For a polynomial $f \in K[x_1, \cdots, x_n]$, its **Jacobian Ideal** is defined as

$$\mathsf{J}(f) := \langle \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \rangle,$$

and the **Mapping-Jacobian Ideal** of f is defined as

$$\mathsf{MJ}(f) := \langle y_1 - \frac{\partial f}{\partial x_1}, \cdots, y_n - \frac{\partial f}{\partial x_n} \rangle,$$

where $y_1, \cdots, y_n$ are newly introduced distinct variables. Note that singular points of f are those also vanishing on all its partial derivatives, and hence

$$\mathsf{Sing}(f) = V(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}) = V(f, \mathsf{J}(f)).$$

Lemma 1. *Isomorphic varieties have the same dimension.*

The Join of Two Varieties.

Definition 1. *Given any two disjoint projective varieties $X, Y \in \mathbb{P}^n$, the join of X and Y, denoted by $Join(X, Y)$, is defined as the union of the lines joining X to Y:*

$$\mathrm{Join}(X, Y) = \bigcup_{x \in X, y \in Y} \overline{x, y}.$$

It is easy to see that this join $Join(X, Y)$ is a subvariety of the Grassmannian $\mathbb{G}(1, n)$ which is proved to be a projective variety, and hence the union of lines $Join(X, Y)$ is also a subvariety of $\mathbb{P}^n$.

Lemma 2 ([?])**.** *Let $X, Y \in \mathbb{P}^n$ be any two disjoint projective varieties,*

$$\mathrm{gdeg}(\mathrm{Join}(X, Y)) = \mathrm{gdeg}(X) \cdot \mathrm{gdeg}(Y).$$

Proof. It suffices to prove this in the special case where X and Y live in complementary linear subspaces $\mathbb{P}^m$ and $\mathbb{P}^{n-m-1} \subset \mathbb{P}^n$, because any join may be realized as the regular projection of such a join.

Let $\dim(X) = k$ and $\dim(Y) = l$. We take Λ_X be a $(m-k)$-dimensional general plane intersecting X transversely. Similarly, let Λ_Y intersect Y transversely. Let $\Lambda^* = \mathrm{Join}(\Lambda_X, \Lambda_Y)$ be the subspace spanned by Λ_X and Λ_Y. Now consider

$$\Lambda^* \cap \mathrm{Join}(X, Y).$$

Note that a point in $\mathrm{Join}(X, Y)$ gives a point in X and a point in Y. Then a point in $\Lambda^* \cap \mathrm{Join}(X, Y)$ should consist of a X-component in Λ_X and a Y-component in Λ_Y. Thus this intersection should have all lines passing through a point in $X \cap \Lambda_X$ and a point in $Y \cap \Lambda_Y$.

Moreover, if Λ_X and Λ_Y intersect X and Y transversely, then the intersection $\Lambda^* \cap \text{Join}(X, Y)$ is generically transverse. Hence it intersects this set of lines in precisely

$$|\Lambda_X \cap X| \cdot |\Lambda_Y \cap Y| = \text{gdeg}(X) \cdot \text{gdeg}(Y).$$

So $\text{gdeg}(\text{Join}(X, Y)) = \text{gdeg}(X) \cdot \text{gdeg}(Y)$. □

In affine case, we can also find a variety analogous to that in projective case.

Definition 2. *Given any two affine varieties $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$, the product of them is defined as*

$$\text{Prod}(X, Y) = X \times Y \subset \mathbb{A}^{n+m}.$$

By definition, $\text{Prod}(X, Y)$ is also an affine variety because $\text{Prod}(X, Y) = V(S_1, S_2)$ with $X = V(S_1)$ and $Y = V(S_2)$. Again using the proof technique similar to that in Lemma 2, we derive the following

Lemma 3. *Let $X \in \mathbb{A}^m$ and $Y \in \mathbb{A}^{n-m}$ be any two disjoint affine varieties,*

$$\text{gdeg}(\text{Prod}(X, Y)) = \text{gdeg}(X) \cdot \text{gdeg}(Y).$$

Proof. Let $\dim(X) = k, \dim(Y) = l$. Let Λ_X be a $(m-k)$-dimensional general plane intersecting X transversely such that $\text{gdeg}(X) = |\Lambda_X \cap X|$ and Λ_Y be a $(n-m-l)$-dimensional general plane intersecting Y transversely such that $\text{gdeg}(Y) = |\Lambda_Y \cap Y|$. Take Λ^* be the general plane spanned by Λ_X and Λ_Y. So the dimension of Λ^* is $n - k - l$.

