Definition : We define a class $\mathcal{C}$ in $\mathbb{F}[x_1, x_2, \cdots]$ as follows :

1. $x_i \in \mathcal{C}$.
2. If $p \in \mathbb{F}[x_1, \cdots, x_n], q \in \mathbb{F}[y_1, \cdots, y_m]$ and $\text{Var}(p) \cap \text{Var}(q) = \phi$ then
   (a) $p + q \in \mathcal{C}$.
   (b) $p \times q \in \mathcal{C}$.
   (c) $\forall m \geq 1 \ : \ p^m \in \mathcal{C}$.

Definition : If $p \in \mathbb{F}[x_1, \cdots, x_n]$ then

$$\text{Jacob}(p) \equiv \left\langle \frac{\partial p}{\partial x_i} \;\middle|\; x_i \in \text{Var}(p) \right\rangle$$

Theorem : If $p \in \mathcal{C}$, then Gröbner(Jacob(p)) = Jacob(p).
Proof : The proof is by induction on the structure of the polynomials in $\mathcal{C}$.
[Basis]
If $p \in \mathcal{C}$ and $p = x_i$ then $\frac{\partial p}{\partial x_j} = 0$, if $i \neq j$, $\frac{\partial p}{\partial x_i} = 1$, if $i = j$. Clearly Jacob$(p) = \langle 1 \rangle$, which is a Gröbner basis. So the basis case holds.

[Additive Case]
If $p, q \in \mathcal{C}$ and $\text{Var}(p) \cap \text{Var}(q) = \phi$, then if $R = p + q$,

$$\frac{\partial R}{\partial x_i} = \frac{\partial P}{\partial x_i}, \text{ if } x_i \in \text{Var}(p)$$
$$\frac{\partial R}{\partial x_i} = \frac{\partial P}{\partial x_i}, \text{ if } x_i \in \text{Var}(q)$$

$$\overline{S\left(\frac{\partial p}{\partial x_i}, \frac{\partial p}{\partial x_j}\right)}^{\text{Jacob}(p+q)} = 0$$

by Induction Hypothesis as the entries of Jacob(p) are in the basis.

$$\text{Similarly } \overline{S\left(\frac{\partial q}{\partial x_i}, \frac{\partial q}{\partial x_j}\right)}^{\text{Jacob}(p+q)} = 0$$

$$\text{Further LM}\left(\frac{\partial p}{\partial x_i}\right) \perp \text{LM}\left(\frac{\partial q}{\partial x_j}\right) \text{ as } \text{Var}(p) \cap \text{Var}(q) = \phi$$

So $\langle \text{Jacob}(p), \text{Jacob}(q) \rangle = \langle \text{Jacob}(R) \rangle$ is a Gröbner basis.

We split the multiplicative case into two analyses as follows :
[Monomial Case]
If $R = x_1^{a_1} \cdots x_n^{a_n}$ then the S-Poly is always zero. Hence Jacob(R) forms a Gröbner basis.

[Multiplicative Case]

If $R \in \mathcal{C}$ is not a monomial then we can express it as one of the following :

$$R = (f + g) \times h \text{ The Var sets are mutually disjoint}$$
$$R = h \times (f + g)$$

Let $f_i = \frac{\partial f}{\partial x_i}, h_j = \frac{\partial h}{\partial x_j}$ We assume that through the induction the following invariant is also maintained for the S-Poly.

$$\begin{aligned} S(f_i h, f h_j) &= \frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f_i h)} f_i h - \frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f h_j)} f h_j \\ &= \alpha_1 f_1 h + \alpha_2 f_2 h + \cdots + \alpha_k f_k h + \beta_1 f h_1 + \beta_2 f h_2 + \cdots + \beta_l f h_l \end{aligned}$$

Such that

$$\frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f h_j)} f h_j = \beta_1 f h_1 + \beta_2 f h_2 + \cdots + \beta_l f h_l$$

