THE USE OF COGNITIVE TASK ANALYSIS TO CAPTURE EXPERT INSTRUCTION IN TEACHING MATHEMATICS

By

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Dedication

This dissertation is dedicated to my best friend, my wife Esther. Thank you for your love and support without which this endeavor would not have been successful. To my children, Alfred, Allan, and Neema – you are amazing and I am blessed to be your dad. This one is for you! You too can do it. To my mom Rebecca Nthambi, though you cannot read and write, your love for education and encouragement has made me the man I am today. To my dad Benjamin Mutie, you taught me hard work and persistence, thank you.

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Abstract

The purpose of this study was to apply Cognitive Task Analysis (CTA) methods to capture expert mathematics instruction in solving quadratic equations. CTA seeks to elicit the highly automated and often-unconscious knowledge experts use to solve difficult problems and perform complex tasks. Students taking algebra find solving and understanding quadratic equations very challenging yet quadratic equations are a major component of building mastery in algebra. Four 8^{th} and 9^{th} grade Algebra teachers, who were qualified as experts using both qualitative and quantitative measures, were interviewed to capture the action and decision steps they use to teach quadratic equations. The individual protocols were then aggregated as a gold standard protocol (GSP) that was reviewed by a fifth senior SME for accuracy and consistency. Overall, there were found to be seven main procedures for solving quadratic equations. However, there was full alignment among the four experts on only seven percent of the action and decision steps, suggesting that multiple experts should be used to capture complex procedures, such as teaching algebra. Moreover, the experts omitted an average of 59.90% of the total action and decision steps, thus supporting previous research finding that experts may omit up to 70% of the critical information required to perform a complex task. The expert knowledge and skills captured may be used to train student teachers in teacher prep-programs and also offer professional development to Algebra teachers for teaching this highly complex subject.

CHAPTER ONE: OVERVIEW OF THE STUDY

Statement of the Problem

The percentage of high-achieving math students in the United States – and most of its individual states – are below those of many of the world's leading industrialized nations (Hanushek, Peterson, & Woessmann, 2011). Hanushek, et al., (2011) observe that unless U.S. schools find the tools to bring students up to the highest level of accomplishment, it places the nation at risk in international economy of the 21st century. There is a need for the United States to maintain its inventive advantage (Nord, Jenkins, Chan, & Kastberg, 2013). Both Hanushek et al. (2011) and Nord et al. (2013) argue that to maintain our inventive advantage, our system of education requires teaching that produces students with advanced mathematics and science skills.

Algebra is a foundation on which advanced mathematics, science, technology, and engineering courses are built (Evan, Gray, & Olchefske, 2006). According to both Gamoran and Hannigan (2000) and Musen (2010), success in Algebra opens doors to more advanced math, a college preparatory high school curriculum, higher college going and graduation rates. Therefore, Algebra is the gateway to success in career and college (Gamoran & Hannigan, 2000; Musen, 2010) and success in algebra is equally linked to career and job readiness and higher earnings once a student has joined the job market (Achieve 2008). The highest level of mathematics reached in high school is a key marker in students' success in college (Adelman, 2006; Evan et al. 2006; and Musen, 2010). In fact Evan et al. (2006) makes a vital observation in that passing algebra no later than 9th grade significantly increases the chances of the student graduating from high school, going to college, and graduating from college.

Algebra students find it challenging to solve and understand quadratic equations (Vaiyavutjamai, Ellerton, & Clements, 2005). Seeing the significance of quadratic equations in algebra and mathematics in general, Vaiyavutjamai et al. (2005) wonders why there has been so little research into the teaching and learning of quadratic equations. In fact students that have not mastered and understood how to solve quadratic equations, struggle in their later years in high school because all the other math concepts build on quadratic equations (Chazan & Yerushalmy, 2003; Vaiyavutjamai et al., 2005). Students are taught procedures that provide them solutions (Chazan & Yerushalmy, 2003) and the impression behind the curriculum is that if students have mastered these procedures, then they will be able to apply them in the context of new problems (Vaiyavutjamai et al., 2005).

This problem in student achievement in math is compounded by math teachers' lack of basic understanding of mathematical ideas and procedures of teaching mathematics (Ball, 2000). Ball (2000) noted that teachers do not have the knowledge that matters for teaching and therefore find teaching math difficult. This places a huge burden on Algebra students who are expected to pass this class to have opportunities for higher levels of math in preparation for college and other careers. In order to improve student achievement, schools must attend to the training of teachers because student learning is enhanced by the efforts of teachers who are skillful at teaching it to others (Ball, 2000; Darling-Hammond, 1999). Darling-Hammond (1999) observes that the effects of well-prepared teachers on student achievement are stronger than the influences of poverty, language barriers, and minority status. If we can capture expertise of teaching quadratic equations, then we can improve instruction.

Cognitive task analysis (CTA) is a best practice for capturing expertise. CTA is a methodology that has been used to capture the cognitive processes, decision-making, and judgments that underlie expert behaviors (Yates, 2007). Although there are no published studies that offer evidence on how many mathematics experts must be interviewed in order to capture enough critical information to aid in teaching quadratic equations, Chao and Salvendy (1994) concluded three experts were needed to acquire the optimal vital knowledge and skills needed to solve a complex software-debugging task. As such, this study seeks to use CTA to capture three to four math experts' knowledge and skills for solving quadratic equations.

Purpose of the Study

The purpose of this study is to conduct a CTA with math teachers who have been identified as experts, to capture the action and decision steps they use when teaching solving quadratic equations to 8th and 9th grade students. The following questions will guide this study:

1. What are action and decision steps that expert math teachers recall when they describe how to teach solving quadratic equations in Algebra?
2. What percentage of actions and/or decision steps, when compared to a gold standard, do expert math teachers omit when they describe how to solve quadratic equations in Algebra?