Now we claim that $|\Lambda^* \cap \text{Prod}(X, Y)|$ gives the geometric degree of $\text{Prod}(X, Y)$. Via a similar argument in Lemma 2, we have

$$|\Lambda^* \cap \text{Prod}(X, Y)| = |\Lambda_X \cap X| \cdot |\Lambda_Y \cap Y| = \text{gdeg}(X) \cdot \text{gdeg}(Y).$$

It is left to argue that $\text{gdeg}(\text{Prod}(X, Y)) = |\Lambda^* \cap \text{Prod}(X, Y)|$, that is, the intersection of Λ^* and $\text{Prod}(X, Y)$ is indeed maximum. However, it is easy to see that any $(n - k - l)$-dimensional general plane $\Lambda^{*\prime}$ can be decomposed into Λ'_X of dimension $(m-k)$ and Λ'_Y of dimension $(n-m-l)$, and extra intersections between $\Lambda^{*\prime}$ with $\text{Prod}(X, Y)$ will give extra intersection points in $\Lambda'_X \cap X$ and $\Lambda'_Y \cap Y$. Therefore, by contradiction, $\text{gdeg}(\text{Prod}(X, Y)) = |\Lambda^* \cap \text{Prod}(X, Y)|$. □

Lemma 4. *Let X and Y be of dimensions r_X, r_Y, respectively, and assume that they do not have a common irreducible component. If $r_X = r_Y$, then*

$$\text{gdeg}(X \cup Y) = \text{gdeg}(X) + \text{gdeg}(Y),$$

and if $r_X > r_Y$,

$$\text{gdeg}(X \cup Y) = \text{gdeg}(X).$$

Proof. Let $\overline{X}$ be the projective closure of X. Then $\overline{X} = V_p(I_X^h)$, where I_X^h is the homogenization of the ideal $I_X = I(X)$.

To prove this lemma, we use the definition that $\text{gdeg}(X) := \text{gdeg}(\overline{X})$. We can simply argue that $\text{gdeg}(X \cup Y) = \text{gdeg}(\overline{X \cup Y}) = \text{gdeg}(\overline{X} \cup \overline{Y})$. This is because $(I_X \cdot I_Y)^h = I_X^h \cdot I_Y^h$ (which is easy to see) and then

$$\overline{X \cup Y} = V_p((I_X \cdot I_Y)^h) = V_p(I_X^h \cdot I_Y^h) = V_p(I_X^h) \cup V_p(I_Y^h) = \overline{X} \cup \overline{Y}.$$

Now this lemma becomes to prove $\text{gdeg}(\overline{X} \cup \overline{Y}) = \text{gdeg}(\overline{X}) + \text{gdeg}(\overline{Y})$ if $r_X = r_Y$ or $\text{gdeg}(\overline{X} \cup \overline{Y}) = \text{gdeg}(\overline{X})$ if $r_X > r_Y$. This can be proved using Hilbert polynomial, and the details can be found in [**?**]. □

Measuring "Entangleness."

Lemma 5. *Given $f \in K[x_1, \cdots, x_n]$ and $g \in K[w_1, \cdots, w_m]$ be two polynomials of disjoint variables, then*

$$\mathsf{Sing}(fg) = \mathsf{Sing}(f) \cup \mathsf{Sing}(g).$$

Proof. Note that when f and g are of disjoint variables

$$\mathsf{Sing}(fg) = V(fg, g\frac{\partial f}{\partial x_1}, \cdots, g\frac{\partial f}{\partial x_n}, f\frac{\partial g}{\partial w_1}, \cdots, f\frac{\partial g}{\partial w_m}).$$

We first show that

$$\mathsf{Sing}(fg) \subset (\mathsf{Sing}(f) \cup \mathsf{Sing}(g)).$$

If a point $P \in \mathsf{Sing}(fg)$, at least $P \in V(f)$ or $V(g)$. If $P \in V(f)$, $f(P) = 0$.

$$(g\frac{\partial f}{\partial x_i})(P) = g(P)\frac{\partial f}{\partial x_i}(P) = 0$$

implies $g(P) = 0$ or else $\frac{\partial f}{\partial x_i}(P) = 0$ for all i. That is, either

$$P \in (V(f) \cap V(g)) \quad \text{or} \quad P \in \mathsf{Sing}(f).$$

Since f and g are of disjoint variables, $V(f) \cap V(g)) \subset \mathsf{Sing}(f)$. Similarly, if $P \in V(g)$, we have $P \in \mathsf{Sing}(g)$. Therefore, $P \in (\mathsf{Sing}(f) \cup \mathsf{Sing}(g))$.

