so that it is now the dividend of our division problem (A)

2.1.2. Demonstrate that 2x3=6, and working backward, this is related to the equation 6÷3=2 (A)

2.1.3. Pose division problem with a quotient greater than one, such as 12÷2 (A, C)

2.1.3.1. IF students derive correct answer, 6, THEN step 2.1.4 (A)

2.1.3.2. IF students seem unclear, THEN step 2.1 (A)

2.1.4. Ask students, “What is the meaning of this problem?” (A, C)

2.1.5. Direct students to discuss this with a neighbor (C)

2.1.6. Check for understanding by calling on students randomly (A, C)

2.1.6.1.1. IF student explains that she decided how many times 2 fits into 12, THEN go to step 2.1.8 (A)

2.1.6.1.2. IF student says she broke the dividend into the number of groups indicated by the divisor and determined how many were in each group THEN step 2.1.6.2 (A)

2.1.6.2. Realize that this is a pattern of student thought that needs to be rewired (A)

2.1.6.3. Explain that because multiplication is commutative there are two ways to think about division (A)

2.1.6.3.1. Explain that one way is to break the dividend into the number of groups indicated by the divisor and then determine the size of each group (A)

2.1.6.3.2. Explain that this is a method that we are going to avoid (A)

2.1.6.3.3. Explain that the other method is to determine how many times the divisor fits into the dividend, just as we do in long division and then show an example on board (A, C)

2.1.6.3.4. Establish the expectation that this is the way students need to begin to think about division (A, C)

2.1.7. Pose new problem, such as 12÷2 (A, C)

2.1.7.1. Ask students to explain what the meaning of this problem is (A,C)

2.1.7.1.1. IF student says she broke the dividend into the number of groups indicated by the divisor and determined how many were in each group THEN step 2.1.6.2 (A)

2.1.7.1.2. IF student says she determined how many times the divisor fit into the dividend, THEN step 2.1.8 (A)

2.1.8. Ask students to formulate a real-life example of the problem (C)

2.1.9. Direct students to discuss this with a partner (C)

2.1.10. Check for understanding by calling on students randomly (C)

2.1.11. Highlight correct examples, such as, "If Juan has $12, how many $2 bills would be needed to match that amount?" (C)

2.2. Continue to reinforce that the goal of division is not to determine how many equal groups are indicated by the divisor, but rather, to determine how many times the divisor fits into the dividend, as in long division (A, C)

2.3. Conclude lesson with a similar problem, to which students must provide a written answer (A, C)

2.4. Review all answers (A, C)

2.4.1. IF more than 3 answers indicate misunderstanding, THEN begin next class session with a review of the concept of the meaning of division (A, C)

2.4.2. IF only 2 or 3 answers indicate misunderstanding, THEN sit with these students during next class session and provide remediation (A, C)

2.5. Pose division problem with a quotient less than one, such as 1÷2 (C)

2.5.1. Ask students, "What is the meaning of this problem?" (A,C)

2.5.2. Direct students to discuss this with a neighbor (C)

2.5.3. Check for understanding by calling on students randomly (C)

2.5.4. Entertain a discussion centering on the concept that the problem involves determining how many groups of 2 are in 1 (A, C)

2.5.5. Ask students to formulate a real-life example of the problem (C)

2.5.6. Direct students to discuss this with a partner (C)

2.5.7. Check for understanding by calling on students randomly (C)

2.5.8. Highlight correct examples, such as, "If Juan has 1 bagel, how many groups of 2 bagels are in that 1 bagel?" (C)

2.6. Pose another similar problem, such as 1÷3 (C)

2.6.1. Ask students, "What is the meaning of this problem?" (A, C)

2.6.2. Direct students to discuss this with a neighbor (C)

2.6.3. Check for understanding by calling on students randomly (C)

2.6.4. Entertain a discussion centering on the concept that the problem involves determining how many groups of 3 are in 1 (A, C)

2.6.5. Ask students to formulate a real-life example of the problem (C)

2.6.6. Direct students to discuss this with a partner (C)

2.6.7. Check for understanding by calling on students randomly (C)

2.6.8. Highlight correct examples, such as, "If Rachel has 1 bagel, how many groups of 3 bagels are in that 1 bagel (C)

2.7. Conclude lesson with a similar problem, to which students must provide a written answer (C)

2.8. Review all answers (C)

2.8.1. IF a majority of answers indicate misunderstanding, THEN begin next class session with a review of the concept of the meaning of division (C)

2.8.2. IF only 2 or 3 answers indicate misunderstanding, THEN sit with these students during next class session and provide remediation (C)

3. Teach operations with integers

3.1. Teach integer operations using a number line (B)

3.1.1. Demonstrate addition of a negative: 4 + -2 involves starting at 4 and going back 2 on number line to 2 (B)

3.1.2. Demonstrate addition of a positive: -2 + 7 involves starting at -2 and going forward 7 to 5 (B)
3.1.3. Demonstrate subtraction of a positive: -4 – 2 involves starting at -4 and going back 2 to -6 (B)
3.1.4. Demonstrate subtraction of a negative: -4 – (-2) is the opposite of -4 -2, so we go forward 2 to -2 (B)
3.1.5. Assign similar problems for homework (B)
3.1.6. Visually inspect one or two key problems on following day using clipboard (B)
3.1.6.1. IF student demonstrates understanding, THEN go to step 3.2 (B)
3.1.6.2. IF student's answers do not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)
3.2. Teach integer operations using manipulatives (B)
3.2.1. Distribute integer chips (B)
3.2.2. Explain that the black side indicates positive and the red side indicates negative (B)
3.2.3. Demonstrate addition (B)
3.2.3.1. Show that -4 + 2 can be represented as four red chips and two black chips (B)
3.2.3.2. Match one red chip and one black chip to indicate zero, based on the inverse property of addition (B)
3.2.3.3. Repeat for the other black chip to yield a total of two zeroes, which yields a sum of -2 (B)
3.2.4. Demonstrate subtraction (B)
3.2.4.1. Show that -4 – 2 can be represented as four red chips (B)
3.2.4.2. Explain that it's impossible to match negatives and positives because you only have 4 black chips (B)
3.2.4.3. Explain that identity property of addition says that adding zero to a number results in no change to that number (B)
3.2.4.4. Demonstrate placing two zero pairs (one black and one red chip) next to the four red chips (B)
3.2.4.5. Demonstrate that you can now subtract the two positive (black) chips and the resulting value is -6 (B)
3.2.5. Assign similar problems for homework (B)
3.2.6. Visually inspect one or two key problems on following day using clipboard (B)
3.2.6.1. IF student demonstrates understanding, THEN go to step 3.2.7
3.2.6.2. IF student's answers do not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)
3.2.7. Demonstrate multiplication (B)
3.2.7.1. Show that 3 x -2 can be represented as three groups of two red chips, which is -6 (B)
3.2.7.2. Show that -3 x -2 would be the opposite (additive inverse) of 3 x -2 (B)
3.2.7.3. Demonstrate that we can derive the opposite of 3 x -2 by simply flipping the integer chips over, yielding 6 positives, which signifies that -3 x -2 equals positive 6 (B)
3.2.8. Demonstrate division (B)
3.2.8.1. Show that -6 ÷ 2 can be represented as six red chips (B)

3.2.8.2. Ask, "How many groups of positive 2 can I make out of these 6 negatives?" (B)
3.2.8.3. Explain that the answer is -3 (B)
3.2.8.4. Show next that -6 ÷ -2 would be the opposite (additive inverse) of -6 ÷ 2 by flipping the six red tiles (-6) over and then showing that the answer is 3 *positive* groups (B)
3.2.8.5. Assign similar problems for homework (B)
3.2.8.6. Visually inspect one or two key problems on following day using clipboard (B)
3.2.8.6.1. IF student demonstrates understanding, THEN go to step 4
3.2.8.6.2. IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)
3.2.8.6.2.1. IF student demonstrates understanding, THEN go to step 3 (B)
3.2.8.6.2.2. IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)

4. Teach number domains
4.1. Explain that before we can begin operations with fractions, we need to understand the different types of numbers (B)
4.2. Review that whole numbers are 0,1,2,3,4…to infinity (B)
4.3. Review that integers are whole numbers and their opposites, such as 1 and -1, 2 and -2, etc. (B)
4.4. Teach that in between the whole numbers and integers are other numbers (B)
4.5. Explain that these are called rational numbers, defined as any number that can be written as a fraction (B)
4.6. Explain that whole numbers and integers are also rational numbers, because they can also be written as fractions (B)
4.7. Explain that decimals are also rational numbers, because they can be written as fractions (B)
4.7.1. Display the example of 0.7 (B)
4.7.2. Ask students to write on their whiteboards the name of the place the 7 occupies (B)
4.7.2.1. IF student understands it is in the tenths place, THEN go to step 4.7.3 (B)
4.7.2.2. If student does not understand it is in the tenths place, check in with student before end of class to provide brief review (B)
4.7.3. Ask students to write how 0.7 is read on their whiteboards (B)
4.7.3.1. IF student understands it is read as "seven tenths", THEN go to step 4.7.4 (B)
4.7.3.2. IF student does not understand it is read as "seven tenths", then check in with student before end of class to provide brief review (B)
4.7.4. Ask students to write 0.7 as a fraction on their whiteboards (B)
4.7.4.1. IF student understands it is written as $\frac{7}{10}$ THEN go to step 4.7.5 (B)
4.7.4.2. IF student does not understand it is written as $\frac{7}{10}$ THEN check in with student before end of class and provide brief review (B)
4.7.5. Ask students to write 0.03 as a fraction on their whiteboards (B)

4.7.5.1. IF student understands it is written as $\frac{3}{100}$ then go to step 4.7.6 (B)

4.7.5.2. IF student does not understand it is written as $\frac{3}{100}$ THEN check in with her before end of class and provide brief review (B)

4.7.6. Ask students to write 0.004 as a fraction on their whiteboards (B)

4.7.6.1. IF student understands it is written as $\frac{4}{1000}$ THEN go to step 4.7.7 (B)

4.7.6.2. IF student does not understand it is written as $\frac{4}{1000}$ THEN check in with her before end of class and provide brief review (B)

4.7.7. Review that the place of the number on the right in any decimal determines what the denominator will be when it is converted to a fraction (B)

4.8. Review procedure for converting a fraction to a decimal (B)

4.8.1. Review that we need a power of 10 in the denominator in order to convert many fractions to a decimal (B)

4.8.2. Demonstrate that a fraction such as $\frac{3}{5}$ needs to be converted to an equivalent fraction with a power of 10, in this case 10, in the denominator (B)

4.8.3. Demonstrate multiplying $\frac{3}{5}$ x $\frac{2}{2}$ to yield $\frac{6}{10}$ (B)

4.8.4. Demonstrate that this converts to the decimal 0.6 (B)

4.8.5. Assign similar problems (B)

4.8.6. Visually check for understanding (B)

4.8.6.1. IF students demonstrate understanding, THEN go to step 4.8.7 (B)

4.8.6.2. IF student does not demonstrate understanding, THEN check in with her before end of class to provide brief review (B)

4.8.7. Review that some fractions, such as $\frac{1}{16}$, have denominators that cannot be converted to a power of 10 (B)

4.8.8. Review that with fractions such as these, we can divide the numerator by the denominator, such that 1÷ 16 = .0625 (B)

4.8.9. Assign similar problems (B)

4.8.9.1. IF student demonstrates understanding, THEN go to step 4.8.10 (B)

4.8.9.2. IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)

4.8.10. Review that some fractions, such as $\frac{1}{3}$ yield non-terminating, repeating decimals (B)

4.8.11. Demonstrate that, again, we can divide the numerator by the denominator, such that $1 \div 3 = 0.333\overline{3}$, which repeats and does not terminate, as indicated by the superscript (B)

4.8.12. Assign similar problems (B)

4.8.12.1. IF student demonstrates understanding, THEN go to step 4.8.13 (B)

4.8.12.2. IF student does not demonstrate understanding, THEN check in with her before end of class and provide brief review (B)

4.8.13. Assign homework problems for decimal/fraction concepts (B)

4.8.14. Visually inspect one or two key problems on following day using clipboard (B)

4.8.14.1. IF student demonstrates understanding, THEN go to step 4.9 (B)

4.8.14.2. IF student does not demonstrate understanding, THEN check in with her before end of class to check for understanding (B)

4.8.14.2.1. IF student demonstrates understanding, THEN go to step 4.9 (B)
4.8.14.2.2. IF student does not demonstrate understanding, THEN direct her to come to class during Muir Time and provide remediation (B)
4.9. Explain that not all rational numbers can be integers or whole numbers (B)
4.10. Explain that a number that cannot be written as a fraction (a non-repeating, non-terminating decimal, such as $\sqrt{12}$), is an irrational number (B)

5. Review representations of fractions
5.1. Explain that, although this review involves fractions as parts of a whole, later we will be dealing with fraction concepts involving ratios and proportions (B)
5.2. Remind students that the concept of fractions as parts of a whole requires us to split up that whole (B)
5.3. Distribute fraction circle manipulatives (B)
5.4. Ask students to represent $\frac{3}{8}$ (B)
5.4.1. Visually check for understanding (B)
5.4.1.1. IF student demonstrates understanding, THEN go to step 5.5 (B)
5.4.1.2. IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)
5.5. Ask students to represent $\frac{1}{2}$ (B)
5.5.1. Visually check for understanding (B)
5.5.1.1. IF student demonstrates understanding, THEN go to step 5.6 (B)
5.5.1.2. IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)
5.6. Ask students to show another way to make $\frac{1}{2}$ (B)
5.7. Explain to students that representing $\frac{1}{2}$ by its equivalents, such as $\frac{2}{4}$ or $\frac{3}{6}$, exemplifies the identity property of multiplication, which states that multiplying any number by a form of 1 yields that same number (B)
5.8. Ask students, "How do you write an equivalent fraction for $\frac{1}{2}$?" (B)
5.9. Remind students that you do that by multiplying by a form of 1, such as $\frac{2}{2}$, resulting in the equivalent fraction, $\frac{2}{4}$ (B)
5.10. Assign similar problems (B)
5.10.1. Visually check for understanding (B)
5.10.1.1. IF student demonstrates understanding, THEN go to step 5.11 (B)
5.10.1.2. IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)
5.11. Distribute fraction strip manipulatives (B)
5.12. Ask students, in groups, or with partners to represent $\frac{1}{2}$ (B)
5.12.1. Visually check for understanding (B)
5.12.1.1. IF student demonstrates understanding, THEN go to step 5.13 (B)
5.12.1.2. IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)
5.13. Ask students to demonstrate as many ways as possible to represent $\frac{1}{2}$ (B)
5.13.1. Visually check for understanding (B)

5.13.1.1. IF student demonstrates understanding, THEN go to step 5.14 (B)
5.13.1.2. IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)
5.14. Remind students that dividing any number by a form of 1 is identical to the identity property of multiplication: the result in both cases is the original number (B)
5.15. Distribute fraction circle manipulatives (B)
5.16. Ask students to represent a fraction such as $\frac{4}{8}$ in simplest terms (B)
5.17. Visually check for understanding (B)
5.17.1. IF student demonstrates understanding, THEN go to step 6 (B)
5.17.2. IF student does not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)
5.17.2.1. IF student demonstrates understanding of all elements of step 5, THEN go to step 6 (B)
5.17.2.2. IF student does not demonstrate understanding of all elements of step 4, THEN direct her to come to class during Muir time and provide remediation (B)

6. Review addition and subtraction of fractions and mixed numbers
6.1. Review addition and subtraction with like denominators (A, B)
6.2. Distribute manipulative fraction strips and circles (A, B)
6.2.1. Ask students to model a problem such as $\frac{1}{2} + \frac{1}{2}$ with strips and circles (B)
6.2.2. Ask students to demonstrate that the answer, two halves, is visually equivalent to one whole (B)
6.2.3. Ask students to divide by a fraction equal to 1, such as $\frac{2}{2}$, to also yield one whole (A, B)
6.2.4. Ask students to model a problem such as $\frac{3}{8} + \frac{1}{8}$ (B)
6.2.5. Ask students to demonstrate that the answer, four eighths, is visually equivalent to $\frac{1}{2}$ (B)
6.2.6. Ask students to divide by a fraction equal to 1, such as $\frac{4}{4}$, to also yield $\frac{1}{2}$ (A, B)
6.2.6.1. IF students demonstrate understanding, then go to step 6.2.7 (B)
6.2.6.2. IF student appears unclear, then check in with her before end of class to provide brief review (B)
6.2.7. Ask students to model a problem, such as $\frac{5}{8} - \frac{1}{8}$ (B)
6.2.8. Ask students to demonstrate that the answer is visually equivalent to $\frac{1}{2}$ (B)
6.2.9. Ask students to divide by a fraction equal to 1, such as $\frac{4}{4}$, to also yield $\frac{1}{2}$ (A, B)
6.2.9.1. IF students demonstrate understanding, then go to step 6.3 (B)
6.2.9.2. IF student appears unclear, THEN check in with her before end of class to provide brief review (B)
6.3. Review algorithm for adding and subtracting fractions (A, B)
6.3.1. Ask students to add two fractions with like denominators, such as $\frac{1}{4} + \frac{1}{4}$ (A, B)
6.3.2. Visually check that students are adding numerators to yield $\frac{2}{4}$ (A, B)