Methodology of the Study

The methodology of this study was to conduct a Cognitive Task Analysis to determine the action and decision steps expert algebra teachers' use while teaching solving quadratic equations to 8th and 9th grade students. The teachers came from two unified school districts in Southern California identified as experts through students' achievement data on State Standardized Tests and by their peers. Five SMEs were chosen, four to participate in semi-structured interviews and the fifth to verify the data collected form the SMEs on the steps to solve quadratic equations. The CTA followed a five-step process of:

1) A preliminary phase to build general familiarity mostly known as "bootstrapping;"

2) The identification of declarative and procedural knowledge and any hierarchical relationships in the application of these knowledge types;
3) Knowledge elicitation through semi-structured interviews;
4) Data analysis involving coding, inter-rater reliability, and individual SME protocol verification; and
5) The development of a gold standard protocol that was used to analyze and determine expert omissions and eventually to be used for training of novice algebra teachers.

Definition of Terms

The following are the definition of terms related to cognitive task analysis as suggested by Zepeda-McZeal (2014).

Adaptive expertise: When experts can rapidly retrieve and accurately apply appropriate knowledge and skills to solve problems in their fields or expertise; to possess cognitive flexibility in evaluating and solving problems (Gott, Hall, Pokomy, Dibble, & Glaser, 1993; Hatano & Inagaki, 2000)

Automaticity: An unconscious fluidity of task performance following sustained and repeated execution results in automated mode of functioning (Anderson, 1996; Ericsson, 2004).

Automated knowledge: Knowledge about how to do something: operates outside of conscious awareness due to repetition of task (Wheatley & Wegner, 2001)

Cognitive load: Simultaneous demands placed on working memory during information processing that can present challenges to learners (Sweller, 1988).

Cognitive tasks: Tasks that require mental effort and engagement to perform (Clark & Estes, 1996).

Cognitive task analysis: Knowledge elicitation techniques for extracting implicit and explicit knowledge from multiple experts for use in instruction and instructional design (Clark et al., 2008; Schraagen, Chipman, & Shalin, 2000).

Conditional knowledge: Knowledge about why and when to do something; a type of procedural knowledge to facilitate the strategic application of declarative and procedural knowledge to problem solve (Paris, Lipson, & Wixson, 1983).

Declarative knowledge: Knowledge about why or what something is; information that is accessible in long-term memory and consciously observable in working memory (Anderson, 1996; Clark & Elen, 2006).

Expertise: The point at which an expert acquires knowledge and skills essential for consistently superior performance and complex problem solving in a domain; typically develops after a minimum of 10 years of deliberate practice or repeated engagement in domain-specific tasks (Ericsson, 2004).

Procedural knowledge: Knowledge about how and when something occurs; acquired through instruction or generated through repeated practice (Anderson, 1982; Clark & Estes, 1996).

Subject matter expert: An individual with extensive experience in a domain who can perform tasks rapidly and successfully; demonstrates consistent superior performance or ability to solve complex problems (Clark, Feldon, van Merriënboer, Yates, & Early, 2008).

Organization of the Study

Chapter Two of this study reviews the literature in two parts; the first part of the literature review assesses the relevant literature associated to mathematics student performance and achievement in K-12 in the United States while the second part concentrates on literature relevant to Cognitive Task Analysis as a knowledge elicitation method for subject matter expertise. Chapter Three addresses the methodology of this study and how the approach to the research answers the research questions. Chapter Four analyses the collected data and results of the study. This chapter also compares these results in relation to the research questions. Chapter Five discusses the findings, the implications of the findings and CTA, limitations of the study, and implications for future research.

CHAPTER TWO: LITERATURE REVIEW

United States Math Performance

Countries all over the world have been participating in common international assessments of mathematics and science, the Trends in International Mathematics and Science Study (TIMSS) and Program in International Student Assessment (PISA) (Hanushek & Woessmann, 2010). According to Bishop (1992) and Hanushek and Woessmann (2010) these assessments provide countries with data that help them understand both the significance of low achievement and its impact of skills on economic and social outcomes. The proportion of U.S. students performing at proficient levels is lower than most of the world's leading industrialized countries (Fleischman, Hopstock, Pelczar, & Shelley, 2010; Hanushek et al., 2011; Nord, Jenkins, Chan, & Kastberg, 2013Va; Perterson, Woessmann, Hanushek, & Lastra-Andon, 2011). Students in the United States are not just underperforming because of the many English learners in United States' schools; "only 8% of white students in the U.S. class of 2009 scored at the advanced level, a percentage that was less than the share of advanced students in 24 other countries regardless of their ethnic background" (Hanushek, Peterson, & Woessmann, 2011, p.5). Hanushek et al. (2011) lament that the inability of American schools to bring students up to the advanced level of achievement in mathematics is much more deep-rooted. Within the 50 states, student achievement at the advanced level in mathematics varies significantly, but all do poorly when compared internationally (Hanushek et al., 2011). Therefore it is not a question of some individual states performing at higher levels being offset by the low achievement of other states, it is a question of the United States not preparing its students to learn and master the skills to perform competitively amongst other developed countries.

In 2013, Nord et al. reported that 9 percent of 15-year-old students in the United States scored at proficient level 5 or above in PISA assessment, which was lower than the Organization for Economic Cooperation and Development (OECD) average of 13%. The percentage of 15-year-old students that scored below the baseline proficient level 2 was reported at 26 percent, which was higher than the OECD average of 23 percent (Nord et al., 2013). Hanushek et al. found that the percentage of students in the U.S. Class of 2009 who were proficient in PISA's math assessment was well below that of most countries the U.S. normally compares itself. The average math scores in 2012 in the U.S. were not significantly different from the average scores in 2003, 2006 and 2009 (Nord et al., 2013).