We assume $R = (f + g) \times h$,

$$\begin{aligned} \frac{\partial R}{\partial x_i} &= f_i h, \text{ if } x_i \in \text{Var}(f) \\ \frac{\partial R}{\partial x_i} &= g_i h, \text{ if } x_i \in \text{Var}(g) \\ \frac{\partial R}{\partial x_i} &= (f + g) h, \text{ if } x_i \in \text{Var}(h) \end{aligned}$$

Note that these are the entries in Jacob(R), and we have to show they form a Gröbner basis.
Now consider $S(f_i h, (f+g) h_j)$, as $\text{Var}(p) \cap \text{Var}(q) = \phi$, either $\text{LT}(f) \prec \text{LT}(g)$ or $\text{LT}(f) \succ \text{LT}(g)$. We assume that $\text{LT}(f) \succ \text{LT}(g)$ in the following.

$$S(f_i h, (f + g) h_j) = \frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f_i h)} f_i h - \frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f h_j)} (f + g) h_j,$$

(as $\text{LM}((f + g) h_j) = \text{LM}(f h_j)$)

$$= \frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f_i h)} f_i h - \frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f h_j)} f h_j - \frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f h_j)} g h_j$$

By Inductive Hypothesis

$$\frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f h_j)} f h_j = \beta_1 f h_1 + \beta_2 f h_2 + \cdots + \beta_l f h_l$$

Which implies that

$$\beta_1 g h_1 + \beta_2 g h_2 + \cdots + \beta_l g h_l = \frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f h_j)} g h_j$$

as $\mathbb{F}[x_1, \cdots, x_n]$ is an integral domain.

$$\text{Hence } \overline{S(f_i h, (f + g) h_j)}^{\text{Jacob}((f+g)h)} = 0.$$

$$\text{Note that } \beta_1 (f + g) h_1 + \beta_2 (f + g) h_2 + \cdots + \beta_l (f + g) h_l = \frac{\text{lcm}(\text{LM}(f_i h), \text{LM}(f h_j))}{\text{LT}(f h_j)} (f + g) h_j$$

so the induction goes through.
Now in the case that $LT(f) \prec LT(g)$, we have

$$\begin{aligned} S(f_i h, (f+g)h_j) &= \frac{\mathrm{lcm}(\mathrm{LM}(f_i h), \mathrm{LM}(g h_j))}{\mathrm{LT}(f_i h)} f_i h - \frac{\mathrm{lcm}(\mathrm{LM}(f_i h), \mathrm{LM}(g h_j))}{\mathrm{LT}(g h_j)} (f+g)h_j \\ &= LM(f_i g) \frac{\mathrm{lcm}(LM(h), LM(h_j))}{LT(f_i) LT(h)} f_i h - LM(f_i g) \frac{\mathrm{lcm}(LM(h), LM(h_j))}{LT(g) LT(h_j)} (f+g)h_j \\ &= LM(g) \frac{\mathrm{lcm}(LM(h), LM(h_j))}{LT(h)} f_i h - LM(f_i) \frac{\mathrm{lcm}(LM(h), LM(h_j))}{LT(h_j)} (f+g)h_j \end{aligned}$$

Assuming the polynomials are monic this is true, as $\mathrm{LT}(f) = \mathrm{LM}(f)$ in that case.

[Ken : Finally this is the case which has to be tackled. Since the other cases are symmetric.]

[Powering Case] If $R = p^m$ for some $m \geq 1$, we have $\mathrm{Jacob}(p^m) = p^{m-1}(Jacob(p))$. Clearly if $\langle f \rangle$ is a principal ideal and I is an ideal for which G_I is a Gröbner basis then we have $f \times G_I = \langle f \times g | g \in G_I \rangle$ is also a Gröbner basis as all the S-Poly are now $f \times S(g_i, g_j)$ where $g_i, g_j \in G_I$ which are zero by Induction. Hence $\mathrm{Jacob}(p^m)$ is also a Gröbner basis.