6.3.3. Ask students to divide by a fraction equal to 1, such as $\frac{2}{2}$, to put in simplest terms, $\frac{1}{2}$ (A, B)

6.3.4. Ask students to subtract two fractions with like denominators, such as $\frac{3}{6}$ - $\frac{1}{6}$ (B)

6.3.5. Ask students to divide by a fraction equal to 1, such as $\frac{2}{2}$, to put in simplest terms, $\frac{1}{3}$ (A, B)

6.3.5.1. Assign similar problems for homework (B)

6.3.5.2. Visually inspect one or two key problems on following day using clipboard (B)

6.3.5.2.1. IF student demonstrates understanding, THEN go to step 6.4 (B)

6.3.5.2.2. IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)

6.3.5.2.2.1. IF student demonstrates understanding, THEN go to step 6.4 (B)

6.3.5.2.2.2. IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)

6.4. Review addition and subtraction of fractions with unlike denominators (A, B, C)

6.4.1. Demonstrate adding two fractions with unlike denominators, such as $\frac{1}{2}$ + $\frac{1}{4}$ (A, B, C)

6.4.2. Review that we can't add them unless the denominators are the same (A, B, C)

6.4.3. Review that the identify property of multiplication means that we can multiply $\frac{1}{2}$ by a form of one, $\frac{2}{2}$ to yield a fraction with a denominator of 4, $\frac{2}{4}$ (A, B)

6.4.4. Review that we can now add $\frac{2}{4}$ + $\frac{1}{4}$ = $\frac{3}{4}$ (A, B, C)

6.4.5. Assign similar problems (A, B, C)

6.4.5.1. IF students demonstrate understanding, then go to step 6.4.6 (B)

6.4.5.2. IF student appears unclear, then check in with her before end of class to provide brief review (B)

6.4.6. Demonstrate subtracting two fractions with unlike denominators, such as $\frac{1}{2}$ - $\frac{1}{4}$ (B)

6.4.7. Review that the identify property of multiplication means that we can multiply $\frac{1}{2}$ by a form of one, $\frac{2}{2}$ to yield a fraction with a denominator of 4, $\frac{2}{4}$ (A,B)

6.4.8. Review that we can now subtract $\frac{2}{4}$ - $\frac{1}{4}$ = $\frac{1}{4}$ (B)

6.4.9. Assign similar problems (B)

6.4.9.1. IF students demonstrate understanding, then go to step 6.4.10 (B)

6.4.9.2. IF student appears unclear, then check in with her before end of class to provide brief review (B)

6.4.10. Assign addition and subtraction of fractions with unlike denominators for homework (B)

6.4.11. Visually inspect one or two key problems on following day using clipboard (B)

6.4.11.1. IF student demonstrates understanding, THEN go to step 6.5 (B)

6.4.11.2. IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)

6.4.11.2.1. IF student demonstrates understanding, THEN go to step 6.5 (B)
6.4.11.2.2. IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)

6.5. Review addition and subtraction of mixed numbers with like denominators (B)
6.5.1. Demonstrate $2\frac{1}{2} + 4\frac{1}{2}$ using hand-drawn circles to represent the wholes and fractions (B)
6.5.2. Review that the two fractions add to one whole, which we must add to the other six wholes, to yield 7 (B)
6.5.3. Demonstrate $1\frac{3}{5} + 2\frac{4}{5}$ using hand-drawn circles (B)
6.5.4. Review that this yields 3 wholes and seven-fifths (B)
6.5.5. Review through use of fraction strips that $\frac{7}{5}$ is the same as one whole and $\frac{2}{5}$ (B)
6.5.6. Demonstrate that we now add the 1 whole to the other 3 whole to yield 4 whole, and then add $\frac{2}{5}$, resulting in $4\frac{2}{5}$ (B)
6.5.7. Demonstrate with fraction strips the corollary of this concept: $1\frac{2}{5}$ can be converted to the improper fraction, $\frac{7}{5}$ (B)
6.5.8. Demonstrate $6\frac{1}{8} - 4\frac{5}{8}$ using hand-drawn circles (B)
6.5.9. Review that one whole can be represented by any number over itself, such as $\frac{6}{6}$ or $\frac{7}{7}$ or $\frac{8}{8}$ (B)
6.5.10. Review that we need to borrow one whole from the 6, which we can represent as $\frac{8}{8}$ (B)
6.5.11. Review that we must add the $\frac{8}{8}$ we borrowed to the $\frac{1}{8}$ to yield $\frac{9}{8}$ (B)
6.5.12. Demonstrate that we can now subtract to yield $1\frac{4}{8}$ (B)
6.5.13. Review that we need to simplify the $\frac{4}{8}$ by dividing by a form of one, $\frac{4}{4}$, yielding $1\frac{1}{2}$ (B)
6.5.14. Assign similar problems (B)
6.5.15. Visually check for understanding (B)
6.5.15.1. IF students demonstrate understanding, then go to step 6.5.16 (B)
6.5.15.2. IF student appears unclear, then check in with her before end of class to provide brief review (B)
6.5.16. Assign addition and subtraction of mixed numbers with like denominators for homework (B)
6.5.17. Visually inspect one or two key problems on following day using clipboard (B)
6.5.17.1. IF student demonstrates understanding, THEN go to step 6.6 (B)
6.5.17.2. IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)
6.5.17.2.1. IF student demonstrates understanding, THEN go to step 6.6 (B)
6.5.17.2.2. IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)

6.6. Review addition and subtraction of mixed numbers with unlike denominators (B)

- 6.6.1. Demonstrate $1\frac{1}{2} + 2\frac{1}{4}$ using hand-drawn circles (B)
- 6.6.2. Review that we cannot add fractions unless we have the same denominators (A, B, C)
- 6.6.3. Review that we can multiply by a form of one, $\frac{2}{2}$, to yield $1\frac{2}{4}$ (B)
- 6.6.4. Demonstrate adding $1\frac{2}{4} + 2\frac{1}{4}$ to yield $3\frac{3}{4}$ (B)
- 6.6.5. Demonstrate subtracting $3\frac{1}{2} - 1\frac{1}{4}$ using hand-drawn circles (B)
- 6.6.6. Review that we cannot subtract fractions unless we have the same denominators (B)
- 6.6.7. Review that we can multiply by a form of one, $\frac{2}{2}$, to yield $3\frac{2}{4}$ (A, B)
- 6.6.8. Demonstrate subtracting $3\frac{2}{4} - 1\frac{1}{4}$ to yield $2\frac{1}{4}$ (B)
- 6.6.9. Assign similar problems (B)
- 6.6.10. Visually check for understanding (B)
  - 6.6.10.1. IF students demonstrate understanding, then go to step 6.6.11 (B)
  - 6.6.10.2. IF student appears unclear, then check in with her before end of class to provide brief review (B)
- 6.6.11. Assign addition and subtraction of mixed numbers with unlike denominators for homework (B)
- 6.6.12. Visually inspect one or two key problems on following day using clipboard (B)
  - 6.6.12.1. IF student demonstrates understanding, THEN go to step 6.7 (B)
  - 6.6.12.2. IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)
    - 6.6.12.2.1. IF student demonstrates understanding, THEN go to step 6.7 (B)
    - 6.6.12.2.2. IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)

6.7. Review that because rational numbers can be written as both fractions and decimals, we can add and subtract fractions and mixed numbers by converting them to decimals (B)

- 6.7.1. Demonstrate this for fraction addition using an example such as $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ (B)
- 6.7.2. Review that $\frac{1}{4}$ needs to be converted to a fraction with a denominator that is a power of 10 (B)
- 6.7.3. Demonstrate that multiplying $\frac{1}{4}$ by $\frac{25}{25}$ yields $\frac{25}{100}$, which in decimal form is .25 (B)
- 6.7.4. Review that to add .25 + .25 we must line up the decimals and places, then add, yielding a sum of .5 (B)
- 6.7.5. Demonstrate that .5 is the same as five tenths, or $\frac{5}{10}$, which can be reduced to simplest terms as $\frac{1}{2}$ (B)
- 6.7.6. Demonstrate that we can derive the same answer, .5, by converting our answer in $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ (B)
- 6.7.7. Demonstrate that if we convert $\frac{1}{2}$ to a fraction with a power of 10 in the denominator, we get $\frac{5}{10}$ (B)
- 6.7.8. Review that $\frac{5}{10}$ is the same as .5 (B)

6.7.9. Reinforce idea that regardless of whether we add rational numbers as fractions or decimals, the answer is the same (B)
6.7.10. Assign similar problems(B)
   6.7.10.1. IF students demonstrates understanding, THEN go to step 6.7.11 (B)
   6.7.10.2. IF student does not demonstrate understanding, THEN check in with her before end of class and provide brief review (B)
6.7.11. Demonstrate same concept for subtraction using an example such as $\frac{3}{4}$ - $\frac{1}{4}$ = $\frac{2}{4}$, or $\frac{1}{2}$ (B)
6.7.12. Demonstrate that multiplying $\frac{3}{4}$ and $\frac{1}{4}$ by $\frac{25}{25}$ yields $\frac{75}{100}$ and $\frac{25}{100}$, which in decimal form are .75 and .25 (B)
6.7.13. Review that to subtract .75 - .25 we must line up the decimals and places, then subtract, yielding a difference of .5 (B)
6.7.14. Demonstrate that .5 is the same as five tenths, or $\frac{5}{10}$, which can be reduced to simplest terms as $\frac{1}{2}$ (B)
6.7.15. Demonstrate that we can derive the same answer, .5, by converting our answer in $\frac{3}{4}$ - $\frac{1}{4}$ = $\frac{1}{2}$ (B)
6.7.16. Demonstrate that if we convert $\frac{1}{2}$ to a fraction with a power of 10 in the denominator, we get $\frac{5}{10}$ (B)
6.7.17. Review that $\frac{5}{10}$ is the same as .5 (B)
6.7.18. Reinforce idea that regardless of whether we subtract rational numbers as fractions or decimals, the answer is the same (B)
6.7.19. Assign similar problems (B)
   6.7.19.1. IF students demonstrates understanding, THEN go to step 6.7.20 (B)
   6.7.19.2. IF student does not demonstrate understanding, THEN check in with her before end of class and provide brief review (B)
6.7.20. Demonstrate same concept for addition of mixed numbers using an example such as $1\frac{1}{4}$ + $1\frac{1}{4}$ = $2\frac{2}{4}$, or $2\frac{1}{2}$ (B)
6.7.21. Review that $\frac{1}{4}$ needs to be converted to a fraction with a denominator that is a power of 10 (B)
6.7.22. Demonstrate that multiplying $\frac{1}{4}$ by $\frac{25}{25}$ yields $\frac{25}{100}$, which in decimal form is .25 (B)
6.7.23. Review that to add 1.25 + 1.25 we must line up the decimals and places, then add, yielding a sum of 2.5 (B)
6.7.24. Demonstrate that 2.5 is the same as two and five tenths, or $2\frac{5}{10}$, which can be reduced to simplest terms as $2\frac{1}{2}$ (B)
6.7.25. Demonstrate that we can derive the same answer, 2.5, by converting our answer in $1\frac{1}{4}$ + $1\frac{1}{4}$ = $2\frac{1}{2}$ (B)
6.7.26. Demonstrate that if we convert $2\frac{1}{2}$ to a fraction with a power of 10 in the denominator, we get $2\frac{5}{10}$ (B)

6.7.27. Review that $2\frac{5}{10}$ is the same as 2.5 (B)
6.7.28. Reinforce idea that regardless of whether we add rational numbers as fractions or decimals, the answer is the same (B)
6.7.29. Assign similar problems (B)
6.7.29.1. IF students demonstrates understanding, THEN go to step 6.7.30 (B)
6.7.29.2. IF student does not demonstrate understanding, THEN check in with her before end of class and provide brief review(B)
6.7.30. Demonstrate same concept for subtraction of mixed numbers using an example such as $2\frac{3}{4}$ - $1\frac{1}{4}$ = $1\frac{2}{4}$, or $1\frac{1}{2}$ (B)
6.7.31. Demonstrate that multiplying $\frac{3}{4}$ and $\frac{1}{4}$ by $\frac{25}{25}$ yields $\frac{75}{100}$ and $\frac{25}{100}$, which in decimal form are .75 and .25 (B)
6.7.32. Review that to subtract 2.75 - 1.25 we must line up the decimals and places, then subtract, yielding a difference of 1.5 (B)
6.7.33. Demonstrate that 1.5 is the same as one and five tenths, or $1\frac{5}{10}$, which can be reduced to simplest terms as $1\frac{1}{2}$ (B)
6.7.34. Demonstrate that we can derive the same answer, 1.5, by converting our answer in $2\frac{3}{4}$ - $1\frac{1}{4}$ = $1\frac{1}{2}$ (B)
6.7.35. Demonstrate that if we convert $1\frac{1}{2}$ to a fraction with a power of 10 in the denominator, we get $1\frac{5}{10}$ (B)
6.7.36. Review that $1\frac{5}{10}$ is the same as 1.5 (B)
6.7.37. Reinforce idea that regardless of whether we subtract rational numbers as fractions or decimals, the answer is the same (B)
6.7.38. Assign similar problems (B)
6.7.38.1. IF students demonstrates understanding, THEN go to step 6.7.39 (B)
6.7.38.2. IF student does not demonstrate understanding, THEN check in with her before end of class and provide brief review (B)
6.7.39. Assign homework problems for decimal/fraction equivalence (B)
6.7.40. Visually inspect one or two key problems on following day using clipboard (B)
6.7.40.1. IF student demonstrates understanding, THEN go to step 7
6.7.40.2. IF student does not demonstrate understanding, THEN check in with her before end of class to check for understanding (B)
6.7.40.2.1. IF student demonstrates understanding, THEN go to step 7 (B)
6.7.40.2.2. IF student does not demonstrate understanding, THEN direct her to come to class during Muir Time and provide remediation (B)

7. Review multiplication of fractions and mixed numbers
7.1. Review multiplication of fractions (B)
7.1.1. Demonstrate multiplying a fraction by a whole number, such as $\frac{1}{3}$ x 2 (B)
7.1.2. Demonstrate we can approach this through repeated addition (B)
7.1.3. Review that $\frac{1}{3}$ x 2 means two one-thirds, or $\frac{1}{3}$ + $\frac{1}{3}$ = $\frac{2}{3}$ (B)
7.1.4. Demonstrate that we can also convert the whole number, 2, to a fraction by giving the 2 a denominator, 1, yielding $\frac{2}{1}$ (B)