With high unemployment in the United States, many are wondering whether our schools are preparing students effectively for the job market of the 21st century (Peterson et al., 2011). As President Barack Obama said in his 2011 State of the Union address, "We know what it takes to compete for the jobs and industries of our time. We need to out-innovate, out-educate, and out-build the rest of the world" (as cited in Peterson et al., 2011). In affirming the president's view, Peterson et al. (2011) observe:

> The United States could enjoy a remarkable increment in its annual GDP growth per capita by enhancing the math proficiency of U.S. students. Increasing the percentage of proficient students to the levels attained in Canada and Korea would increase the annual U.S. growth rate by 0.9 percentage points and 1.3 percentage points, respectively. Since long-term average annual growth rates hover around 2 and 3 percentage points, that increment would lift growth rates by between 30 and 50 percent (p. viii).

According to Peterson et al. (2011), when this is translated into dollar terms, these percentage increases in the annual U.S. growth amount to nothing less than 75 trillion dollars over a period of 80 years. Therefore those who say mathematics performance does not matter are clearly wrong and furthermore there is strong evidence that mathematics competence in high school is a major predictor of potential earnings and economic stability in the future than other skills acquired in high school (Bishop, 1992; Hanushek et al., 2011).

Job Market Consequences

With the growing economy, the demands of college faculty and employers for graduates with advanced math skills are increasing (Musen, 2010) and there is concern that future workers will not have the necessary skills they need to succeed in the 21st century economy (Evan et al. 2006). Musen (2010) acknowledges that the United States has made "a significant shift from a manufacturing- and agriculture- based economy to a knowledge- and service- based economy (p. 3). Consequently demand for highly qualified workers will continue to rise while at the same time the high unemployment rate will likely continue because those seeking for work are not qualified.

"Employers in manufacturing, high tech, health care, and other fields are struggling to find employees with the skills necessary to function well and meet expectations" (Achieve, 2008, p. 10) and this has long-term implications for the U.S. economy. According to Achieve (2008) a labor market whose qualifications are not keeping up with the rest of the world impedes the capacity for the U.S. to compete with other nations. Lack of skills has severe consequences for a country's overall growth and productivity (Hanushek et al., 2011; see also Achieve, 2008; Evan et al., 2006; Musen, 2010). To this end, the United States must invest in its K-12 education system by providing highly qualified teachers especially for algebra, which is the building block for advanced mathematics, science, engineering, and technology. "Algebra is not simply a means to an end; it is a gatekeeper" (Evan et al., 2006, p.9).

Algebra as a Foundational Course

This country needs to radically increase the percentage of students leaving high school with skills that are competitive by increasing the number of students who achieve proficiency in algebra in their middle school and early high school years (Evan et al., 2006). In fact, Evan et al. notes, "… successfully passing algebra early in a student's career – no later than 9^{th} grade – greatly improves the chances of the student graduating from high school, going to college, and graduating from college" (p. 9). History tells us that algebra was never a regular course offered in high school. It was not until the Massachusetts' act of 1827 (the first high school law in America) algebra was introduced as a compulsory course in high schools of every town in the state with a population of more than 500 families (Overn, 1937). Overn (1937) noted that as high schools became widespread in the country in the nineteenth century, algebra became a regular course. As algebra became a regular course in high school curriculum, "… its position was greatly strengthened by the fact that one college after another added elementary algebra to its admission requirements" (Overn, 1937, p.374). Therefore algebra has been linked to college entrance requirements for a long time and its importance cannot be overemphasized.

Many students are frustrated by algebra and see it as a monster that haunts and follows them everywhere. The following quotation captures so well how algebra has been regarded for many years:

> If there is a heaven for school subjects, algebra will never go there. It is the one subject in the curriculum that has kept children from finishing high school, from developing their special interests and from enjoying much of their home study work. It has caused more

> family rows, more tears, more heartaches, and more sleepless nights than any other school subject.
>
> Algebra is required in practically every course except those courses which are frankly dumb-bell courses. It is a requirement for graduation; it is a requirement for college entrance … (Anonymous editorial writer in a metropolitan newspaper as cited in Reeve, 1936, p.2).

According to Musen (2010), students that are successful in algebra have the opportunity to take more advanced math courses and college preparatory high school curriculum. The academic strength of a high school student's curriculum counts more than anything else in providing impetus toward completing a bachelor's degree (Adelman, 2006; Musen, 2010). Moses and Cobb (2001) through their book, Radical Equations: Civil Rights from Mississippi to the Algebra Project saw algebra not just as a gatekeeper but as a civil rights issue:

> So algebra, once solely in place as a gatekeeper for higher math and the priesthood who gained access to it, now is the gatekeeper for citizenship; and people who don't have it are like the people who couldn't read and write in the industrial age. … [Algebra has become] … a barrier to citizenship (p.14).

Moses and Cobb (2001) are reaffirming the belief that all students should learn algebra; making math literacy and economic access a civil rights issue of our time. Schools have to commit to every student to gain math literacy instead of weeding all but the brightest students out of advanced math (Moses & Cobb, 2001).

Algebra matters (Rose & Betts, 2004). In other words, Rose and Betts (2004) believe that "a curriculum that includes algebra is systematically related to higher earnings for graduates a decade after graduation" (p.510). Several studies (Adelman, 2006; Evan et al., 2006; Gamoran &

Hannigan, 2000; Moses & Cobb, 2001; Musen, 2010; Smith, 1996) have all concluded that algebra is the gateway to success in career and college. Vaiyavutjamai, Ellerton, and Clements (2005) established that students taking algebra find solving and understanding quadratic equations very challenging yet quadratic equations are a major component of building mastery in algebra.