Conversely, we show

$$(\mathsf{Sing}(f) \cup \mathsf{Sing}(g)) \subset \mathsf{Sing}(fg).$$

If $P \in \mathsf{Sing}(f)$, we have

$$f(P) = \frac{\partial f}{\partial x_i}(P) = 0.$$

With these,

$$(fg)(P) = g(P)\frac{\partial f}{\partial x_i}(P) = f(P)\frac{\partial g}{\partial w_i}(P) = 0.$$

So $P \in \mathsf{Sing}(fg)$. Similarly, if $P \in \mathsf{Sing}(g)$ then $P \in \mathsf{Sing}(fg)$.

Putting everything together, we have

$$\mathsf{Sing}(fg) = \mathsf{Sing}(f) \cup \mathsf{Sing}(g).$$

□

Lemma 6. *Given $f(X) \in K[x_1, \cdots, x_n]$, by extending the affine space and introducing new variable t, then*

$$\operatorname{gdeg}\left(g(W)t - 1, y_1 - g\frac{\partial f}{\partial x_1}, \cdots, y_n - g\frac{\partial f}{\partial x_n}\right) = \operatorname{gdeg}\left(g(W)t - 1, \mathsf{MJ}(f)\right).$$

Proof. Without loss of generality, assume working in $\mathbb{A}^{2n+m+1}$ with

$$f \in K[x_1, \cdots, x_n, y_1, \cdots, y_n, w_1, \cdots, w_m, t].$$

Note that

$$V\left(g(W)t - 1, g(W)(y_1 - \frac{\partial f}{\partial x_1}), \cdots, g(W)(y_n - g\frac{\partial f}{\partial x_n})\right) = V\left(g(W)t - 1, \mathsf{MJ}(f)\right).$$

Hence it suffices to prove

$$\operatorname{gdeg}\left(g(W)t - 1, y_1 - g\frac{\partial f}{\partial x_1}, \cdots, y_n - g\frac{\partial f}{\partial x_n}\right) = \operatorname{gdeg}\left(g(W)t - 1, g(W)(y_1 - \frac{\partial f}{\partial x_1}), \cdots, g(W)(y_n - \frac{\partial f}{\partial x_n})\right).$$

Let

$$X = V\left(g(W)t - 1, g(W)(y_1 - \frac{\partial f}{\partial x_1}), \cdots, g(W)(y_n - \frac{\partial f}{\partial x_n})\right),$$
$$Y = V\left(g(W)t - 1, y_1 - g\frac{\partial f}{\partial x_1}, \cdots, y_n - g\frac{\partial f}{\partial x_n}\right).$$

Now define ϕ to be the following morphism from X to Y:

$$y_i \mapsto y_i \cdot g(W),$$

and $\phi^{-1} : Y \to X$ to be:

$$y_i \mapsto \frac{y_i}{g(W)}.$$

□

Theorem 7. *Let $f \in K[x_1, \cdots, x_n]$ and $g \in K[w_1, \cdots, w_m]$ be two polynomials of disjoint variables. The following are true:*

(a) $\mathrm{gdeg}(\mathsf{J}(fg)) \le \mathrm{gdeg}(f) \cdot \mathrm{gdeg}(g)$,

(b) $\mathrm{gdeg}(\mathsf{MJ}(fg)) \ge \mathrm{gdeg}(\mathsf{MJ}(f)) \cdot \mathrm{gdeg}(\mathsf{MJ}(g))$,

(c) $\mathrm{gdeg}(\mathsf{Sing}(fg)) = \mathrm{gdeg}(\mathsf{Sing}(f)) + \mathrm{gdeg}(\mathsf{Sing}(g))$.