7.1.5. Demonstrate that $\frac{1}{3}$ x 2 is actually $\frac{1}{3}$ x $\frac{2}{1}$, or $\frac{2}{3}$ (B)
7.1.6. Review that in multiplication of fractions, the algorithm requires that we multiply the numerators by the numerators and the denominators by the denominators (B)
7.1.7. Extend this understanding to multiplication of fractions by fractions
7.1.8. Demonstrate that a problem such as $\frac{1}{3}$ x $\frac{1}{2}$ also involves multiplying the numerators by the numerators and the denominators by the denominators, yielding $\frac{1}{6}$ (B)
7.1.9. Assign similar problems (B)
7.1.10. Visually check for understanding (B)
    7.1.10.1. IF students demonstrate understanding, then go to step 7.1.11 (B)
    7.1.10.2. IF student appears unclear, then check in with her before end of class to provide brief review (B)
7.1.11. Assign multiplication of fractions for homework (B)
7.1.12. Visually inspect one or two key problems on following day using clipboard (B)
    7.1.12.1. IF student demonstrates understanding, THEN go to step 7.1.13 (B)
    7.1.12.2. IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)
        7.1.12.2.1. IF student demonstrates understanding, THEN go to step 7.1.13 (B)
        7.1.12.2.2. IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)
7.1.13. Demonstrate multiplying a mixed number by a mixed number, such as $2\frac{1}{4}$ x $2\frac{1}{6}$ (B)
7.1.14. Review that we must convert each mixed number into an improper fraction before we can multiply (B)
7.1.15. Review that $2\frac{1}{4}$ can be represented as $\frac{9}{4}$ and $2\frac{1}{6}$ can be represented as $\frac{13}{6}$ (B)
7.1.16. Demonstrate that we multiply the numerators by the numerators and the denominators by the denominators, yielding $\frac{117}{24}$ (B)
7.1.17. Review that we need to simplify, yielding 4 whole and 21 twenty-fourths (B)
7.1.18. Review that $\frac{21}{24}$ can be simplified by dividing by a form of one, $\frac{3}{3}$, yielding $\frac{7}{8}$ (B)
7.1.19. Display completed product, $4\frac{7}{8}$ (B)
7.1.20. Assign similar problems (B)
7.1.21. Visually check for understanding (B)
    7.1.21.1. IF students demonstrate understanding, then go to step 7.1.22 (B)
    7.1.21.2. IF student appears unclear, then check in with her before end of class to provide brief review (B)
7.1.22. Assign multiplication of mixed numbers for homework (B)
7.1.23. Visually inspect one or two key problems on following day using clipboard (B)
    7.1.23.1. IF student demonstrates understanding, THEN go to step 8 (B)
    7.1.23.2. IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)
        7.1.23.2.1. IF student demonstrates understanding, THEN go to step 8 (B)
        7.1.23.2.2. IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)

8. Teach division of whole numbers by fractions
    8.1. Hold up a whole piece of paper and pose question that represents the problem $1\div \frac{1}{2}$, such as "If this piece of paper represents a whole cake, how can I divide it by one-halves?" (B, C)
    8.2. Direct students to discuss the question (B, C)
    8.3. Circulate to check for understanding (B, C)
    8.4. Call on volunteers who were able to understand that the solution is two halves (B, C)
    8.5. Distribute fraction strips (C)
    8.6. Relate the scenario, "If I have one piece of paper, how many halves are there in that one piece?" (C)
        8.6.1. Ask students to work with a partner to model scenario with manipulatives (C)
        8.6.2. Circulate among students to determine whether they are creating concrete representations of $1 \div \frac{1}{2} = 2$ (C)
        8.6.3. Check for understanding by calling on students randomly (C)
        8.6.4. Validate correct answers (C)
        8.6.5. Demonstrate a correct representation on whiteboard using magnetic fraction strips (A, C)
        8.6.6. Ask students in pairs to formulate a real-life example of the problem, $1 \div \frac{1}{2} = 2$ (C)
        8.6.7. Circulate among students to check for understanding (C)
        8.6.8. Call on volunteers who were able to formulate correct examples, such as, "If I have a whole pizza, how many halves are contained therein?" (C)
        8.6.9. Ask students to work with a partner to explain what the problem $1 \div \frac{1}{2} = 2$ means (C)
        8.6.10. Circulate to check for understanding (C)
        8.6.11. Call on volunteers who were able to articulate that the problem means, "How many one-halves are contained in one whole" (A, B, C)
    8.7. Direct students to work with a partner to create a real-life example of the problem, $1\div \frac{1}{4}$ (C)
    8.8. Circulate among students to check for understanding (C)
    8.9. Call on volunteers who were able to articulate examples, such as, "If I have one dollar, how many quarters are contained in that dollar?" (C)
    8.10. Ask students to work with a partner to explain what the problem $1\div \frac{1}{4}$ means (C)
    8.11. Circulate to check for understanding (C)
    8.12. Call on volunteers who were able to articulate that the problem means, "How many one-fourths are contained in one whole" (A, B, C)
        8.12.1. Ask students to work with a partner to model scenario with manipulatives (C)
        8.12.2. Circulate among students to determine whether they are creating concrete representations of $1 \div \frac{1}{4} = 4$ (C)
        8.12.3. Check for understanding by calling on students randomly (C)
        8.12.4. Validate correct answers (C)
        8.12.5. Demonstrate a correct representation on whiteboard using magnetic fraction strips (C)

8.12.5.1. IF teacher observation indicates student understands how to create concrete representations of problems involving whole numbers divided by fractions, THEN step 8.13 (C)

8.12.5.2. IF teacher observation indicates student does not understand, THEN provide remediation with similar examples after class (C)

8.13. Distribute math paper and pencils

8.14. Relate the scenario, “If I have one pizza and want to share it among 3 people, how much will each person get?” (C)

8.14.1. Direct students to work with a partner to draw either a number line or area model representation of the scenario (C)

8.14.2. Circulate among students to determine whether they are creating pictorial representations of $1 \div \frac{1}{3} = 3$ (C)

8.14.3. Ask students to share their drawings with other neighbors (C)

8.14.4. Check for understanding by calling on students randomly, asking them to describe their drawings (C)

8.14.5. Validate correct answers that represent $1 \div \frac{1}{3} = 3$ (C)

8.14.6. Demonstrate a correct representation on whiteboard by drawing both an area model and a number line depicting $1 \div \frac{1}{3} = 3$ (C)

8.15. Relate next scenario, “If I have one bagel and want to share it among 4 people, how much will each person get?” (C)

8.15.1. Direct students to work with a partner to draw either a number line or area model representation of the scenario (C)

8.15.2. Circulate among students to determine whether they are creating pictorial representations of $1 \div \frac{1}{4} = 4$ (C)

8.15.3. Ask students to share their drawings with other neighbors (C)

8.15.4. Check for understanding by calling on students randomly, asking them to describe their drawings (C)

8.15.5. Validate correct answers that represent $1 \div \frac{1}{4} = 4$ (C)

8.15.6. Demonstrate a correct representation on whiteboard by drawing both an area model and a number line depicting $1 \div \frac{1}{4} = 4$ (C)

8.15.6.1. IF teacher observation indicates student understands how to create pictorial representations of problems involving whole numbers divided by fractions, THEN step 8.16 (C)

8.15.6.2. IF teacher observation indicates student does not understand, THEN provide remediation after class (C)

8.16. Ask students to consider their answers to previous problems, such as $1 \div \frac{1}{2} = 2$, $1 \div \frac{1}{3} = 3$, and $1 \div \frac{1}{4} = 4$ (C)

8.16.1. Ask students to pair with a partner (C)

8.16.2. Ask students to look for patterns in these problems (A, C)

8.16.3. Ask students to formulate a procedural rule that can be followed (A, C)

8.16.4. Circulate among students to check for understanding (A, C)

8.16.5. Ask students that were able to formulate a rule to share it with the class (A, C)

8.16.6. Validate that the rule is to multiply the whole number by the reciprocal of the divisor (A, B, C)
8.16.7. Assign similar problems (A, C)
8.16.8. Direct students to solve using the procedural rule and to write an explanation of the procedure they used (A, C)
8.16.9. Circulate among students, and check written explanations for understanding (A, C)
8.16.9.1. IF student demonstrates understanding, THEN step 9 (A, C)
8.16.9.2. IF student does not demonstrate understanding, THEN provide remediation after class (A, C)

9. Teach division of fractions by whole numbers
9.1. Display an apple that is cut into two pieces, asking, “If I want to share one of these halves of an apple between two people, how much will each person get?” (C)
9.2. Call on student volunteers (C)
9.3. Confirm correct answers by demonstrating that the answer is one-fourth of the whole apple for each person (C)
9.4. Distribute fraction strips (C)
9.5. Relate the same scenario, “If I have one half an apple and want to share it between two people, how much will each person get?” (C)
9.5.1. Ask students to work with a partner to model scenario with manipulatives (C)
9.5.2. Circulate among students to determine whether they are creating concrete representations of $\frac{1}{2} \div 2 = \frac{1}{4}$ (C)
9.5.3. Check for understanding by calling on students randomly (C)
9.5.4. Validate correct answers (C)
9.5.5. Demonstrate a correct representation on whiteboard using magnetic fraction strips (A, C)
9.6. Relate next scenario, “If I have one half an apple and want to share it among three people, how much will each person get?” (C)
9.7. Display an apple that is cut into two pieces, asking, “If I want to share one of these halves of an apple among three people, how much will each person get?” (C)
9.8. Call on student volunteers (C)
9.9. Confirm correct answers by demonstrating that the answer is one-sixth of the whole apple for each person (A, C)
9.9.1. Ask students to work with a partner to model scenario with manipulatives (C)
9.9.2. Circulate among students to determine whether they are creating concrete representations of $\frac{1}{2} \div 3 = \frac{1}{6}$ (C)
9.9.3. Check for understanding by calling on students randomly (C)
9.9.4. Validate correct answers (C)
9.9.5. Demonstrate a correct representation on whiteboard using magnetic fraction strips (A, C)
9.9.5.1. IF teacher observation indicates student understands how to create concrete representations of problems involving fractions divided by whole numbers, THEN step 9.10 (C)
9.9.5.2. IF teacher observation indicates student does not understand, THEN provide remediation after class (C)
9.10. Distribute math paper and pencils (C)

9.11. Relate the scenario, "If I have one third of a candy bar and want to share it between 2 people, how much will each person get?" (C)
9.12. Draw both an area model and a number line representation on whiteboard of problem (C)
9.13. Demonstrate how both of these representations are similar to the previous fraction strip representation (C)
9.13.1. Direct students to work with a partner to draw either a number line or area model representation of the scenario (C)
9.13.2. Circulate among students to determine whether they are creating pictorial representations of $\frac{1}{3} \div 2$ (C)
9.13.3. Ask students to share their drawings with other neighbors (C)
9.13.4. Check for understanding by calling on students randomly, asking them to describe their drawings (C)
9.13.5. Validate correct answers that represent $\frac{1}{3} \div 2 = \frac{1}{6}$ (C)

9.14. Relate next scenario, "If I have one fourth of a bagel and want to share it between 2 people, how much will each person get?" (C)
9.14.1. Direct students to work with a partner to draw either a number line or area model representation of the scenario (C)
9.14.2. Circulate among students to determine whether they are creating pictorial representations of $\frac{1}{4} \div 2$ (C)
9.14.3. Ask students to share their drawings with other neighbors (C)
9.14.4. Check for understanding by calling on students randomly, asking them to describe their drawings (C)
9.14.5. Validate correct answers that represent $\frac{1}{4} \div 2 = \frac{1}{8}$ (C)
9.14.6. Demonstrate a correct representation on whiteboard by drawing both an area model and a number line depicting $\frac{1}{4} \div 2 = \frac{1}{8}$ (C)
9.14.6.1. IF teacher observation indicates student understands how to create pictorial representations of problems involving fractions divided by whole numbers, THEN step 9.15 (C)
9.14.6.2. IF teacher observation indicates student does not understand, THEN provide remediation after class (C)
9.15. Ask students to consider their answers to previous problems, such as $\frac{1}{2} \div 2 = \frac{1}{4}$, $\frac{1}{2} \div 3 = \frac{1}{6}$, and $\frac{1}{4} \div 2 = \frac{1}{8}$ (C)
9.15.1. Ask students to pair with a partner (C)
9.15.2. Ask students to look for patterns in the previous problems (A, C)
9.15.3. Ask students to formulate a procedural rule that can be followed (A,C)
9.15.4. Circulate among students to check for understanding (A, C)
9.15.5. Ask students that were able to formulate a rule to share it with the class (A, C)
9.15.6. Validate that the rule is to multiply the fraction by the reciprocal of the whole number (A, B, C)
9.15.7. Assign similar problems, such as $\frac{1}{5} \div 2$ (A,C)

9.15.8. Direct students to solve using fraction strips, then a pictorial representation, and then to validate using the procedural rule, and include a written description of the written description (A, C)

9.15.9. Circulate among students and check strips, pictures, and written descriptions for understanding (A, C)

9.15.9.1. IF student demonstrates understanding, THEN step 10 (A, C)

9.15.9.2. IF student does not demonstrate understanding, THEN provide remediation after class (A, C)

10. Teach division of fractions by fractions

10.1. Teach concrete representation (C)

10.2. Distribute fractions strips (C)

10.2.1. Pose problem such as $\frac{3}{4} \div \frac{1}{4}$ (C)

10.2.2. Remind students that the problem involves determining how many fourths are in three fourths (A, B, C)

10.2.3. Use magnetic fraction strips on whiteboard to demonstrate that the answer is 3: there are 3 one-fourths in three-fourths (C)

10.2.4. Direct students to model the same problem at their desks (C)

10.2.5. Circulate among students to check for understanding (C)

10.2.6. Pose similar problem, such as $\frac{1}{2} \div \frac{1}{4}$ (C)

10.2.7. Direct students to model problem with fraction strips or linking cubes (C)

10.2.8. Circulate among students to check for understanding (C)

10.2.9. Demonstrate on board that the answer is 2: there are two one-fourths in one half (C)

10.2.10. Pose similar problems (130-131)

10.2.11. Circulate among students to check for understanding (77-78)

10.2.11.1. IF student demonstrates understanding, THEN step 10.3 (C)

10.2.11.2. IF students does not demonstrate understanding, THEN provide remediation after class (C)

10.3. Teach pictorial representation (B, C)

10.3.1. Pose problem, such as $\frac{3}{4} \div \frac{1}{4}$ (B, C)

10.3.2. Remind students that the problem involves determining how many fourths are in three fourths (A, B, C)

10.3.3. Draw an area model and a number line on whiteboard to demonstrate that there are three one-fourths in three fourths (B, C)

10.3.4. Direct students to model the same problem at their desks (B, C)

10.3.5. Circulate among students to check for understanding (B, C)

10.3.6. Direct students to validate the answer using the procedural rule (C)

10.3.7. Circulate among students to check for understanding (B, C)

10.3.8. Call on volunteers who were able to multiply three-fourths by the reciprocal of one-fourth, 4, to yield 3 (B, C)

10.3.9. Pose similar problem, such as $\frac{1}{2} \div \frac{1}{4}$ (C)

10.3.10. Direct students to model problem with fraction strips, and then area models or number lines, and then validate with the procedural rule (C)

10.3.11. Circulate among students to check for understanding (C)

10.3.12. Demonstrate on board that there are two fourths in one half (C)

10.3.13. Pose similar problems (C)
10.3.14. Circulate among students to check for understanding (C)
10.3.14.1. IF student demonstrates understanding, THEN step 10.4 (C)
10.3.14.2. IF students does not demonstrate understanding, THEN provide remediation after class (C)
10.4. Teach complex fraction approach (B)
10.4.1. Remind students that division can be represented through fractions: the dividend can be represented as numerator, and the divisor as denominator (B)
10.4.2. Demonstrate that $\frac{3}{4} \div \frac{1}{4}$ can be represented as a complex fraction, in the form $\frac{\frac{3}{4}}{\frac{1}{4}}$ (B)
10.4.3. Remind students that anything divided by one is itself (B)
10.4.4. Remind students that the identity property states that multiplying a value by 1 does not change that value (B)
10.4.5. Demonstrate that multiplying $\frac{\frac{3}{4}}{\frac{1}{4}}$ x $\frac{\frac{4}{1}}{\frac{4}{1}}$ is both multiplying by 1 and also going to create a 1 in the denominator, based on the inverse property of multiplication (690-B)
10.4.6. Draw the outline of a “1” around $\frac{\frac{4}{1}}{\frac{4}{1}}$ (B)
10.4.7. Remind students that because the denominator is 1, we are left with $\frac{3}{4}$ x $\frac{4}{1}$ (B)
10.4.8. Remind students that the algorithm for multiplying fractions, by which we multiply the numerators by numerators and denominators by denominators, means that the product is going to be $\frac{12}{4}$ (B)
10.4.9. Remind students to reduce by dividing by a form of 1, in this case $\frac{4}{4}$ (B)
10.4.10. Explain that the answer is 3, which is the same answer derived through the pictorial method (B)
10.4.11. Assign similar problems asking students to solve using complex fractions (B)
10.4.12. Visually check for understanding (B)
10.4.12.1. IF students demonstrate understanding, then go to step 10.4.13 (B)
10.4.12.2. IF student appears unclear, then check in with her before end of class to provide brief review (B)
10.4.13. Review that converting a fraction division problem into a complex fraction results in multiplying the first fraction by the reciprocal of the second fraction, a fact we also discovered when we divided whole numbers by fractions and fractions by whole numbers using manipulatives and pictorial representations (A, B, C)
10.4.14. Administer assessment in which students must solve similar problems and also write an explanation of why they can derive the same answer by simply multiplying by the reciprocal (B)
10.4.14.1. IF student answers demonstrate understanding, THEN go to step 10.5 (B)
10.4.14.2. IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)