Quadratic Equations

Researchers that have studied the teaching and learning of algebra have established that in order to have a rich understanding of the function concept that is the basis of the quadratic equations one must know how to represent functions in different ways (Vaiyavutjamai, 2009). In fact, Vaiyavutjamai (2009) continues, "… teachers often emphasize procedural skills more than the links between representations" (p. 1) which does not build the conceptual understanding that students need to solve quadratic equations. Chaysuwan's (1996) study of 661 grade 9 students in Bangkok reported that 70 percent of students' responses to standard quadratic equations tasks were incorrect immediately after participating in lessons on quadratic equations (as cited in Vaiyavutjamai et al., 2005). In a study spanning three countries, Vaiyavutjamai et al. (2005) established that more than 50 percent of students in Thailand and Brunei that were involved in the study were confused with respect to the concept of a variable as it manifested itself in quadratic equations. The United States, which was the third country in the study, 41 percent of second year university students in the study were confused with the concept of two solutions and that $\sqrt{}$ symbol means "the positive square root of." Students find it difficult to understand and solve quadratic equations (Eraslan, 2005; Vaiyavutjamai, 2009; Vaiyavutjamai et al., 2005).

Furthermore Zaslavsky's (1997) study of 800 10th and 11th grade students in Israel from eight high schools found that students could not differentiate between a quadratic function and a quadratic equation after they had completed the study of functions. Students treated a quadratic function as though it was a quadratic equation. Zaslavsky (1997) concluded that the relation between quadratic functions and quadratic equations seemed to hamper students' understanding of both quadratic functions and equations. In another study Vaiyavutjamai and Clements (2006) revealed that many students in their study that got the correct solutions had grave misconceptions about what quadratic equations were. "There answers were correct but, from a mathematical point of view, they did not know what they were talking about" (p. 73). Therefore, with quadratic equations remaining as important as they are, research is needed to inform teachers about how students think and what needs to be done to assist teachers to improve their students' concepts of a variable in the context of quadratic equations (Vaiyavutjamai & Clements, 2006; Vaiyavutjamai et al., 2005). Students must possess the knowledge of solving quadratic equations to gain fluency in algebra. This requires direct and systematic instruction on the recognition of and interaction with variables.

Teachers' Knowledge in Teaching Mathematics

While many factors contribute to a student's academic success, access to teachers' with the knowledge to teach mathematics contributes to gains in students' mathematics achievement (Boston, 2012; Hill, Rowan, & Ball, 2005). According to Hill et al. (2005) teachers' knowledge for teaching mathematics positively predicted student gains in mathematics. In this study, what was being measured was the relationship between the knowledge teachers were assumed to have for teaching mathematics, not just computational aptitude or courses taken. Knowledge for teaching mathematics goes beyond the courses math teachers take or basic mathematical skills (Hill et al., 2005; see also Boston, 2012; Leinhardt, 1989). Shechtman, Roschelle, Haertel, and Knudsen (2010) conducted a study in which they collected data in 125 seventh-grade and 56 eighth-grade classrooms where they investigated the relationship between teachers' mathematics knowledge for teaching and student achievement among other variables. They found that teachers' mathematical knowledge for teaching was associated with student achievement in mathematics.

Also teachers' knowledge of mathematics demonstrated strong links with the mathematical quality of instruction in their classroom (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008; Shechtman et al., 2010) and in turn showed a positive association to student achievement (Shechtman et al., 2010). Hill et al. (2008) further shows that there is a significant relationship between teachers' knowledge of mathematics, how they know it, and what they do in the classroom in the context of instruction. "Effective teaching is teaching that produces high levels of student performance skill" (Leinhardt, 1989, p. 52). Overall, these studies show that teachers play an important role in student achievement and more so when the teachers are knowledgeable in their content and the art of teaching. In fact, according to Darling –Hammond (2007), teacher expertise is one of the primary factors and single most important predictor influencing student achievement gains and therefore teachers who lack preparation in either subject matter or teaching methods are significantly less effective in producing student learning gains than those who are fully prepared and certified. The weight of substantial evidence indicates that teachers who have had more preparation for teaching are more confident and successful with students than those who have had little or none (Darling-Hammond, 2000; Darling-Hammond, 2007). All teachers must be provided with stronger understanding of how students learn and develop, how a variety of curricular and instructional strategies can address their needs, and how changes in school and classroom practices can support student growth and achievement (Darling-Hammond, 2007). Darling-Hammond (2004) observed that states and school districts that have focused on broader notions of accountability include investments in teacher knowledge and skill, organization of schools to support teacher and student learning, and systems of assessment that drive curriculum reform and teaching improvements.

Expertise in Mathematics

Borko and Livingston's (1989) study found that "Expert teachers have larger, better-integrated stores of facts, principles, and experiences upon which to draw as they engage in planning, reflection, and other forms of pedagogical reasoning" (p. 475) while novice math teachers seem not to have those same skills which are major components of learning to teach. Like experts in other fields, expert teachers have well-formed schemas that provide an outline for the reasonable analysis of information (Westerman, 1991; see also Borko & Livingston, 1989; Livingston & Borko, 1989). Mathematics teacher education is mostly concerned with the content knowledge required to teach mathematics (Liljedahl et al., 2009). Liljedahl et al. made this observation:

> Teacher education is a unique enterprise. The reason for this is that the *what* is also the *how.* That is, what we teach is also how we teach. As such, pre-service teachers have a unique experience. What they are learning is also how they are learning. Through their experiences as student teachers they are both student and teacher, and through the constant shifting between student and teacher they are given the opportunity to not only acquire the knowledge that they will require to become effective teachers, but also are given the opportunity to recast their initial (pre-conceived) beliefs about what it means to be a teacher, what it means to teach, what it means to learn, and even what it means for something to be mathematics. (p. 29)

Ultimately, what is at stake here is the teacher they eventually become, what they will teach and how they will teach mathematics especially quadratic equations to build long lasting conceptual understanding of their students. Darling-Hammond (1999) observes that the effects of well-prepared teachers on student achievement are stronger than the influences of poverty, language barriers, and minority status. The U.S. school system must provide high quality teaching and learning to all students. Schools that provide professional development to build the capacity of its teachers are better equipped to meet the needs of diverse learners (Darling-Hammond, 2004).