Proof.
(a). Since f and g are of disjoint variables,

$$\mathrm{gdeg}(\mathsf{J}(fg)) = \mathrm{gdeg}(g\frac{\partial f}{\partial x_1}, \cdots, g\frac{\partial f}{\partial x_n}, f\frac{\partial g}{\partial w_1}, \cdots, f\frac{\partial g}{\partial w_m}).$$

Now via Bézout's Theorem,

$$\mathrm{gdeg}(\mathsf{J}(fg)) \le \mathrm{gdeg}(g\frac{\partial f}{\partial x_1}, \cdots, g\frac{\partial f}{\partial x_n}) \cdot \mathrm{gdeg}(f\frac{\partial g}{\partial w_1}, \cdots, f\frac{\partial g}{\partial w_m}).$$

Note that

$$V(g\frac{\partial f}{\partial x_1}, \cdots, g\frac{\partial f}{\partial x_n}) = V(\mathsf{J}(f)) \cup V(g),$$

and then by Lemma 4, we have

$$\mathrm{gdeg}(g\frac{\partial f}{\partial x_1}, \cdots, g\frac{\partial f}{\partial x_n}) = \mathrm{gdeg}(g)$$

because $V(g)$ is a hypersurface and thus $\dim(V(g)) > \dim(V(\mathsf{J}(f)))$. Similarly, we also get

$$\mathrm{gdeg}(f\frac{\partial g}{\partial w_1}, \cdots, f\frac{\partial g}{\partial w_m}) = \mathrm{gdeg}(f).$$

Therefore,

$$\mathrm{gdeg}(\mathsf{J}(fg)) \le \mathrm{gdeg}(f) \cdot \mathrm{gdeg}(g).$$

(b). □

Theorem 8.

$$\mathrm{gdeg}(\mathsf{MJ}(fg)) \le \Big(\mathrm{gdeg}(g(W)s-1, \mathsf{MJ}(f)) + \deg(g(W))\Big) \cdot \Big(\mathrm{gdeg}(f(X)t-1, \mathsf{MJ}(g)) + \deg(f(W))\Big)$$

Proof.

$$\begin{aligned} & V(y_1 - g\frac{\partial f}{\partial x_1}, \cdots, y_n - g\frac{\partial f}{\partial x_n}) \\ = {} & V(g(W)s-1, y_1 - g\frac{\partial f}{\partial x_1}, \cdots, y_n - g\frac{\partial f}{\partial x_n}) \cup V(g(W), y_1 = 0, \cdots, y_n = 0) \end{aligned}$$

By isomorphism,

$$\dim(V(g(W)s-1, y_1-g\frac{\partial f}{\partial x_1}, \cdots, y_n-g\frac{\partial f}{\partial x_n})) = \dim(V(g(W)s-1, y_1-\frac{\partial f}{\partial x_1}, \cdots, y_n-\frac{\partial f}{\partial x_n})),$$

and simple calculations gives

$$\dim(V(g(W)s-1, y_1-\frac{\partial f}{\partial x_1}, \cdots, y_n-\frac{\partial f}{\partial x_n})) = \dim(g(W), y_1=0, \cdots, y_n=0).$$

Since

$$\mathrm{gdeg}(g(W)s-1, y_1-g\frac{\partial f}{\partial x_1}, \cdots, y_n-g\frac{\partial f}{\partial x_n}) = \mathrm{gdeg}(g(W)s-1, y_1-\frac{\partial f}{\partial x_1}, \cdots, y_n-\frac{\partial f}{\partial x_n}),$$

and by Lemma 4,

$$\begin{aligned}
&\mathrm{gdeg}(y_1-g\frac{\partial f}{\partial x_1}, \cdots, y_n-g\frac{\partial f}{\partial x_n}) \\
=\ &\mathrm{gdeg}(g(W)s-1, y_1-\frac{\partial f}{\partial x_1}, \cdots, y_n-\frac{\partial f}{\partial x_n}) + \mathrm{gdeg}(g(W), y_1=0, \cdots, y_n=0) \\
=\ &\mathrm{gdeg}(g(W)s-1, \mathsf{MJ}(f)) + \deg(g(W)) \\
=\ &\mathrm{gdeg}(g(W)s-1)\cdot\mathrm{gdeg}(\mathsf{MJ}(f)) + \mathrm{gdeg}(g(W)).
\end{aligned}$$

□

References