10.5. Teach alternate method for dividing fractions by fractions, which involves converting the fractions to decimals, and then dividing the decimals (B)

10.5.1. Demonstrate conceptual nature of process with an example using currency

10.5.1.1. Demonstrate using the example $\frac{1}{2} \div \frac{1}{4}$ (B)

10.5.1.2. Review that we must first convert both $\frac{1}{2}$ and $\frac{1}{4}$ to equivalent fractions with denominators that are powers of 10 or 100 (B)

10.5.1.3. Demonstrate multiplying $\frac{1}{2}$ and $\frac{1}{4}$ by $\frac{5}{5}$ and $\frac{25}{25}$ respectively, to yield $\frac{5}{10}$ and $\frac{25}{100}$ (B)

10.5.1.4. Review that these new fractions convert to decimals of .50 and .25 (B)

10.5.1.5. Connect these two values to the concept of money by explaining that .50 can be viewed as fifty cents and .25 can be viewed as twenty five cents (B)

10.5.1.6. Demonstrate that .50 ÷ .25 is equivalent to asking, "How many quarters make up fifty cents?" (B)

10.5.1.7. Demonstrate that the answer is "2" (B)

10.5.2. Demonstrate procedural process (B)

10.5.2.1. Review that $\frac{1}{2} \div \frac{1}{4}$ can be represented by decimals as .50 ÷ .25 (B)

10.5.2.2. Review that we can represent this in long division as $.25\overline{).50}$ (B)

10.5.2.3. Review that we must shift both the decimal point in the divisor and the decimal point in the dividend two places to the right before we can divide, yielding $25\overline{)50}$ (B)

10.5.2.4. Demonstrate that the quotient becomes 2, the same answer we obtained in the money example (B)

10.5.2.5. Assign similar problems (B)

10.5.2.6. Visually check for understanding (B)

10.5.2.6.1. IF student demonstrates understanding, THEN step 11 (B)

10.5.2.6.2. IF student does not demonstrate understanding, THEN direct her to come to class during Muir Time and provide remediation (B)

11. Teach division of mixed numbers by mixed numbers

11.1. Pose problem such as $2\frac{1}{2} \div 1\frac{1}{4}$ (A, B, C)

11.2. Direct students to create a concrete representation with fraction strips (C)

11.2.1. Circulate among students to check for understanding (C)

11.2.2. Ask students with correct answers to recreate their representations on whiteboard with magnetic fraction strips (C)

11.2.3. Validate that correct answer is 2 (C)

11.2.4. Assign similar problems (C)

11.2.4.1. IF student demonstrates understanding, THEN step 11.2.5 (C)

11.2.4.2. IF student does not demonstrate understanding, THEN provide remediation after class (C)

11.2.5. Direct students to create a pictorial representation of same problem (B, C)

11.2.6. Circulate among students to check for understanding (B, C)

11.2.7. Ask students with correct answers to recreate their area models, circle models, or number lines on whiteboard (B, C)

11.2.8. Validate that correct drawings depict $2\frac{1}{2} \div 1\frac{1}{4} = 2$ (B, C)

11.2.9. Assign similar problems (B, C)

11.2.9.1. IF student demonstrates understanding, THEN step 11.3 (B, C)

11.2.9.2. IF student does not demonstrate understanding, THEN provide remediation after class (B, C)

11.3. Direct students to solve problem using the complex fraction method (B)

11.3.1. Remind students that we need to convert each term to an improper fraction, yielding $\frac{5}{2} \div \frac{5}{4}$ (B)

11.3.2. Demonstrate that we can convert this to a complex fraction, $\frac{\frac{5}{2}}{\frac{5}{4}}$ (B)

11.3.3. Remind students that we can multiply by $\frac{\frac{4}{5}}{\frac{4}{5}}$ to change the denominator to 1 (B)

11.3.4. Remind students that converting to a complex fraction and then converting the denominator to 1 is same as multiplying by the reciprocal (B)

11.3.5. Assign similar problems (B)

11.3.6. Visually check for understanding (B)

11.3.6.1. IF students demonstrate understanding, then go to step 11.4 (B)

11.3.6.2. IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)

11.4. Direct students to solve problem using the procedural method (A, B, C)

11.4.1. Circulate among students to check for understanding (A, B, C)

11.4.2. Ask students with correct answers to recreate their algorithms on whiteboard (A, B, C)

11.4.3. Validate that correct procedure involves converting $2\frac{1}{2}$ to $\frac{5}{2}$ and $1\frac{1}{4}$ to $\frac{5}{4}$, and then multiplying $\frac{5}{2}$ by the reciprocal of $\frac{5}{4}$ to yield $\frac{5}{2}$ x $\frac{4}{5} = \frac{20}{10}$ or 2 (A, B, C)

11.4.4. Assign similar problems (A, B, C)

11.4.4.1. IF student demonstrates understanding, THEN step 12 (A, B, C)

11.4.4.2. IF student does not demonstrate understanding, THEN provide remediation after class (A, B, C)

12. Teach division of fraction word problems

12.1.1. Pose problem such as, "If a recipe calls for $\frac{1}{2}$ cup of flour for one batch of bread, and I have $\frac{3}{4}$ cup of flour, how many batches of the recipe can I make?" (C)

12.1.2. Direct students to model the problem with manipulatives (C)

12.1.3. Circulate among students to check for understanding (C)

12.1.4. Ask students with correct representations to recreate their representations on whiteboard with magnetic fraction strips (C)

12.1.5. Validate correct concrete representation of $\frac{3}{4} \div \frac{1}{2}$ (C)

12.1.6. Direct students to model the problem pictorially (C)

12.1.7. Circulate among students to check for understanding (C)

12.1.8. Ask students with correct representations to recreate their representations on whiteboard with number lines or area models (C)

12.1.9. Validate correct pictorial representation of $\frac{3}{4} \div \frac{1}{2}$ (C)

12.1.10. Direct students to solve the problem with the previously discovered procedure (C)

12.1.11. Circulate among students to check for understanding (C)

12.1.12. Ask students with correct procedure to write their number sentences on the board (C)

12.1.13. Validate correct number sentence is $\frac{3}{4} \div \frac{1}{2} = \frac{3}{4}$ x $\frac{2}{1} = \frac{6}{4} = 1\frac{1}{2}$ (C)

12.1.14. Assign similar problems, asking students to create both a concrete and pictorial representation, and to solve using the number sentence procedure (C)

12.1.15. Circulate among students to check for understanding (C)

12.1.15.1. IF student demonstrates understanding, THEN end task (C)

12.1.15.2. IF student does not demonstrate understanding, THEN provide remediation after class (C)

Appendix F
Incremental Coding Spreadsheet

Spreadsheet Analysis: Gold Standard Protocol Procedures, Action and Decision Steps

			SME			Steps		Alignment
Step	Type	Final Gold Standard Protocol Analysis	A	B	C	A	D	1, 2, 3
		Procedure 1: Review concept of multiplication				40	6	
1	A	1.1 Explain that to model multiplication, we will construct rectangles (A)	1	0	0			1
2	A	1.1.1 Introduce notion of rectangles as area models for multiplication (A)	1	0	0			1
3	A	1.1.2 Review concept that area models are arrays with rows and columns (A)	1	0	0			1
4	A	1.1.2.1 Explain that the rows and columns correspond to the two multiplicands	1	0	0			1
5	A	1.1.3 Introduce the example of 2 x 3 (A)	1	0	0			1

6	A	1.1.3.1 Model how to draw the rows: 2 (first multiplicand) rectangles on top of each other (234-5) and that the 2 rectangles need to be long enough to accommodate the columns that correspond to the other multiplicand, 3 (A)	1	0	0			1
7	A	1.1.3.2 Ask, "What do we multiply those 2 rows by?" (A)	1	0	0			1
8	A	1.1.3.3 Call on students whose hands are not raised (A)	1	0	0			1
9	D	1.1.3.3.1 IF student has correct answer, THEN go to step 1.1.3.4 (A)	1	0	0			1
10	D	1.1.3.3.2 IF student has incorrect answer, THEN call on another student whose hand isn't raised, until a student gives the correct answer (A)	1	0	0			1
11	A	1.1.3.4 Check for understanding with discussion (A)	1	0	0			1
12	A	1.1.4 Model how to divide those two boxes by 3, the other multiplicand, so that there are 3 columns (A)	1	0	0			1

13	A	1.1.4.1 Demonstrate how to count the number of boxes created by the 2 rows and 3 columns, which is 6 (A)	1	0	0			1
14	A	1.1.4.2 Reinforce the point of the exercise that 2 rows by 3 columns is a graphical representation of 2x3=6 (A)	1	0	0			1
15	A	1.2 Repeat above sequence with a new multiplication problem with unequal multiplicands; students independently create area models corresponding to that problem (A)	1	0	0			1
16	A	1.3 Wander around and check visually (A)	1	0	0			1
17	A	1.4 Ask students, following several additional problems area models with unequal multiplicands, “What shape have we been creating?” (A)	1	0	0			1
18	D	1.4.1 IF students demonstrate understanding, THEN step 1.5 (A)	1	0	0			1
19	D	1.4.2 IF students are unclear, THEN give them a worksheet for homework (A)	1	0	0			1
20	A	1.5 Reinforce that all examples have been rectangles (A)	1	0	0			1

21	A	1.6 Ask, "What does it mean to be a rectangle?" (A)	1	0	0				1
22	A	1.7 Ask, "How do you know a rectangle when you see one?" (A)	1	0	0				1
23	A	1.8 Challenge students to create a definition of a rectangle (2 pairs of parallel sides, the opposites being equal in length, and 4 right angles) (A)	1	0	0				1
24	A	1.9 Reinforce that what you have built is a rectangle (A)	1	0	0				1
25	A	1.10 Introduce several multiplication problems with 2 equal multiplicands (A)	1	0	0				1
26	A	1.11 Ask students to create one area model with unequal multiplicands and then one area model with equal multiplicands (A)	1	0	0				1
27	A	1.12 Check visually (A)	1	0	0				1

28	A	1.12.1 Build a connection by asking, “What plane figure is created when all sides are equal?” (A)	1	0	0			1
29	A	1.12.2 Ask, after student response of “square” is elicited, “Is a square not also a rectangle?” (A)	1	0	0			1
30	A	1.12.3 Reinforce concept that a square is a rectangle with 4 equal sides (A)	1	0	0			1
31	A	1.12.4Display visual on board during discussion. (A)	1	0	0			1
32	A	1.12.5 Display all squares whose sides range in length from 1 to 10 units. (A)	1	0	0			1
33	A	1.12.6 Discuss that squares are a subset of rectangles (A)	1	0	0			1
34	A	1.12.7 Discuss here that all squares are rectangles (A)	1	0	0			1
35	D	1.12.7.1 IF students are struggling to conceptualize these understandings, problems. THEN give students a worksheet (A)	1	0	0			1

36	D	1.12.7.2 IF students understand concepts, THEN step1.13	1	0	0				1
37	A	1.13 Introduce a new problem with unequal multiplicands (e.g. 3x4) (A)	1	0	0				1
38	A	1.13.1 Ask students to compare their models with a neighbor (A)	1	0	0				1
39	A	1.13.2 Ask students to decide if their area model looks like their neighbor's (A)	1	0	0				1
40	A	1.13.2.1 Showcase students whose models differ (e.g. 3x4 area models versus 4x3 area models) (A)	1	0	0				1
41	A	1.13.2.2 Circulate and ask students with different models to draw their models on board and then have a discussion (A)	1	0	0				1
42	A	1.13.2.3 Ask, "If our multiplication problem is 3x4, how many boxes do we expect in our area model," the answer, of course to which is 12 (A)	1	0	0				1
43	A	1.13.2.4 Challenge students to pair with someone else with a differently shaped area model and establish that both have the same number of boxes (A)	1	0	0				1

44	A	1.13.2.5 Explain that 3x4 and 4x3 area models, although they are shaped differently, yield the same number of boxes (A)	1	0	0			1
45	A	1.13.2.5.1 Equate this with the notion of groups: a 3x4 area model is the same multiplication problem as a 4x3 area model (A)	1	0	0			1
46	A	1.13.2.5.2 Establish a connection to the understanding that “3 groups of 4 is the same idea as 4 groups of 3” and explain that this relationship between multiplicands is known as the commutative property of multiplication (A)	1	0	0			1
		Procedure 2: Review concepts of division				42	10	
47	A	2.1 Explain that division is the inverse of multiplication so we have to work backwards in applying multiplication to division (A)	1	0	0			1
48	A	2.1.1 Explain this involves changing the product of our multiplication problem so that it is now the dividend of our division problem (A)	1	0	0			1
49	A	2.1.2 Demonstrate that 2x3=6, and working backward, this is related to the equation 6÷3=2 (A)	1	0	0			1

50	A	2.1.3 Pose division problem with a quotient greater than one, such as 12÷2 (A, C)	1	0	1			2
51	D	2.1.3.1 IF students derive correct answer, 6, THEN step 2.1.4 (A)	1	0	0			1
52	D	2.1.3.2 IF students seem unclear, THEN step 2.1 (A)	1	0	0			1
53	A	2.1.4 Ask students, "What is the meaning of this problem?" (A, C)	1	0	1			2
54	A	2.1.5 Direct students to discuss this with a neighbor (C)	0	0	1			1
55	A	2.1.6 Check for understanding by calling on students randomly (A, C)	1	0	1			2
56	D	2.1.6.1.1 IF student explains that she decided how many times 2 fits into 12, THEN go to step 2.1.8 (A)	1	0	0			1

57	D	2.1.6.1.2 IF student says she broke the dividend into the number of groups indicated by the divisor and determined how many were in each group THEN step 2.1.6.2 (A)	1	0	0			1
58	A	2.1.6.2 Realize that this is a pattern of student thought that needs to be rewired (A)	1	0	0			1
59	A	2.1.6.3 Explain that because multiplication is commutative there are two ways to think about division (A)	1	0	0			1
60	A	2.1.6.3.1 Explain that one way is to break the dividend into the number of groups indicated by the divisor and then determine the size of each group (A)	1	0	0			1
61	A	2.1.6.3.2 Explain that this is a method that we are going to avoid (A)	1	0	0			1
62	A	2.1.6.3.3 Explain that the other method is to determine how many times the divisor fits into the dividend, just as we do in long division and then show an example on board (A, C)	1	0	1			2
63	A	2.1.6.3.4 Establish the expectation that this is the way students need to begin to think about division (A, C)	1	0	1			2

64	A	2.1.7 Pose new problem, such as 12÷2 (A, C)	1	0	1			2
65	A	2.1.7.1 Ask students to explain what the meaning of this problem is (A,C)	1	0	1			2
66	D	2.1.7.1.1 IF student says she broke the dividend into the number of groups indicated by the divisor and determined how many were in each group THEN step 2.1.6.2 (A)	1	0	0			1
67	D	2.1.7.1.2 IF student says she determined how many times the divisor fit into the dividend, THEN step 2.1.8 (A)	1	0	0			1
68	A	2.1.8 Ask students to formulate a real-life example of the problem (C)	0	0	1			1
69	A	2.1.9 Direct students to discuss this with a partner (C)	0	0	1			1
70	A	2.1.10 Check for understanding by calling on students randomly (C)	0	0	1			1

71	A	2.1.11 Highlight correct examples, such as, “If Juan has $12, how many $2 bills would be needed to match that amount?” (C)	0	0	1			1
72	A	2.2 Continue to reinforce that the goal of division is not to determine how many equal groups are indicated by the divisor, but rather, to determine how many times the divisor fits into the dividend, as in long division (A, C)	1	0	1			2
73	A	2.3 Conclude lesson with a similar problem, to which students must provide a written answer (A, C)	1	0	1			2
74	A	2.4 Review all answers (A, C)	1	0	1			2
75	D	2.4.1 IF more than 3 answers indicate misunderstanding, THEN begin next class session with a review of the concept of the meaning of division (A, C)	1	0	1			2
76	D	2.4.2 IF only 2 or 3 answers indicate misunderstanding, THEN sit with these students during next class session and provide remediation (A, C)	1	0	1			2
77	A	2.5 Pose division problem with a quotient less than one, such as 1÷2 (C)	0	0	1			1