Professional Development to Improve Teacher Learning and Student Achievement

Professional development that focuses on concrete tasks of teaching is vital to enhancing teacher expertise and improving instructional practices that ultimately support an increase in student learning outcomes. Darling-Hammond and Richardson (2009) note that the most useful professional development emphasizes active teaching, assessment, observation, and reflection rather than abstract discussions. Effective professional development is intensive, ongoing, and connected to the teaching practice; focuses on the teaching and learning of specific academic content; is connected to other school initiatives; and builds strong working relationships among teachers (Darling-Hammond, Wei, Andree, Richardson & Orphanos, 2009). While teachers typically need substantial professional development in a given area (close to 50 hours) to improve their skills and their students' learning, most professional development opportunities in the U.S. are much shorter (Darling-Hammond, et al., 2009). Penuel, Fishman, Yamaguchi and Gallagher (2007) emphasize that professional development that is of longer duration and time span is more likely to contain the kinds of learning opportunities necessary for teachers to integrate new knowledge into practice. Garet, Porter, Desimone, Birman, and Yoon (2001) in their study: *What makes professional development effective? Results from a national sample of teachers,* noted that professional development may help contribute to a shared professional culture, in which teachers in a school or teachers who teach the same subject develop a common understanding of instructional goals, methods, problems, and solutions. Their study also indicated that professional development that focused on academic subject matter (content), gave teachers opportunities for practical work (active learning), and was incorporated into the daily life of the school (coherence), was more likely to produce enhanced knowledge and skills. In another study of mathematics teaching in California based on data on teachers' professional development experiences and school-level data on student achievement on a mathematics test administered statewide, Cohen and Hill (2000) found that controlling for the characteristics of students enrolled, average mathematics achievement was higher in schools in which teachers had participated in extensive professional development focusing on teaching specific mathematics content, compared to the achievement in schools where teachers had not. Their study also found that participation in professional development focusing on general pedagogy was not related to student achievement. Darling-Hammond et al. (2009) adds that improving professional development and collaborative learning opportunities for educators is a crucial step in transforming schools and improving academic achievement for all students. Therefore, if we capture expertise in teaching solving quadratic equations, then we can offer targeted professional development that would assist in improving instruction.

Cognitive Task Analysis

Cognitive task analysis (CTA) refers to a set of methods for capturing expertise. CTA is a methodology that has been used to capture the cognitive processes, decision-making, and judgments that underlie expert behaviors (Yates, 2007). Although there are no published studies that offer evidence on how many mathematics experts must be interviewed in order to capture enough critical information to aid in teaching quadratic equations, Chao and Salvendy (1994) concluded three experts were needed to acquire the optimal vital knowledge and skills needed to solve a complex software-debugging task. This study seeks to use CTA to capture math experts' knowledge and skills for solving quadratic equations.

Experts are frequently called upon for their knowledge and skills to teach, to inform curriculum content and instructional materials, and to mentor and coach others to perform complex tasks and solve difficult problems. The purpose of education is to replicate knowledge. According to Jackson (1985), traditions in education started with the expert and novice model in which the objective was for the novice to imitate the expert. The expert knows what precedes what in the range of steps and she devotedly follows such an order when deciding what her student is to learn at any one time (Jackson, 1985). However, current research shows that experts may omit up to 70 percent of the critical knowledge and skills novices need to replicate expert performance (Feldon & Clark, 2006). In their study, Feldon and Clark found that self-reports by experts were incomplete and inaccurate. In fact, as Jackson (1985) inquired, if teachers do not cover it all, then what do they leave out? According to several studies (Clark, Feldon, van Merriënboer, Yates, & Early, 2008; Feldon & Clark, 2006; see also Bedard & Chi 1992; Feldon, 2007), experts often omit critical knowledge and skills during self-report because they have automated their knowledge and skills through repeated practice so that it becomes unconscious and unavailable for recall.

Clark et al. (2008) noted that declarative knowledge is recalled from long-term memory and is consciously available in working memory. Declarative knowledge alone is not sufficient for performance. Procedural knowledge is required for skill performance and, as skills are continuously practiced, automaticity is attained (Clark et al., 2008). Since automated knowledge is outside the consciousness of the expert, CTA has been shown to be an effective method for capturing both the conscious and automated knowledge experts uses to perform complex skills and solve difficult problems (Feldon & Clark, 2006). To further understand why CTA is effective, the following sections examine the types of knowledge, the nature of automaticity, and the characteristics of expertise.

Knowledge Types

The procedure of teaching, learning, and assessing skills requires the transmission of knowledge from the expert (teacher) to the novice (student). The key to a learning experience that is productive is the achievement of the vital components of knowledge: declarative (*why* and *what* it is), procedural (*when* and *how* to do it) and the conditions under which to perform a procedural task.