78	A	2.5.1 Ask students, “What is the meaning of this problem?” (A,C)	1	0	1			2
79	A	2.5.2 Direct students to discuss this with a neighbor (C)	0	0	1			1
80	A	2.5.3 Check for understanding by calling on students randomly (C)	0	0	1			1
81	A	2.5.4 Entertain a discussion centering on the concept that the problem involves determining how many groups of 2 are in 1 (A, C)	1	0	1			2
82	A	2.5.5 Ask students to formulate a real-life example of the problem (C)	0	0	1			1
83	A	2.5.6 Direct students to discuss this with a partner (C)	0	0	1			1
84	A	2.5.7 Check for understanding by calling on students randomly (C)	0	0	1			1
85	A	2.5.8 Highlight correct examples, such as, “If Juan has 1 bagel, how many groups of 2 bagels are in that 1 bagel?” (C)	0	0	1			1

86	A	2.6 Pose another similar problem, such as 1÷3 (C)	0	0	1			1
87	A	2.6.1 Ask students, "What is the meaning of this problem?" (A, C)	1	0	1			2
88	A	2.6.2 Direct students to discuss this with a neighbor (C)	0	0	1			1
89	A	2.6.3 Check for understanding by calling on students randomly (C)	0	0	1			1
90	A	2.6.4 Entertain a discussion centering on the concept that the problem involves determining how many groups of 3 are in 1 (A, C)	1	0	1			2
91	A	2.6.5 Ask students to formulate a real-life example of the problem (C)	0	0	1			1
92	A	2.6.6 Direct students to discuss this with a partner (C)	0	0	1			1
93	A	2.6.7 Check for understanding by calling on students randomly (C)	0	0	1			1

94	A	2.6.8 Highlight correct examples, such as, "If Rachel has 1 bagel, how many groups of 3 bagels are in that 1 bagel (C)	0	0	1			1
95	A	2.7 Conclude lesson with a similar problem, to which students must provide a written answer (C)	0	0	1			1
96	A	2.8 Review all answers (C)	0	0	1			1
97	D	2.8.1 IF a majority of answers indicate misunderstanding, THEN begin next class session with a review of the concept of the meaning of division (C)	0	0	1			1
98	D	2.8.2 IF only 2 or 3 answers indicate misunderstanding, THEN sit with these students during next class session and provide remediation (C)	0	0	1			1
		Procedure 3: Teach operations with integers				33	8	
99	A	3.1 Teach integer operations using a number line (B)	0	1	0			1

100	A	3.1.1 Demonstrate addition of a negative: 4 + -2 involves starting at 4 and going back 2 on number line to 2 (B)	0	1	0				1
101	A	3.1.2 Demonstrate addition of a positive: -2 + 7 involves starting at -2 and going forward 7 to 5 (B)	0	1	0				1
102	A	3.1.3 Demonstrate subtraction of a positive: -4 – 2 involves starting at -4 and going back 2 to -6 (B)	0	1	0				1
103	A	3.1.4 Demonstrate subtraction of a negative: -4 – (-2) is the opposite of -4 -2, so we go forward 2 to -2 (B)	0	1	0				1
104	A	3.1.5 Assign similar problems for homework (B)	0	1	0				1
105	A	3.1.6 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0				1
106	D	3.1.6.1 IF student demonstrates understanding, THEN go to step 3.2 (B)	0	1	0				1
107	D	3.1.6.2 IF student's answers do not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)	0	1	0				1

108	A	3.2 Teach integer operations using manipulatives (B)	0	1	0			1
109	A	3.2.1 Distribute integer chips (B)	0	1	0			1
110	A	3.2.2 Explain that the black side indicates positive and the red side indicates negative (B)	0	1	0			1
111	A	3.2.3 Demonstrate addition (B)	0	1	0			1
112	A	3.2.3.1 Show that -4 + 2 can be represented as four red chips and two black chips (B)	0	1	0			1
113	A	3.2.3.2 Match one red chip and one black chip to indicate zero, based on the inverse property of addition (B)	0	1	0			1
114	A	3.2.3.3 Repeat for the other black chip to yield a total of two zeroes, which yields a sum of -2 (B)	0	1	0			1
115	A	3.2.4 Demonstrate subtraction (B)	0	1	0			1

116	A	3.2.4.1 Show that -4 – 2 can be represented as four red chips (B)	0	1	0			1
117	A	3.2.4.2 Explain that it's impossible to match negatives and positives because you only have 4 black chips (B)	0	1	0			1
118	A	3.2.4.3 Explain that identity property of addition says that adding zero to a number results in no change to that number (B)	0	1	0			1
119	A	3.2.4.4 Demonstrate placing two zero pairs (one black and one red chip) next to the four red chips (B)	0	1	0			1
120	A	3.2.4.5 Demonstrate that you can now subtract the two positive (black) chips and the resulting value is -6 (B)	0	1	0			1
121	A	3.2.5 Assign similar problems for homework (B)	0	1	0			1
122	A	3.2.6 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0			1
123	D	3.2.6.1 IF student demonstrates understanding, THEN go to step 3.2.7 (B)	0	1	0			1

124	D	3.2.6.2 IF student's answers do not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)	0	1	0			1
125	A	3.2.7 Demonstrate multiplication (B)	0	1	0			1
126	A	3.2.7.1 Show that 3 x -2 can be represented as three groups of two red chips, which is -6 (B)	0	1	0			1
127	A	3.2.7.2 Show that -3 x -2 would be the opposite (additive inverse) of 3 x -2 (B)	0	1	0			1
128	A	3.2.7.3 Demonstrate that we can derive the opposite of 3 x -2 by simply flipping the integer chips over, yielding 6 positives, which signifies that -3 x -2 equals positive 6 (B)	0	1	0			1
129	A	3.2.8 Demonstrate division (B)	0	1	0			1
130	A	3.2.8.1 Show that -6 ÷ 2 can be represented as six red chips (B)	0	1	0			1

131	A	3.2.8.2 Ask, "How many groups of positive 2 can I make out of these 6 negatives?" (B)	0	1	0			1
132	A	3.2.8.3 Explain that the answer is -3 (B)	0	1	0			1
133	A	3.2.8.4 Show next that -6 ÷ -2 would be the opposite (additive inverse) of -6 ÷ 2 by flipping the six red tiles (-6) over and then showing that the answer is 3 positive groups (B)	0	1	0			1
134	A	3.2.8.5 Assign similar problems for homework (B)	0	1	0			1
135	A	3.2.8.6 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0			1
136	D	3.2.8.6.1 IF student demonstrates understanding, THEN go to step 4 (B)	0	1	0			1
137	D	3.2.8.6.2 IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)	0	1	0			1

138	D	3.2.8.6.2.1 IF student demonstrates understanding, THEN go to step 3 (B)	0	1	0			1
139	D	3.2.8.6.2.2 IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1
		Procedure 4: Teach number domains				31	20	
140	A	4.1 Explain that before we can begin operations with fractions, we need to understand the different types of numbers (B)	0	1	0			1
141	A	4.2 Review that whole numbers are 0,1,2,3,4…to infinity (B)	0	1	0			1
142	A	4.3 Review that integers are whole numbers and their opposites, such as 1 and -1, 2 and -2, etc. (B)	0	1	0			1
143	A	4.4 Teach that in between the whole numbers and integers are other numbers (B)	0	1	0			1
144	A	4.5 Explain that these are called rational numbers, defined as any number that can be written as a fraction (B)	0	1	0			1

145	A	4.6 Explain that whole numbers and integers are also rational numbers, because they can also be written as fractions (B)	0	1	0			1
146	A	4.7 Explain that decimals are also rational numbers, because they can be written as fractions (B)	0	1	0			1
147	A	4.71 Display the example of 0.7 (B)	0	1	0			1
148	A	4.72 Ask students to write on their whiteboards the name of the place the 7 occupies (B)	0	1	0			1
149	D	4.7.2.1 IF student understands it is in the tenths place, THEN go to step 4.7.3 (B)	0	1	0			1
150	D	4.7.2.2 If student does not understand it is in the tenths place, check in with student before end of class to provide brief review (B)	0	1	0			1
151	A	4.73 Ask students to write how 0.7 is read on their whiteboards (B)	0	1	0			1
152	D	4.7.3.1 IF student understands it is read as “seven tenths”, THEN go to step 4.7.4 (B)	0	1	0			1

153	D	4.7.3.2 IF student does not understand it is read as “seven tenths”, then check in with student before end of class to provide brief review (B)	0	1	0			1
154	A	4.7.4 Ask students to write 0.7 as a fraction on their whiteboards (B)	0	1	0			1
155	D	4.7.4.1 IF student understands it is written as 7/10 THEN go to step 4.7.5 (B)	0	1	0			1
156	D	4.7.4.2 IF student does not understand it is written as 7/10 THEN check in with student before end of class and provide brief review (B)	0	1	0			1
157	A	4.7.5 Ask students to write 0.03 as a fraction on their whiteboards (B)	0	1	0			1
158	D	4.7.5.1 IF student understands it is written as 3/100 then go to step 4.7.6 (B)	0	1	0			1
159	D	4.7.5.2 IF student does not understand it is written as 3/100 THEN check in with her before end of class and provide brief review (B)	0	1	0			1
160	A	4.7.6 Ask students to write 0.004 as a fraction on their whiteboards (B)	0	1	0			1

161	D	4.7.6.1 IF student understands it is written as 4/1000 THEN go to step 4.7.7 (B)	0	1	0			1
162	D	4.7.6.2 IF student does not understand it is written as 4/1000 THEN check in with her before end of class and provide brief review (B)	0	1	0			1
163	A	4.7.7 Review that the place of the number on the right in any decimal determines what the denominator will be when it is converted to a fraction (B)	0	1	0			1
164	A	4.8 Review procedure for converting a fraction to a decimal (B)	0	1	0			1
165	A	4.8.1 Review that we need a power of 10 in the denominator in order to convert many fractions to a decimal (B)	0	1	0			1
166	A	4.8.2 Demonstrate that a fraction such as 3/5 needs to be converted to an equivalent fraction with a power of 10, in this case 10, in the denominator (B)	0	1	0			1
167	A	4.8.3 Demonstrate multiplying 3/5 x 2/2 to yield 6/10 (B)	0	1	0			1

168	A	4.8.4 Demonstrate that this converts to the decimal 0.6 (B)	0	1	0			1
169	A	4.8.5 Assign similar problems (B)	0	1	0			1
170	A	4.8.6 Visually check for understanding (B)	0	1	0			1
171	D	4.8.6.1 IF students demonstrate understanding, THEN go to step 4.8.7 (B)	0	1	0			1
172	D	4.8.6.2 IF student does not demonstrate understanding, THEN check in with her before end of class to provide brief review (B)	0	1	0			1
173	A	4.8.7 Review that some fractions, such as 1/16 , have denominators that cannot be converted to a power of 10 (B)	0	1	0			1
174	A	4.8.8 Review that with fractions such as these, we can divide the numerator by the denominator, such that 1÷ 16 = .0625 (B)	0	1	0			1
175	A	4.8.9 Assign similar problems (B)	0	1	0			1

176	D	4.8.9.1 IF student demonstrates understanding, THEN go to step 4.8.10 (B)	0	1	0			1
177	D	4.8.9.2 IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)	0	1	0			1
178	A	4.8.10 Review that some fractions, such as 1/3 yield non-terminating, repeating decimals (B)	0	1	0			1
179	A	4.8.11 Demonstrate that, again, we can divide the numerator by the denominator, such that 1 ÷ 3 = 0.3333 ?, which repeats and does not terminate, as indicated by the superscript (B)	0	1	0			1
180	A	4.8.12 Assign similar problems (B)	0	1	0			1
181	D	4.8.12.1 IF student demonstrates understanding, THEN go to step 4.8.13 (B)	0	1	0			1
182	D	4.8.12.2 IF student does not demonstrate understanding, THEN check in with her before end of class and provide brief review (B)	0	1	0			1

183	A	4.8.13 Assign homework problems for decimal/fraction concepts (B)	0	1	0			1
184	A	4.8.14 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0			1
185	D	4.8.14.1 IF student demonstrates understanding, THEN go to step 4.9 (B)	0	1	0			1
186	D	4.8.14.2 IF student does not demonstrate understanding, THEN check in with her before end of class to check for understanding (B)	0	1	0			1
187	D	4.8.14.2.1 IF student demonstrates understanding, THEN go to step 4.9 (B)	0	1	0			1
188	D	4.8.14.2.2 IF student does not demonstrate understanding, THEN direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1
189	A	4.9 Explain that not all rational numbers can be integers or whole numbers (B)	0	1	0			1

190	A	4.10 Explain that a number that cannot be written as a fraction (a non-repeating, non-terminating decimal, such as square root of 12, is an irrational number (B)	0	1	0			1
		Procedure 5: Review representations of fractions				22	14	
191	A	5.1 Explain that, although this review involves fractions as parts of a whole, later we will be dealing with fraction concepts involving ratios and proportions (B)	0	1	0			1
192	A	5.2 Remind students that the concept of fractions as parts of a whole requires us to split up that whole (B)	0	1	0			1
193	A	5.3 Distribute fraction circle manipulatives (B)	0	1	0			1
194	A	5.4 Ask students to represent 3/8 (B)	0	1	0			1
195	A	5.4.1 Visually check for understanding (B)	0	1	0			1

196	D	5.4.1.1 IF student demonstrates understanding, THEN go to step 5.5 (B)	0	1	0			1
197	D	5.4.1.2 IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)	0	1	0			1
198	A	5.5 Ask students to represent 1/2 (B)	0	1	0			1
199	A	5.5.1 Visually check for understanding (B)	0	1	0			1
200	D	5.5.1.1 IF student demonstrates understanding, THEN go to step 5.6 (B)	0	1	0			1
201	D	5.5.1.2 IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)	0	1	0			1
202	A	5.6 Ask students to show another way to make 1/2 (B)	0	1	0			1

203	A	5.7 Explain to students that representing 1/2 by its equivalents, such as 2/4 or 3/6, exemplifies the identity property of multiplication, which states that multiplying any number by a form of 1 yields that same number (B)	0	1	0				1
204	A	5.8 Ask students, “How do you write an equivalent fraction for 1/2 ?” (B)	0	1	0				1
205	A	5.9 Remind students that you do that by multiplying by a form of 1, such as 2/2, resulting in the equivalent fraction, 2/4 (B)	0	1	0				1
206	A	5.10 Assign similar problems (B)	0	1	0				1
207	A	5.10.1 Visually check for understanding (B)	0	1	0				1
208	D	5.10.1.1 IF student demonstrates understanding, THEN go to step 5.11 (B)	0	1	0				1
209	D	5.10.1.2 IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)	0	1	0				1

210	A	5.11 Distribute fraction strip manipulatives (B)	0	1	0				1
211	A	5.12 Ask students, in groups, or with partners to represent 1/2 (B)	0	1	0				1
212	A	5.12.1 Visually check for understanding (B)	0	1	0				1
213	D	5.12.1.1 IF student demonstrates understanding, THEN go to step 5.13 (B)	0	1	0				1
214	D	5.12.1.2 IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)	0	1	0				1
215	A	5.13 Ask students to demonstrate as many ways as possible to represent 1/2 (B)	0	1	0				1
216	A	5.13.1 Visually check for understanding (B)	0	1	0				1
217	D	5.13.1.1 IF student demonstrates understanding, THEN go to step 5.14 (B)	0	1	0				1

218	D	5.13.1.2 IF student does not demonstrate understanding, THEN check in with student before end of class to provide brief review (B)	0	1	0			1
219	A	5.14 Remind students that dividing any number by a form of 1 is identical to the identity property of multiplication: the result in both cases is the original number (B)	0	1	0			1
220	A	5.15 Distribute fraction circle manipulatives (B)	0	1	0			1
221	A	5.16 Ask students to represent a fraction such as 4/8 in simplest terms (B)	0	1	0			1
222	A	5.17 Visually check for understanding (B)	0	1	0			1
223	D	5.17.1 IF student demonstrates understanding, THEN go to step 6 (B)	0	1	0			1
224	D	5.17.2 IF student does not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)	0	1	0			1
225	D	5.17.2.1 IF student demonstrates understanding of all elements of step 5, THEN go to step 6 (B)	0	1	0			1