Declarative Knowledge

Declarative is knowledge that is controlled and can be changed abruptly in the working memory (Clark & Estes, 1996). Declarative knowledge is explicit knowledge about facts, of why or that something is. It is overt and comprised of information about "*why*, *what* and *that*." According to Anderson (1982) the declarative stage of acquiring automated procedural knowledge is where the domain knowledge is directly embodied in procedures for performing the skill although by itself declarative knowledge is insufficient to execute skilled performance. When teaching new concepts and procedures to students such as teaching how to solve quadratic equations, it is necessary to focus on the required declarative knowledge so they are able to understand and comprehend the steps needed to solve the problem. Declarative knowledge, knowing *why* and *what* something is, clears the path for and supports the attainment of the *how* and *when* something is to be performed as procedural knowledge (Anderson, 1982; Anderson & Fincham, 1994; Anderson & Schunn, 2000; Clark & Estes, 1996).

Procedural and Conditional Knowledge

Declarative, procedural, and conditional knowledge are required for completing complex tasks and are acquired as one transitions from novice to expert. Procedural knowledge is the knowledge about "*when* and *how*" to perform a task (Ambrose, Bridges, DiPietro, Lovett, & Norman, 2010; Anderson, 1982; Anderson & Krathwohl, 2001). According to Clark and Estes (1996) procedural knowledge is difficult to learn and fast to execute. It requires practice and feedback but once it becomes a high level of expertise or automated, it is very difficult to change (Anderson, 1993; Clark & Estes, 1996). Procedural knowledge accounts for 70 to 90 percent of the total knowledge adults have (Clark & Elen, 2006). Paris, Lipson and Wilson (1983) noted that conditional knowledge is a form of procedural knowledge. Conditional knowledge is knowing *when*; provides the circumstances or rational for various actions including value judgments, and helps modulate procedural and declarative knowledge (Paris et al., 1983). With repetition and practice, both declarative and procedural knowledge become stronger and performance becomes more fluid, consistent, and automated.

Automaticity

Through repeated performance and deliberate practice of a task, declarative and procedural knowledge becomes automated and unconscious in nature and the speed in performing the task increases while the amount of active mental effort decreases (Feldon, 2007). Clark and Elen (2006) emphasize that the automation process is advantageous to expertise as it supports the capacity to respond to novel problems with speed, accuracy, and consistency within an expert's domain. Automated processes often initiate without prompting and once SMEs initiate, automated processes run to completion without being available for conscious monitoring (Clark, 1999; Feldon, 2007). Automaticity frees up the working memory by unconsciously processing and running procedures (Wheatley & Wegner, 2001) and with repeated practice, cognitive tasks becomes fluid and automatic and SMEs are able to deploy strategies to solve problems with ease (Clark, 1999). Automaticity enables SMEs to perform tasks requiring declarative and procedural knowledge unconsciously freeing up working memory to address novelty, however due to the unconscious nature of automaticity it is resistant to change (Clark, 2008c; Wheatley & Wegner, 2001) and difficult to modify, eliminate, or express to others using concrete language and examples (Clark & Elen, 2006; Clark & Estes, 1996; Kirschner, Sweller, & Clark, 2006).

Expertise

Characteristics of Experts

The characteristics of expertise include extensive and highly structured knowledge of the domain, effective strategies for solving problems within the domain, and expanded working memory that utilizes elaborated schemas to organize information effectively for rapid storage, retrieval, and manipulation (Bedard & Chi, 1992; Ericsson & Lehmann, 1996; Feldon, 2007). In fact, Chi (2006) defines an expert as a distinguished or brilliant journeyman, highly regarded by peers, whose judgments are uncommonly accurate and reliable, whose performance shows consummate skill and economy of effort, and who can deal with certain types of rare or tough cases. Also, an expert is one who has special skills or knowledge derived from extensive experience sub-domains. According to Bedard and Chi (2006), what sets the expert apart from novice is that experts have developed schemas allowing them to efficiently organize information so it is quickly and efficiently retrieved with minimal effort. An expert can see beyond function and simple schemas, they create mental models while novices are more literal and predictable. According to Ericsson and Lehmann (1996) the ability of experts to exceed usual capacity limitations is important because it demonstrates how acquired skills can supplant critical limits within a specific type of activity. Extensive evidence indicates that experts are able to attend to and process much more domain-relevant information in working memory that is possible for novices (Ericsson & Lehmann, 1996; Feldon, 2007; see also Glaser & Chi, 1988). Therefore, as will be described later in this review, by using CTA, the expertise of subject matter experts can be captured and taught to novice learners to begin to build their own expertise.

Building Expertise

Expertise, by its nature, is acquired as a result of continuous and deliberate practice in solving problems in a domain. Expert performance continues to improve as a function of more experience, coupled with purposeful practice (Alexander, 2003; Ericsson, 2004; Ericsson & Charness, 1994; see also Ericsson, Krampe, & Tesch-Römer, 1993); the challenge for aspiring expert performers is to avoid the arrested development associated with automaticity and to acquire cognitive skills to support their continued learning and improvement (Ericsson, 2004). According to Ericsson (see also Ericsson & Charness, 1994), deliberate practice is therefore designed to improve aspects of performance in a manner that assures that attained changes can be successfully integrated into representative performance. However, Ericsson et al., (1993) warn that although experts outperform novices, research has shown that expertise does not transfer to domains unfamiliar to the expert. Thus, the domain-specific nature of expert's superior performance implies that acquired knowledge and skill are important to attainment of expert performance. Once we conceive of expert performance as mediated by complex integrated systems of representations for the execution, monitoring, planning, and analyses of performance, it becomes clear that its acquisition requires an orderly and deliberate approach (Ericsson, 2004). Therefore, by engaging in purposeful practice and problem solving, a novice learner develops over time (usually 10 years) more efficient schema, knowledge, skills and decision steps.