226	D	5.17.2.2 IF student does not demonstrate understanding of all elements of step 4, THEN direct her to come to class during Muir time and provide remediation (B)	0	1	0			1
		Procedure 6: Review addition and subtraction of fractions and mixed numbers				103	40	
227	A	6.1 Review addition and subtraction with like denominators (A, B)	1	1	0			2
228	A	6.2 Distribute manipulative fraction strips and circles (A, B)	1	1	0			2
229	A	6.2.1 Ask students to model a problem such as 1/2 + 1/2 with strips and circles (B)	0	1	0			1
230	A	6.2.2 Ask students to demonstrate that the answer, two halves, is visually equivalent to one whole (B)	0	1	0			1
231	A	6.2.3 Ask students to divide by a fraction equal to 1, such as 2/2 , to also yield one whole (A, B)	1	1	0			2
232	A	6.2.4 Ask students to model a problem such as 3/8 + 1/8 (B)	0	1	0			1

233	A	6.2.5 Ask students to demonstrate that the answer, four eighths, is visually equivalent to 1/2 (B)	0	1	0			1
234	A	6.2.6 Ask students to divide by a fraction equal to 1, such as 4/4 , to also yield 1/2 (A, B)	1	1	0			2
235	D	6.2.6.1 IF students demonstrate understanding, then go to step 6.2.7 (B)	0	1	0			1
236	D	6.2.6.2 IF student appears unclear, then check in with her before end of class to provide brief review (B)	0	1	0			1
237	A	6.2.7 Ask students to model a problem, such as 5/8 - 1/8 (B)	0	1	0			1
238	A	6.2.8 Ask students to demonstrate that the answer is visually equivalent to 1/2 (B)	0	1	0			1
239	A	6.2.9 Ask students to divide by a fraction equal to 1, such as 4/4 , to also yield 1/2 (A, B)	1	1	0			2
240	D	6.2.9.1 IF students demonstrate understanding, then go to step 6.3 (B)	0	1	0			1

241	D	6.2.9.2 IF student appears unclear, THEN check in with her before end of class to provide brief review (B)	0	1	0			1
242	A	6.3 Review algorithm for adding and subtracting fractions (A, B)	1	1	0			2
243	A	6.3.1 Ask students to add two fractions with like denominators, such as 1/4 + 1/4 (A, B)	1	1	0			2
244	A	6.3.2 Visually check that students are adding numerators to yield 2/4 (A, B)	1	1	0			2
245	A	6.3.3 Ask students to divide by a fraction equal to 1, such as 2/2 , to put in simplest terms, 1/2 (A, B)	1	1	0			2
246	A	6.3.4 Ask students to subtract two fractions with like denominators, such as 3/6 - 1/6 (B)	0	1	0			1
247	A	6.3.5 Ask students to divide by a fraction equal to 1, such as 2/2 , to put in simplest terms, 1/3 (A, B)	1	1	0			2
248	A	6.3.5.1 Assign similar problems for homework (B)	0	1	0			1

249	A	6.3.5.2 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0			1
250	D	6.3.5.2.1 IF student demonstrates understanding, THEN go to step 6.4 (B)	0	1	0			1
251	D	6.3.5.2.2 IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)	0	1	0			1
252	D	6.3.5.2.2.1 IF student demonstrates understanding, THEN go to step 6.4 (B)	0	1	0			1
253	D	6.3.5.2.2.2 IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1
254	A	6.4 Review addition and subtraction of fractions with unlike denominators (A, B, C)	1	1	1			3
255	A	6.4.1 Demonstrate adding two fractions with unlike denominators, such as 1/2 + 1/4 (A, B, C)	1	1	1			3
256	A	6.4.2 Review that we can't add them unless the denominators are the same (A, B, C)	1	1	1			3

257	A	6.4.3 Review that the identify property of multiplication means that we can multiply 1/2 by a form of one, 2/2 to yield a fraction with a denominator of 4, 2/4 (A, B)	1	1	0			2
258	A	6.4.4 Review that we can now add 2/4 + 1/4 = 3/4 (A, B, C)	1	1	1			3
259	A	6.4.5 Assign similar problems (A, B, C)	1	1	1			3
260	D	6.4.5.1 IF students demonstrate understanding, then go to step 6.4.6 (B)	0	1	0			1
261	D	6.4.5.2 IF student appears unclear, then check in with her before end of class to provide brief review (B)	0	1	0			1
262	A	6.4.6 Demonstrate subtracting two fractions with unlike denominators, such as 1/2 - 1/4 (B)	0	1	0			1
263	A	6.4.7 Review that the identify property of multiplication means that we can multiply 1/2 by a form of one, 2/2 to yield a fraction with a denominator of 4, 2/4 (A,B)	1	1	0			2
264	A	6.4.8 Review that we can now subtract 2/4 - 1/4 = 1/4 (B)	0	1	0			1

265	A	6.4.9 Assign similar problems (B)	0	1	0			1
266	D	6.4.9.1 IF students demonstrate understanding, then go to step 6.4.10 (B)	0	1	0			1
267	D	6.4.9.2 IF student appears unclear, then check in with her before end of class to provide brief review (B)	0	1	0			1
268	A	6.4.10 Assign addition and subtraction of fractions with unlike denominators for homework (B)	0	1	0			1
269	A	6.4.11 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0			1
270	D	6.4.11.1 IF student demonstrates understanding, THEN go to step 6.5 (B)	0	1	0			1
271	D	6.4.11.2 IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)	0	1	0			1
272	D	6.4.11.2.1 IF student demonstrates understanding, THEN go to step 6.5 (B)	0	1	0			1

273	D	6.4.11.2.2 IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1
274	A	6.5 Review addition and subtraction of mixed numbers with like denominators (B)	0	1	0			1
275	A	6.5.1 Demonstrate 21/2 + 41/2 using hand-drawn circles to represent the wholes and fractions (B)	0	1	0			1
276	A	6.5.2 Review that the two fractions add to one whole, which we must add to the other six wholes, to yield 7 (B)	0	1	0			1
277	A	6.5.3 Demonstrate 13/5 + 24/5 using hand-drawn circles (B)	0	1	0			1
278	A	6.5.4 Review that this yields 3 wholes and seven-fifths (B)	0	1	0			1
279	A	6.5.5 Review through use of fraction strips that 7/5 is the same as one whole and 2/5 (B)	0	1	0			1
280	A	6.5.6 Demonstrate that we now add the 1 whole to the other 3 whole to yield 4 whole, and then add 2/5 , resulting in 42/5 (B)	0	1	0			1

281	A	6.5.7 Demonstrate with fraction strips the corollary of this concept: 12/5 can be converted to the improper fraction, 7/5 (B)	0	1	0			1
282	A	6.5.8 Demonstrate 61/8 - 45/8 using hand-drawn circles (B)	0	1	0			1
283	A	6.5.9 Review that one whole can be represented by any number over itself, such as 6/6 or 7/7 or 8/8 (B)	0	1	0			1
284	A	6.5.10 Review that we need to borrow one whole from the 6, which we can represent as 8/8 (B)	0	1	0			1
285	A	6.5.11 Review that we must add the 8/8 we borrowed to the 1/8 to yield 9/8 (B)	0	1	0			1
286	A	6.5.12 Demonstrate that we can now subtract to yield 14/8 (B)	0	1	0			1
287	A	6.5.13 Review that we need to simplify the 4/8 by dividing by a form of one, 4/4 , yielding 11/2 (B)	0	1	0			1
288	A	6.5.14 Assign similar problems (B)	0	1	0			1

289	A	6.5.15 Visually check for understanding (B)	0	1	0			1
290	D	6.5.15.1 IF students demonstrate understanding, then go to step 6.5.16 (B)	0	1	0			1
291	D	6.5.15.2 IF student appears unclear, then check in with her before end of class to provide brief review (B)	0	1	0			1
292	A	6.5.16 Assign addition and subtraction of mixed numbers with like denominators for homework (B)	0	1	0			1
293	A	6.5.17 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0			1
294	D	6.5.17.1 IF student demonstrates understanding, THEN go to step 6.6 (B)	0	1	0			1
295	D	6.5.17.2 IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)	0	1	0			1
296	D	6.5.17.2.1 IF student demonstrates understanding, THEN go to step 6.6 (B)	0	1	0			1

297	D	6.5.17.2.2 IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1
298	A	6.6 Review addition and subtraction of mixed numbers with unlike denominators (B)	0	1	0			1
299	A	6.6.1 Demonstrate 11/2 + 21/4 using hand-drawn circles (B)	0	1	0			1
300	A	6.6.2 Review that we cannot add fractions unless we have the same denominators (A, B, C)	1	1	1			3
301	A	6.6.3 Review that we can multiply by a form of one, 2/2 , to yield 12/4 (B)	0	1	0			1
302	A	6.6.4 Demonstrate adding 12/4 + 21/4 to yield 33/4 (B)	0	1	0			1
303	A	6.6.5 Demonstrate subtracting 31/2 - 11/4 using hand-drawn circles (B)	0	1	0			1
304	A	6.6.6 Review that we cannot subtract fractions unless we have the same denominators (B)	0	1	0			1
305	A	6.6.7 Review that we can multiply by a form of one, 2/2 , to yield 32/4 (A, B)	1	1	0			2

306	A	6.6.8 Demonstrate subtracting 32/4 - 11/4 to yield 21/4 (B)	0	1	0			1
307	A	6.6.9 Assign similar problems (B)	0	1	0			1
308	A	6.6.10 Visually check for understanding (B)	0	1	0			1
309	A	6.6.10.1 IF students demonstrate understanding, then go to step 6.6.11 (B)	0	1	0			1
310	D	6.6.10.2 IF student appears unclear, then check in with her before end of class to provide brief review (B)	0	1	0			1
311	D	6.6.11 Assign addition and subtraction of mixed numbers with unlike denominators for homework (B)	0	1	0			1
312	A	6.6.12 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0			1
313	D	6.6.12.1 IF student demonstrates understanding, THEN go to step 6.7 (B)	0	1	0			1
314	D	6.6.12.2 IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)	0	1	0			1

315	D	6.6.12.2.1 IF student demonstrates understanding, THEN go to step 6.7 (B)	0	1	0			1
316	D	6.6.12.2.2 IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1
317	A	6.7 Review that because rational numbers can be written as both fractions and decimals, we can add and subtract fractions and mixed numbers by converting them to decimals (B)	0	1	0			1
318	A	6.7.1 Demonstrate this for fraction addition using an example such as 1/4 + 1/4 = 1/2 (B)	0	1	0			1
319	A	6.7.2 Review that 1/4 needs to be converted to a fraction with a denominator that is a power of 10 (B)	0	1	0			1
320	A	6.7.3 Demonstrate that multiplying 1/4 by 25/25 yields 25/100 , which in decimal form is .25 (B)	0	1	0			1
321	A	6.7.4 Review that to add .25 + .25 we must line up the decimals and places, then add, yielding a sum of .5 (B)	0	1	0			1

322	A	6.7.5 Demonstrate that .5 is the same as five tenths, or 5/10 , which can be reduced to simplest terms as 1/2 (B)	0	1	0			1
323	A	6.7.6 Demonstrate that we can derive the same answer, .5, by converting our answer in 1/4 + 1/4 = 1/2 (B)	0	1	0			1
324	A	6.7.7 Demonstrate that if we convert 1/2 to a fraction with a power of 10 in the denominator, we get 5/10 (B)	0	1	0			1
325	A	6.7.8 Review that 5/10 is the same as .5 (B)	0	1	0			1
326	A	6.7.9 Reinforce idea that regardless of whether we add rational numbers as fractions or decimals, the answer is the same (B)	0	1	0			1
327	A	6.7.10 Assign similar problems(B)	0	1	0			1
328	D	6.7.10.1 IF students demonstrates understanding, THEN go to step 6.7.11 (B)	0	1	0			1
329	D	6.7.10.2 IF student does not demonstrate understanding, THEN check in with her before end of class and provide brief review (B)	0	1	0			1

330	A	6.7.11 Demonstrate same concept for subtraction using an example such as 3/4 - 1/4 = 2/4 , or 1/2 (B)	0	1	0			1
331	A	6.7.12 Demonstrate that multiplying 3/4 and 1/4 by 25/25 yields 75/100 and 25/100 , which in decimal form are .75 and .25 (B)	0	1	0			1
332	A	6.7.13 Review that to subtract .75 - .25 we must line up the decimals and places, then subtract, yielding a difference of .5 (B)	0	1	0			1
333	A	6.7.14 Demonstrate that .5 is the same as five tenths, or 5/10 , which can be reduced to simplest terms as 1/2 (B)	0	1	0			1
334	A	6.7.15 Demonstrate that we can derive the same answer, .5, by converting our answer in 3/4 - 1/4 = 1/2 (B)	0	1	0			1
335	A	6.7.16 Demonstrate that if we convert 1/2 to a fraction with a power of 10 in the denominator, we get 5/10 (B)	0	1	0			1
336	A	6.7.17 Review that 5/10 is the same as .5 (B)	0	1	0			1
337	A	6.7.18 Reinforce idea that regardless of whether we subtract rational numbers as fractions or decimals, the answer is the same (B)	0	1	0			1

338	A	6.7.19 Assign similar problems (B)	0	1	0				1
339	D	6.7.19.1 IF students demonstrates understanding, THEN go to step 6.7.20 (B)	0	1	0				1
340	D	6.7.19.2 IF student does not demonstrate understanding, THEN check in with her before end of class and provide brief review (B)	0	1	0				1
341	A	6.7.20 Demonstrate same concept for addition of mixed numbers using an example such as 11/4 + 11/4 = 22/4 , or 21/2 (B)	0	1	0				1
342	A	6.7.21 Review that 1/4 needs to be converted to a fraction with a denominator that is a power of 10 (B)	0	1	0				1
343	A	6.7.22 Demonstrate that multiplying 1/4 by 25/25 yields 25/100 , which in decimal form is .25 (B)	0	1	0				1
344	A	6.7.23 Review that to add 1.25 + 1.25 we must line up the decimals and places, then add, yielding a sum of 2.5 (B)	0	1	0				1
345	A	6.7.24 Demonstrate that 2.5 is the same as two and five tenths, or 25/10 , which can be reduced to simplest terms as 21/2 (B)	0	1	0				1

346	A	6.7.25 Demonstrate that we can derive the same answer, 2.5, by converting our answer in 11/4 + 11/4 = 21/2 (B)	0	1	0			1
347	A	6.7.26 Demonstrate that if we convert 21/2 to a fraction with a power of 10 in the denominator, we get 25/10 (B)	0	1	0			1
348	A	6.7.27 Review that 25/10 is the same as 2.5 (B)	0	1	0			1
349	A	6.7.28 Reinforce idea that regardless of whether we add rational numbers as fractions or decimals, the answer is the same (B)	0	1	0			1
350	A	6.7.29 Assign similar problems (B)	0	1	0			1
351	D	6.7.29.1 IF students demonstrates understanding, THEN go to step 6.7.30 (B)	0	1	0			1
352	D	6.7.29.2 IF student does not demonstrate understanding, THEN check in with her before end of class and provide brief review(B)	0	1	0			1
353	A	6.7.30 Demonstrate same concept for subtraction of mixed numbers using an example such as 23/4 - 11/4 = 12/4 , or 11/2 (B)	0	1	0			1

354	A	6.7.31 Demonstrate that multiplying 3/4 and 1/4 by 25/25 yields 75/100 and 25/100 , which in decimal form are .75 and .25 (B)	0	1	0			1
355	A	6.7.32 Review that to subtract 2.75 - 1.25 we must line up the decimals and places, then subtract, yielding a difference of 1.5 (B)	0	1	0			1
356	A	6.7.33 Demonstrate that 1.5 is the same as one and five tenths, or 15/10 , which can be reduced to simplest terms as 11/2 (B)	0	1	0			1
357	A	6.7.34 Demonstrate that we can derive the same answer, 1.5, by converting our answer in 23/4 - 11/4 = 11/2 (B)	0	1	0			1
358	A	6.7.35 Demonstrate that if we convert 11/2 to a fraction with a power of 10 in the denominator, we get 15/10 (B)	0	1	0			1
359	A	6.7.36 Review that 15/10 is the same as 1.5 (B)	0	1	0			1
360	A	6.7.37 Reinforce idea that regardless of whether we subtract rational numbers as fractions or decimals, the answer is the same (B)	0	1	0			1
361	A	6.7.38 Assign similar problems (B)	0	1	0			1

362	D	6.7.38.1 IF students demonstrates understanding, THEN go to step 6.7.39 (B)	0	1	0			1
363	D	6.7.38.2 IF student does not demonstrate understanding, THEN check in with her before end of class and provide brief review (B)	0	1	0			1
364	A	6.7.39 Assign homework problems for decimal/fraction equivalence (B)	0	1	0			1
365	A	6.7.40 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0			1
366	D	6.7.40.1 IF student demonstrates understanding, THEN go to step 7 (B)	0	1	0			1
367	D	6.7.40.2 IF student does not demonstrate understanding, THEN check in with her before end of class to check for understanding (B)	0	1	0			1
368	D	6.7.40.2.1 IF student demonstrates understanding, THEN go to step 7 (B)	0	1	0			1
369	D	6.7.40.2.2 IF student does not demonstrate understanding, THEN direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1

		Procedure 7: Review multiplication of fractions and mixed numbers				24	12	
370	A	7.1 Review multiplication of fractions (B)	0	1	0			1
371	A	7.1.1 Demonstrate multiplying a fraction by a whole number, such as 1/3 x 2 (B)	0	1	0			1
372	A	7.1.2 Demonstrate we can approach this through repeated addition (B)	0	1	0			1
373	A	7.1.3 Review that 1/3 x 2 means two one-thirds, or 1/3 + 1/3 = 2/3 (B)	0	1	0			1
374	A	7.1.4 Demonstrate that we can also convert the whole number, 2, to a fraction by giving the 2 a denominator, 1, yielding 2/1 (B)	0	1	0			1
375	A	7.1.5 Demonstrate that 1/3 x 2 is actually 1/3 x 2/1 , or 2/3 (B)	0	1	0			1
376	A	7.1.6 Review that in multiplication of fractions, the algorithm requires that we multiply the numerators by the numerators and the denominators by the denominators (B)	0	1	0			1
377	A	7.1.7 Extend this understanding to multiplication of fractions by fractions (B)	0	1	0			1

378	A	7.1.8 Demonstrate that a problem such as 1/3 x 1/2 also involves multiplying the numerators by the numerators and the denominators by the denominators, yielding 1/6 (B)	0	1	0			1
379	A	7.1.9 Assign similar problems (B)	0	1	0			1
380	A	7.1.10 Visually check for understanding (B)	0	1	0			1
381	D	7.1.10.1 IF students demonstrate understanding, then go to step 7.1.11 (B)	0	1	0			1
382	D	7.1.10.2 IF student appears unclear, then check in with her before end of class to provide brief review (B)	0	1	0			1
383	A	7.1.11 Assign multiplication of fractions for homework (B)	0	1	0			1
384	A	7.1.12 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0			1
385	D	7.1.12.1 IF student demonstrates understanding, THEN go to step 7.1.13 (B)	0	1	0			1

386	D	7.1.12.2 IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)	0	1	0			1
387	D	7.1.12.2.1 IF student demonstrates understanding, THEN go to step 7.1.13 (B)	0	1	0			1
388	D	7.1.12.2.2 IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1
389	A	7.1.13 Demonstrate multiplying a mixed number by a mixed number, such as 21/4 x 21/6 (B)	0	1	0			1
390	A	7.1.14 Review that we must convert each mixed number into an improper fraction before we can multiply (B)	0	1	0			1
391	A	7.1.15 Review that 21/4 can be represented as 9/4 and 21/6 can be represented as 13/6 (B)	0	1	0			1
392	A	7.1.16 Demonstrate that we multiply the numerators by the numerators and the denominators by the denominators, yielding 117/24 (B)	0	1	0			1

393	A	7.1.17 Review that we need to simplify, yielding 4 whole and 21 twenty-fourths (B)	0	1	0			1
394	A	7.1.18 Review that 21/24 can be simplified by dividing by a form of one, 3/3 , yielding 7/8 (B)	0	1	0			1
395	A	7.1.19 Display completed product, 47/8 (B)	0	1	0			1
396	A	7.1.20 Assign similar problems (B)	0	1	0			1
397	A	7.1.21 Visually check for understanding (B)	0	1	0			1
398	D	7.1.21.1 IF students demonstrate understanding, then go to step 7.1.22 (B)	0	1	0			1
399	D	7.1.21.2 IF student appears unclear, then check in with her before end of class to provide brief review (B)	0	1	0			1
400	A	7.1.22 Assign multiplication of mixed numbers for homework (B)	0	1	0			1
401	A	7.1.23 Visually inspect one or two key problems on following day using clipboard (B)	0	1	0			1

402	D	7.1.23.1 IF student demonstrates understanding, THEN go to step 8 (B)	0	1	0			1
403	D	7.1.23.2 IF student's answers do not demonstrate understanding, THEN check in with student before end of class to check for understanding (B)	0	1	0			1
404	D	7.1.23.2.1 IF student demonstrates understanding, THEN go to step 8 (B)	0	1	0			1
405	D	7.1.23.2.2 IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1
		Procedure 8: Teach division of whole numbers by fractions				53	6	
406	A	8.1 Hold up a whole piece of paper and pose question that represents the problem 1÷ 1/2, such as "If this piece of paper represents a whole cake, how can I divide it by one-halves?" (B, C)	0	1	1			2
407	A	8.2 Direct students to discuss the question (B, C)	0	1	1			2
408	A	8.3 Circulate to check for understanding (B, C)	0	1	1			2

409	A	8.4 Call on volunteers who were able to understand that the solution is two halves (B, C)	0	1	1			2
410	A	8.5 Distribute fraction strips (C)	0	0	1			1
411	A	8.6 Relate the scenario, “If I have one piece of paper, how many halves are there in that one piece?” (C)	0	0	1			1
412	A	8.6.1 Ask students to work with a partner to model scenario with manipulatives (C)	0	0	1			1
413	A	8.6.2 Circulate among students to determine whether they are creating concrete representations of 1 ÷ 1/2 = 2 (C)	0	0	1			1
414	A	8.6.3 Check for understanding by calling on students randomly (C)	0	0	1			1
415	A	8.6.4 Validate correct answers (C)	0	0	1			1
416	A	8.6.5 Demonstrate a correct representation on whiteboard using magnetic fraction strips (A, C)	1	0	1			2
417	A	8.6.6 Ask students in pairs to formulate a real-life example of the problem, 1 ÷ 1/2 = 2 (C)	0	0	1			1

418	A	8.6.7 Circulate among students to check for understanding (C)	0	0	1			1
419	A	8.6.8 Call on volunteers who were able to formulate correct examples, such as, "If I have a whole pizza, how many halves are contained therein?" (C)	0	0	1			1
420	A	8.6.9 Ask students to work with a partner to explain what the problem 1 ÷ 1/2 = 2 means (C)	0	0	1			1
421	A	8.6.10 Circulate to check for understanding (C)	0	0	1			1
422	A	8.6.11 Call on volunteers who were able to articulate that the problem means, "How many one-halves are contained in one whole" (A, B, C)	1	1	1			3
423	A	8.7 Direct students to work with a partner to create a real-life example of the problem, 1÷ 1/4 (C)	0	0	1			1
424	A	8.8 Circulate among students to check for understanding (C)	0	0	1			1

425	A	8.9 Call on volunteers who were able to articulate examples, such as, "If I have one dollar, how many quarters are contained in that dollar?" (C)	0	0	1			1
426	A	8.10 Ask students to work with a partner to explain what the problem 1÷ 1/4 means (C)	0	0	1			1
427	A	8.11 Circulate to check for understanding (C)	0	0	1			1
428	A	8.12 Call on volunteers who were able to articulate that the problem means, "How many one-fourths are contained in one whole" (A, B, C)	1	1	1			3
429	A	8.12.1 Ask students to work with a partner to model scenario with manipulatives (C)	0	0	1			1
430	A	8.12.2 Circulate among students to determine whether they are creating concrete representations of 1 ÷ 1/4 = 4 (C)	0	0	1			1
431	A	8.12.3 Check for understanding by calling on students randomly (C)	0	0	1			1
432	A	8.12.4 Validate correct answers (C)	0	0	1			1

433	A	8.12.5 Demonstrate a correct representation on whiteboard using magnetic fraction strips (C)	0	0	1			1
434	D	8.12.5.1 teacher observation indicates student understands how to create concrete representations of problems involving whole numbers divided by fractions, THEN step 8.13 (C)	0	0	1			1
435	D	8.12.5.2 IF teacher observation indicates student does not understand, THEN provide remediation with similar examples after class (C)	0	0	1			1
436	A	8.13 Distribute math paper and pencilsC)	0	0	1			1
437	A	8.14 Relate the scenario, “If I have one pizza and want to share it among 3 people, how much will each person get?” (C)	0	0	1			1
438	A	8.14.1 Direct students to work with a partner to draw either a number line or area model representation of the scenario (C)	0	0	1			1
439	A	8.14.2 Circulate among students to determine whether they are creating pictorial representations of 1 ÷ 1/3 = 3 (C)	0	0	1			1

440	A	8.14.3 Ask students to share their drawings with other neighbors (C)	0	0	1			1
441	A	8.14.4 Check for understanding by calling on students randomly, asking them to describe their drawings (C)	0	0	1			1
442	A	8.14.5 Validate correct answers that represent 1 ÷ 1/3 = 3 (C)	0	0	1			1
443	A	8.14.6 Demonstrate a correct representation on whiteboard by drawing both an area model and a number line depicting 1 ÷ 1/3 = 3 (C)	0	0	1			1
444	A	8.15 Relate next scenario, "If I have one bagel and want to share it among 4 people, how much will each person get?" (C)	0	0	1			1
445	A	8.15.1 Direct students to work with a partner to draw either a number line or area model representation of the scenario (C)	0	0	1			1
446	A	8.15.2 Circulate among students to determine whether they are creating pictorial representations of 1 ÷ 1/4 = 4 (C)	0	0	1			1
447	A	8.15.3 Ask students to share their drawings with other neighbors (C)	0	0	1			1

448	A	8.15.4 Check for understanding by calling on students randomly, asking them to describe their drawings (C)	0	0	1			1
449	A	8.15.5 Validate correct answers that represent 1 ÷ 1/4 = 4 (C)	0	0	1			1
450	A	8.15.6 Demonstrate a correct representation on whiteboard by drawing both an area model and a number line depicting 1 ÷ 1/4 = 4 (C)	0	0	1			1
451	D	8.15.6.1 IF teacher observation indicates student understands how to create pictorial representations of problems involving whole numbers divided by fractions, THEN step 8.16 (C)	0	0	1			1
452	D	8.15.6.2 IF teacher observation indicates student does not understand, THEN provide remediation after class (C)	0	0	1			1
453	A	8.16 Ask students to consider their answers to previous problems, such as 1 ÷ 1/2 = 2 , 1 ÷ 1/3 = 3 , and 1 ÷ 1/4 = 4 (C)	0	0	1			1
454	A	8.16.1 Ask students to pair with a partner (C)	0	0	1			1
455	A	8.16.2 Ask students to look for patterns in these problems (A, C)	1	0	1			2

456	A	8.16.3 Ask students to formulate a procedural rule that can be followed (A, C)	1	0	1			2
457	A	8.16.4 Circulate among students to check for understanding (A, C)	1	0	1			2
458	A	8.16.5 Ask students that were able to formulate a rule to share it with the class (A, C)	1	0	1			2
459	A	8.16.6 Validate that the rule is to multiply the whole number by the reciprocal of the divisor (A, B, C)	1	1	1			3
460	A	8.16.7 Assign similar problems (A, C)	1	0	1			2
461	A	8.16.8 Direct students to solve using the procedural rule and to write an explanation of the procedure they used (A, C)	1	0	1			2
462	A	8.16.9 Circulate among students, and check written explanations for understanding (A, C)	1	0	1			2
463	D	8.16.9.1 IF student demonstrates understanding, THEN step 9 (A, C)	1	0	1			2

464	D	8.16.9.2 IF student does not demonstrate understanding, THEN provide remediation after class (A, C)	1	0	1			2
		Procedure 9: Teach division of fractions by whole numbers				45	6	
465	A	9.1 Display an apple that is cut into two pieces, asking, "If I want to share one of these halves of an apple between two people, how much will each person get?" (C)	0	0	1			1
466	A	9.2 Call on student volunteers (C)	0	0	1			1
467	A	9.3 Confirm correct answers by demonstrating that the answer is one-fourth of the whole apple for each person (C)	0	0	1			1
468	A	9.4 Distribute fraction strips (C)	0	0	1			1
469	A	9.5 Relate the same scenario, "If I have one half an apple and want to share it between two people, how much will each person get?" (C)	0	0	1			1

470	A	9.5.1 Ask students to work with a partner to model scenario with manipulatives (C)	0	0	1			1
471	A	9.5.2 Circulate among students to determine whether they are creating concrete representations of 1/2 ÷ 2 = 1/4 (C)	0	0	1			1
472	A	9.5.3 Check for understanding by calling on students randomly (C)	0	0	1			1
473	A	9.5.4 Validate correct answers (C)	0	0	1			1
474	A	9.5.5 Demonstrate a correct representation on whiteboard using magnetic fraction strips (A, C)	0	0	1			1
475	A	9.6 Relate next scenario, "If I have one half an apple and want to share it among three people, how much will each person get?" (C)	0	0	1			1
476	A	9.7 Display an apple that is cut into two pieces, asking, "If I want to share one of these halves of an apple among three people, how much will each person get?" (C)	0	0	1			1

477	A	9.8 Call on student volunteers (C)	0	0	1				1
478	A	9.9 Confirm correct answers by demonstrating that the answer is one-sixth of the whole apple for each person (A, C)	1	0	1				2
479	A	9.9.1 Ask students to work with a partner to model scenario with manipulatives (C)	0	0	1				1
480	A	9.9.2 Circulate among students to determine whether they are creating concrete representations of $1/2 \div 3 = 1/6$ (C)	0	0	1				1
481	A	9.9.3 Check for understanding by calling on students randomly (C)	0	0	1				1
482	A	9.9.4 Validate correct answers (C)	0	0	1				1
483	A	9.9.5 Demonstrate a correct representation on whiteboard using magnetic fraction strips (A, C)	1	0	1				2

484	D	9.9.5.1 IF teacher observation indicates student understands how to create concrete representations of problems involving fractions divided by whole numbers, THEN step 9.10 (C)	0	0	1				1
485	D	9.9.5.2 IF teacher observation indicates student does not understand, THEN provide remediation after class (C)	0	0	1				1
486	A	9.10 Distribute math paper and pencils (C)	0	0	1				1
487	A	9.11 Relate the scenario, “If I have one third of a candy bar and want to share it between 2 people, how much will each person get?” (C)	0	0	1				1
488	A	9.12 Draw both an area model and a number line representation on whiteboard of problem (C)	0	0	1				1
489	A	9.13 Demonstrate how both of these representations are similar to the previous fraction strip representation (C)	0	0	1				1
490	A	9.13.1 Direct students to work with a partner to draw either a number line or area model representation of the scenario (C)	0	0	1				1

491	A	9.13.2 Circulate among students to determine whether they are creating pictorial representations of 1/3 ÷ 2 (C)	0	0	1			1
492	A	9.13.3 Ask students to share their drawings with other neighbors (C)	0	0	1			1
493	A	9.13.4 Check for understanding by calling on students randomly, asking them to describe their drawings (C)	0	0	1			1
494	A	9.13.5 Validate correct answers that represent 1/3 ÷ 2 = 1/6 (C)	0	0	1			1
495	A	9.14 Relate next scenario, “If I have one fourth of a bagel and want to share it between 2 people, how much will each person get?” (C)	0	0	1			1
496	A	9.14.1 Direct students to work with a partner to draw either a number line or area model representation of the scenario (C)	0	0	1			1
497	A	9.14.2 Circulate among students to determine whether they are creating pictorial representations of 1/4 ÷ 2 (C)	0	0	1			1
498	A	9.14.3 Ask students to share their drawings with other neighbors (C)	0	0	1			1