Consequences of Expertise

As new knowledge becomes automated and unconscious, experts are often unable to completely and accurately recall the knowledge and skills that comprise their expertise, negatively impacting instructional efficacy and leading to subsequent difficulties for learners (Chi, 2006; Feldon, 2007). Feldon (2007) observes that automated procedures are deeply rooted and not easy to change and therefore automaticity impairs the development of expertise. Experts regularly cannot articulate their knowledge because much of their knowledge is implied and their overt intuitions can be flawed (Chi, 2006). Evidence (Feldon, 2007) suggests that routine approaches to problems are goal-activated and significantly limits the solution search. This is also made worse because experts tend overestimate their capabilities by being overly confident (Chi, 2006) and therefore fail to articulate relevant cues seen in problem states (Feldon, 2007). Feldon (2007) observed in his study that, extensive practice using procedures to solve problems in a specific domain led experts to automate portions of their skills. Consequently, the most frequently employed elements – presumably those of greatest utility within a domain of experts – were the most difficult to articulate through recall. Therefore, the automaticity of experts impairs their ability to consciously identify many of the decisions they make thereby omitting key details and process information necessary to provide instruction on optimal performance.

Expert Omissions

Experts in an instructional role may unintentionally leave out information that students must master when learning procedural skills. Recent research (Clark, Pugh, Yates, Inaba, Green, and Sullivan, 2011) has shown that when experts describe how they perform a difficult task, they may unintentionally omit up to 70 percent of the critical information novices need to learn to successfully perform the task. According to Clark (2008) this is a serious problem because it forces novices to “fill in the blanks” using less efficient and error-prone trial-and-error methods. Furthermore, as these errors are practiced over time, they become more difficult to “unlearn” and to correct. There are two reasons for this problem (Clark & Estes, 1996). First, as SMEs gain expertise, their skills become automated and the steps of the procedure blend together. Experts perform tasks largely without conscious knowledge as a result of years of practice and experience. This causes experts to omit critical steps when describing a procedure because this information is no longer accessible to conscious processes (Clark & Elen, 2006). Secondly, many SMEs are not able to share the complex thought processes of behavioral execution of skills. Even experts who make an attempt to “think out aloud” during the process of complex problems often omit essential information because their knowledge is automated (Clark & Elen, 2006; Clark & Estes, 1996). Consequently, it is difficult to identify points during a procedure where an expert makes decisions (Clark & Elen, 2006). Clark et al. (2011) further reported that when experts are asked to describe a procedure, they rely on self-recall of specific skills but studies from the field of cognitive psychology suggest that the use of standard self-report or interview protocols to extract descriptions of events, decision making, and problem solving strategies can lead to inaccurate or incomplete reports. In fact, experts often do not recognize these errors because of the automated and unconscious nature of the knowledge described (Wheatley & Wegner, 2001). Experts’ self-reports about their approaches to complex tasks have revealed that they leave out up to 70 percent of critical information (Feldon & Clark, 2006).

Cognitive Task Analysis (CTA)

Definition of CTA

Cognitive task analysis (CTA) is a group of "methods used for studying and describing reasoning and knowledge" (Crandall, Klein, & Hoffmann, 2006, p. 9). CTA has evolved from traditional task analysis methods, and is utilized in order to elicit and clarify expert knowledge within a specific domain. According to Clark et al. (2008) CTA uses a variety of interview and observation strategies to capture a description of the explicit and implicit knowledge that experts use to perform complex tasks. Crandall et al. (2006) adds that CTA is a type of knowledge elicitation, analysis of data, and representation of knowledge tool that captures expert knowledge on the way the mind works. CTA is an extension of traditional task analysis that identifies the knowledge, thought processes and goal structures that underlie observable task performance, as well as overt and covert cognitive functions that form the integrated whole (Chipman, 2000; Clark et al., 2008). Therefore, CTA yields information about the knowledge, thought processes, and goal structures that underlie observable task performance; the explicit and implicit knowledge that is explicated from the CTA can be used to teach, train, assess performance and develop expert systems.

Brief History of CTA

The seeds of CTA were planted as far back as the 1800s and can be found throughout the history of applied psychology, industrial engineering and human factors (Clark & Estes, 1996; Militello & Hoffmann, 2008). And then as recently as the 1980s, CTA methods emerged as a response to capturing cognitive processes as a result of workplace demands and have become refined over the last 20 years due to the demand of modern technology (Militello & Hoffmann, 2008). CTA has been long in evolution and is related to many fields and is now one of the most successful methods of elicitation of expert knowledge that can be used today. Hoffmann and Woods (2000) noted that CTA evolved from traditional task analysis and the study of cognitive engineering, in order to aid in human performance, and improve skill in the use of new technology. According to Annett (2000) the foundations for cognitive psychology took root in the 1950s, and evolved into a definition of CTA in the 1970s with an emphasis on capturing human expertise that was never captured with older methods of task analysis. Therefore, CTA is the advanced task analysis system that fills in that gap.

Cognitive Task Analysis Methodology

A number of researchers have identified the stages through which a typical, ideal cognitive task analysis would proceed. In Chipman, Schraagen and Shalin (2000) view, the ideal model of cognitive task analysis is one that is not subject to resource restrictions, is typified by a series of discrete steps. According to Chipman et al. (2000) and Clark, Feldon, van Merriënboer, Yates, and Early (2008) these discrete stages are: (a) collect preliminary knowledge, (b) identify knowledge representations, (c) apply knowledge elicitation techniques, (d) verify/analyze data elicited, and (e) format results of the analysis as a basis for an expert system or expert cognitive model. Although there are over 100 varieties of cognitive task analysis (Yates, 2007), in a general sense, most varieties follow a five-stage process. Multiple authors have developed taxonomies that categorize these techniques according to a number of criteria.