499	A	9.14.4 Check for understanding by calling on students randomly, asking them to describe their drawings (C)	0	0	1			1
500	A	9.14.5 Validate correct answers that represent 1/4 ÷ 2 = 1/8 (C)	0	0	1			1
501	A	9.14.6 Demonstrate a correct representation on whiteboard by drawing both an area model and a number line depicting 1/4 ÷ 2 = 1/8 (C)	0	0	1			1
502	D	9.14.6.1 IF teacher observation indicates student understands how to create pictorial representations of problems involving fractions divided by whole numbers, THEN step 9.15 (C)	0	0	1			1
503	D	9.14.6.2 IF teacher observation indicates student does not understand, THEN provide remediation after class (C)	0	0	1			1
504	A	9.15 Ask students to consider their answers to previous problems, such as 1/2 ÷ 2 = 1/4 , 1/2 ÷ 3 = 1/6 , and 1/4 ÷ 2 = 1/8 (C)	0	0	1			1
505	A	9.15.1 Ask students to pair with a partner (C)	0	0	1			1

506	A	9.15.2 Ask students to look for patterns in the previous problems (A, C)	1	0	1			2
507	A	9.15.3 Ask students to formulate a procedural rule that can be followed (A,C)	1	0	1			2
508	A	9.15.4 Circulate among students to check for understanding (A, C)	1	0	1			2
509	A	9.15.5 Ask students that were able to formulate a rule to share it with the class (A, C)	1	0	1			2
510	A	9.15.6 Validate that the rule is to multiply the fraction by the reciprocal of the whole number (A, B, C)	1	1	1			3
511	A	9.15.7 Assign similar problems, such as 1/5 ÷ 2 (A,C)	1	0	1			2
512	A	9.15.8 Direct students to solve using fraction strips, then a pictorial representation, and then to validate using the procedural rule, and include a written description of the written description (A, C)	1	0	1			2

3513	A	9.15.9 Circulate among students and check strips, pictures, and written descriptions for understanding (A, C)	1	0	1			2
514	D	9.15.9.1 IF student demonstrates understanding, THEN step 10 (A, C)	1	0	1			2
515	D	9.15.9.2 IF student does not demonstrate understanding, THEN provide remediation after class (A, C)	1	0	1			2
		Procedure 10: Teach division of fractions by fractions				59	10	
516	A	10.1 Teach concrete representation (C)	0	0	1			1
517	A	10.2 Distribute fractions strips (C)	0	0	1			1
518	A	10.2.1 Pose problem such as 3/4÷ 1/4 (C)	0	0	1			1
519	A	20.2.2 Remind students that the problem involves determining how many fourths are in three fourths (A, B, C)	1	1	1			3
520	A	10.2.3 Use magnetic fraction strips on whiteboard to demonstrate that the answer is 3: there are 3 one-fourths in three-fourths (C)	0	0	1			1

521	A	10.2.4 Direct students to model the same problem at their desks (C)	0	0	1			1
522	A	10.2.5 Circulate among students to check for understanding (C)	0	0	1			1
523	A	10.2.6 Pose similar problem, such as 1/2 ÷ 1/4 (C)	0	0	1			1
524	A	10.2.7 Direct students to model problem with fraction strips or linking cubes (C)	0	0	1			1
525	A	10.2.8 Circulate among students to check for understanding (C)	0	0	1			1
526	A	10.2.9 Demonstrate on board that the answer is 2: there are two one-fourths in one half (C)	0	0	1			1
527	A	10.2.10 Pose similar problems (C)	0	0	1			1
528	A	10.2.11 Circulate among students to check for understanding (C)	0	0	1			1
529	D	10.2.11.1 IF student demonstrates understanding, THEN step 10.3 (C)	0	0	1			1

530	D	10.2.11.2 IF students does not demonstrate understanding, THEN provide remediation after class (C)	0	0	1			1
531	A	10.3 Teach pictorial representation (B, C)	0	1	1			2
532	A	10.3.1 Pose problem, such as 3/4÷ 1/4 (B, C)	0	1	1			2
533	A	10.3.2 Remind students that the problem involves determining how many fourths are in three fourths (A, B, C)	1	1	1			3
534	A	10.3.3 Draw an area model and a number line on whiteboard to demonstrate that there are three one-fourths in three fourths (B, C)	0	1	1			2
535	A	10.3.4 Direct students to model the same problem at their desks (B, C)	0	1	1			2
536	A	10.3.5 Circulate among students to check for understanding (B, C)	0	1	1			2
537	A	10.3.6 Direct students to validate the answer using the procedural rule (C)	0	0	1			1
538	A	10.3.7 Circulate among students to check for understanding (B, C)	0	1	1			2
539	A	10.3.8 Call on volunteers who were able to multiply three-fourths by the reciprocal of one-fourth, 4, to yield 3 (B, C)	0	1	1			2

540	A	10.3.9 Pose similar problem, such as 1/2 ÷ 1/4 (C)	0	0	1			1
541	A	10.3.10 Direct students to model problem with fraction strips, and then area models or number lines, and then validate with the procedural rule (C)	0	0	1			1
542	A	10.3.11 Circulate among students to check for understanding (C)	0	0	1			1
543	A	10.3.12 Demonstrate on board that there are two fourths in one half (C)	0	0	1			1
544	A	10.3.13 Pose similar problems (C)	0	0	1			1
545	A	10.3.14 Circulate among students to check for understanding (C)	0	0	1			1
546	D	10.3.14.1 IF student demonstrates understanding, THEN step 10.4 (C)	0	0	1			1
547	D	10.3.14.2 IF students does not demonstrate understanding, THEN provide remediation after class (C)	0	0	1			1

548	A	10.4 Teach complex fraction approach (B)	0	1	0				1
549	A	10.4.1 Remind students that division can be represented through fractions: the dividend can be represented as numerator, and the divisor as denominator (B)	0	1	0				1
550	A	10.4.2 Demonstrate that 3/4 ÷ 1/4 can be represented as a complex fraction, in the form (3/4)/(1/4) (B)	0	1	0				1
551	A	10.4.3 Remind students that anything divided by one is itself (B)	0	1	0				1
552	A	10.4.4 Remind students that the identity property states that multiplying a value by 1 does not change that value (B)	0	1	0				1
553	A	10.4.5 Demonstrate that multiplying (3/4)/(1/4) x (4/1)/(4/1) is both multiplying by 1 and also going to create a 1 in the denominator, based on the inverse property of multiplication (B)	0	1	0				1
554	A	10.4.6 Draw the outline of a “1” around (4/1)/(4/1) (B)	0	1	0				1

555	A	10.4.7 Remind students that because the denominator is 1, we are left with 3/4 x 4/1 (B)	0	1	0				1
556	A	10.4.8 Remind students that the algorithm for multiplying fractions, by which we multiply the numerators by numerators and denominators by denominators, means that the product is going to be 12/4 (B)	0	1	0				1
557	A	10.4.9 Remind students to reduce by dividing by a form of 1, in this case 4/4 (B)	0	1	0				1
558	A	10.4.10 Explain that the answer is 3, which is the same answer derived through the pictorial method (B)	0	1	0				1
559	A	10.4.11 Assign similar problems asking students to solve using complex fractions (B)	0	1	0				1
560	A	10.4.12 Visually check for understanding (B)	0	1	0				1
561	D	10.4.12.1 IF students demonstrate understanding, then go to step 10.4.13 (B)	0	1	0				1

562	D	10.4.12.2 IF student appears unclear, then check in with her before end of class to provide brief review (B)	0	1	0			1
563	A	10.4.13 Review that converting a fraction division problem into a complex fraction results in multiplying the first fraction by the reciprocal of the second fraction, a fact we also discovered when we divided whole numbers by fractions and fractions by whole numbers using manipulatives and pictorial representations (A, B, C)	1	1	1			3
564	A	10.4.14 Administer assessment in which students must solve similar problems and also write an explanation of why they can derive the same answer by simply multiplying by the reciprocal (B)	0	1	0			1
565	D	10.4.14.1 IF student answers demonstrate understanding, THEN go to step 10.5 (B)	0	1	0			1
566	D	10.4.14.2 IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1
567	A	10.5 Teach alternate method for dividing fractions by fractions, which involves converting the fractions to decimals, and then dividing the decimals (B)	0	1	0			1

568	A	10.5.1 Demonstrate conceptual nature of process with an example using currency (B)	0	1	0			1
569	A	10.5.1.1 Demonstrate using the example 1/2 ÷ 1/4 (B)	0	1	0			1
570	A	10.5.1.2 Review that we must first convert both 1/2 and 1/4 to equivalent fractions with denominators that are powers of 10 or 100 (B)	0	1	0			1
571	A	10.5.1.3 Demonstrate multiplying 1/2 and 1/4 by 5/5 and 25/25 respectively, to yield 5/10 and 25/100 (B)	0	1	0			1
572	A	10.5.1.4 Review that these new fractions convert to decimals of .50 and .25 (B)	0	1	0			1
573	A	10.5.1.5 Connect these two values to the concept of money by explaining that .50 can be viewed as fifty cents and .25 can be viewed as twenty five cents (B)	0	1	0			1
574	A	10.5.1.6 Demonstrate that .50 ÷ .25 is equivalent to asking, “How many quarters make up fifty cents?” (B)	0	1	0			1
575	A	10.5.1.7 Demonstrate that the answer is “2” (B)	0	1	0			1

576	A	10.5.2 Demonstrate procedural process (B)	0	1	0			1
577	A	10.5.2.1 Review that 1/2 ÷ 1/4 can be represented by decimals as .50 ÷ .25 (B)	0	1	0			1
578	A	10.5.2.2 Review that we can represent this in long division as .25)(.50) ? (B)	0	1	0			1
579	A	10.5.2.3 Review that we must shift both the decimal point in the divisor and the decimal point in the dividend two places to the right before we can divide, yielding 50 divided by 25 (B)	0	1	0			1
580	A	10.5.2.4 Demonstrate that the quotient becomes 2, the same answer we obtained in the money example (B)	0	1	0			1
581	A	10.5.2.5 Assign similar problems (B)	0	1	0			1
582	A	10.5.2.6 Visually check for understanding (B)	0	1	0			1
583	D	10.5.2.6.1 IF student demonstrates understanding, THEN step 11 (B)	0	1	0			1
584	D	10.5.2.6.2 IF student does not demonstrate understanding, THEN direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1

		Procedure 11: Teach division of mixed numbers by mixed numbers				23	8	0
585	A	11.1 Pose problem such as 21/2 ÷ 11/4 (A, B, C)	1	1	1			3
586	A	11.2 Direct students to create a concrete representation with fraction strips (C)	0	0	1			1
587	A	11.2.1 Circulate among students to check for understanding (C)	0	0	1			1
588	A	11.2.2 Ask students with correct answers to recreate their representations on whiteboard with magnetic fraction strips (C)	0	0	1			1
589	A	11.2.3 Validate that correct answer is 2 (C)	0	0	1			1
590	A	11.2.4 Assign similar problems (C)	0	0	1			1
591	D	11.2.4.1 IF student demonstrates understanding, THEN step 11.2.5 (C)	0	0	1			1
592	D	11.2.4.2 IF student does not demonstrate understanding, THEN provide remediation after class (C)	0	0	1			1
593	A	11.2.5 Direct students to create a pictorial representation of same problem (B, C)	0	1	1			2

594	A	11.2.6 Circulate among students to check for understanding (B, C)	0	1	1			2
595	A	11.2.7 Ask students with correct answers to recreate their area models, circle models, or number lines on whiteboard (B, C)	0	1	1			2
596	A	11.2.8 Validate that correct drawings depict 21/2 ÷ 11/4 = 2 (B, C)	0	1	1			2
597	A	11.2.9 Assign similar problems (B, C)	0	1	1			2
598	D	11.2.9.1 IF student demonstrates understanding, THEN step 11.3 (B, C)	0	1	1			2
599	D	11.2.9.2 IF student does not demonstrate understanding, THEN provide remediation after class (B, C)	0	1	1			2
600	A	11.3 Direct students to solve problem using the complex fraction method (B)	0	1	0			1
601	A	11.3.1 Remind students that we need to convert each term to an improper fraction, yielding 5/2 ÷ 5/4 (B)	0	1	0			1

602	A	11.3.2 Demonstrate that we can convert this to a complex fraction, (5/2)/(5/4) (B)	0	1	0			1
603	A	11.3.3 Remind students that we can multiply by (4/5)/(4/5) to change the denominator to 1 (B)	0	1	0			1
604	A	11.3.4 Remind students that converting to a complex fraction and then converting the denominator to 1 is same as multiplying by the reciprocal (B)	0	1	0			1
605	A	11.3.5 Assign similar problems (B)	0	1	0			1
606	A	11.3.6 Visually check for understanding (B)	0	1	0			1
607	D	11.3.6.1 IF students demonstrate understanding, then go to step 11.4 (B)	0	1	0			1
608	D	11.3.6.2 IF student does not demonstrate understanding, direct her to come to class during Muir Time and provide remediation (B)	0	1	0			1

609	A	11.4 Direct students to solve problem using the procedural method (A, B, C)	1	1	1			3
610	A	11.4.1 Circulate among students to check for understanding (A, B, C)	1	1	1			3
611	A	11.4.2 Ask students with correct answers to recreate their algorithms on whiteboard (A, B, C)	1	1	1			3
612	A	11.4.3 Validate that correct procedure involves converting 21/2 to 5/2 and 11/4 to 5/4 , and then multiplying 5/2 by the reciprocal of 5/4 to yield 5/2 x 4/5 = 20/10 or 2 (A, B, C)	1	1	1			3
613	A	11.4.4 Assign similar problems (A, B, C)	1	1	1			3
614	D	11.4.4.1 IF student demonstrates understanding, THEN step 12 (A, B, C)	1	1	1			3
615	D	11.4.4.2 IF student does not demonstrate understanding, THEN provide remediation after class (A, B, C)	1	1	1			3

		Procedure 12: Teach division of fraction word problems				15	2	
616	A	12.1 Pose problem such as, "If a recipe calls for 1/2 cup of flour for one batch of bread, and I have 3/4 cup of flour, how many batches of the recipe can I make?" (C)	0	0	1			1
617	A	12.1.1 Direct students to model the problem with manipulatives (C)	0	0	1			1
618	A	12.1.2 Circulate among students to check for understanding (C)	0	0	1			1
619	A	12.1.3 Ask students with correct representations to recreate their representations on whiteboard with magnetic fraction strips (C)	0	0	1			1
620	A	12.1.4 Validate correct concrete representation of 3/4 ÷ 1/2 (C)	0	0	1			1
621	A	12.1.5 Direct students to model the problem pictorially (C)	0	0	1			1
622	A	12.1.6 Circulate among students to check for understanding (C)	0	0	1			1
623	A	12.1.7 Ask students with correct representations to recreate their representations on whiteboard with number lines or area models (C)	0	0	1			1

624	A	12.1.8 Validate correct pictorial representation of 3/4 ÷ 1/2 (C)	0	0	1			1
625	A	12.1.9 Direct students to solve the problem with the previously discovered procedure (C)	0	0	1			1
626	A	12.1.10 Circulate among students to check for understanding (C)	0	0	1			1
627	A	12.1.11 Ask students with correct procedure to write their number sentences on the board (C)	0	0	1			1
628	A	12.1.12 Validate correct number sentence is 3/4 ÷ 1/2 = 3/4 x 2/1 = 6/4 = 11/2 (C)	0	0	1			1
629	A	12.1.13 Assign similar problems, asking students to create both a concrete and pictorial representation, and to solve using the number sentence procedure (C)	0	0	1			1
630	A	12.1.14 Circulate among students to check for understanding (C)	0	0	1			1

631	D	12.1.14.1 IF student demonstrates understanding, THEN end task (C)	0	0	1			1
632	D	12.1.14.2 IF student does not demonstrate understanding, THEN provide remediation after class (C)	0	0	1			1

632	Total Action and Decision Steps	**130**	**385**	**227**	490	142
490	Action Steps	110	279	199		
142	Decision Steps	20	106	28		

Total Action and Decision Steps	**20.57%**	**60.92%**	**35.92%**
Action Steps	22.45%	56.94%	40.61%
Decision Steps	14.08%	74.65%	19.72%

Action and Decision Steps Omitted	**502**	**247**	**405**
Action Steps Omitted	380	211	291
Decision Steps Omitted	122	36	114

Action and Decision Steps Omitted	**79.43%**	**39.08%**	**64.08%**
Action Steps Omitted	77.55%	43.06%	59.39%
Decision Steps Omitted	85.92%	25.35%	80.28%

Average	Captured	Omitted
Total Action and Decision Steps	39.14%	60.86%
Action Steps	40.00%	60.00%
Decision Steps	36.15%	63.85%

Highly Aligned	**21**	**3.32%**
Partially Aligned	**68**	**10.76%**

Slightly Aligned	**543**	**85.92%**
	632	100.00%