Taxonomies of Knowledge Elicitation Techniques

Knowledge elicitation is the process of extracting domain specific knowledge that underlies human performance (Cooke, 1999). There are four categories of knowledge elicitation: (a) observations, (b) interviews, (c) process tracing, and (d) conceptual methods (Cooke, 1999). According to Cooke (1994, 1999), knowledge elicitation begins with observing task performance within the domain of interest and provides a general conceptualization of the domain and any constraints or issues to be addressed in the later phases. While interviews are the most frequently used elicitation method, process tracing is the most often used method to elicit procedural information, such as conditional rules used in decision making (Cooke, 1999). Cooke (1994, 1999) added that conceptual methods elicit and represent conceptual structure in the form of domain-related concepts and their interrelations. This method is mainly used to elicit knowledge to improve user interface design, guide development of training programs, and understand expert-novice differences. Wei and Salvendy (2004) identified formal models as a fourth family of knowledge elicitation technique. Since these techniques are based on processes, such taxonomies/typologies may make it difficult for analysts to choose an appropriate CTA approach, especially when the desired result is a particular type of knowledge. In order to elicit accurate and complete expert knowledge descriptions, Cooke (1994, 1999) proposed using multiple knowledge elicitation techniques to capture rich representations of the task.

Pairing Knowledge Elicitation with Knowledge Analysis

Since the current classification schemes organize CTA methods by process rather than the desired outcome or application, practitioners find it difficult to select an optimal method for their specific purpose (or knowledge outcome). Therefore, Yates (2007) identified the most frequently used CTA methods and the knowledge types associated with the respective methods and outcomes (product approach versus the existing process approach). Although data analysis and knowledge representation are considered as two distinct techniques of CTA, they are often linked with elicitation methods (Yates, 2007). Additionally, since both share common characteristics, data analysis and knowledge representation are often combined into a single category in a classification scheme. Crandall, Klein, and Hoffman (2006) noted that many knowledge elicitation methods have analytical processes and representational formats embedded within the method. Therefore, it may be more appropriate to examine CTA as a pairing of knowledge elicitation with an analysis/representation technique (Yates, 2007). The results of Yate's (2007) study revealed that the most frequently used CTA method pairings included standardized methods and informal methods and the application of these methods were associated more with declarative knowledge than procedural knowledge. Also, this study found that standardized methods (protocol analysis and conceptual methods) appeared to provide greater consistency in the results than informal models (observations and interviews). CTA relies on the use of both elicitation and analysis/representation methods. For efficiency and optimal use of CTA, CTA methods need to be classified in terms of desired outcome rather than process.

Effectiveness of CTA

Cognitive task analysis has proven to be an effective method for capturing the explicit observable behaviors, as well as the tacit, unobservable knowledge of experts. According to Hoffman and Militello (2009) the use of CTA identifies the explicit and implicit knowledge of experts to use for training; the knowledge elicited from experts includes domain content, concepts and principles, experts' schemas, reasoning and heuristics, mental models and sense making. Data captured from CTA supports effective and efficient training and instructional activities in complex systems. Particularly for domains that emphasize technical-functional capabilities, such as engineers or military personnel, simply listing the action steps for a particular procedure or task is not an adequate way to train (Means & Gott, 1988). Even if context is captured, traditional methods such as asking experts to list steps or making observations, do not accurately account for abstract knowledge in experts. Therefore, use of CTA is useful for educators to identify the subtle skills, perceptual differences, and procedures that may be left out during instruction (Crandall et al., 2006). Compared to other strategies, Cognitive Task Analysis is more effective at capturing the unconscious, complex cognitive action and decision steps of experts.

Research has shown instruction using Cognitive Task Analysis is more cost effective and efficient than other instructional models (Clark, Feldon, van Merriënboer, Yates, & Early, 2008). This research found that in a number of studies reviewed, CTA-based instruction resulted in higher achievement compared to non CTA-based instruction. Clark et al. (2008) noted the importance of CTA was based on compelling evidence that experts are not fully aware of about 70% of their own actions, decisions and mental analysis of tasks and so are unable to explain them fully even when they intend to support professional training of novices. Therefore, according to Clark et al. (2008) CTA methods attempt to overcome this problem by employing observational and interview strategies that allow knowledge elicitors to capture more accurate and complete descriptions of how experts succeed at complex tasks and this in turn reduces the total training days by nearly a half. Flynn’s (2012) research found that much of the literature emphasized the degree to which the CTA methodology positively impacted costs and efficiencies; however, more meaningful, was the degree to which the method elicited expert knowledge that could be translated into guided instructional or Guided Experimental Learning (Clark, 2004). The use of CTA in instruction and training has been proven to be positively related to cost savings due to reduced training times with comparable learning outcomes (Clark, 2011).

Benefits of CTA for Instruction

Studies that have applied Cognitive Task Analysis to capture knowledge and deliver instruction have uncovered several benefits and useful design strategies as compared to other forms of instruction. According to Hoffman and Militello (2009) CTA is able to identify the explicit and implicit knowledge of experts to use for training and technology. The authors indicate that the knowledge that can be elicited from experts includes domain content, concepts and principles, experts' schemas, reasoning and heuristics, mental models and sense making. In fact these authors observed that data captured from CTA supported effective and efficient training and instructional activities in complex systems. Others like Crandall et al. (2006) noted that CTA could be used for training in a variety of ways, such as "cognitive training requirements, scenario design, cognitive feedback, and on-the-job training" (p. 196). CTA has proven to be an effective method for eliciting the nuances in expert knowledge, such as decision points and perspectives, resulting in a variety of instructional strategies utilizing the outcomes of CTA (Crandall et al., 2006; Hoffman & Militello, 2009; Means & Gott, 1998).

Studies across a variety of domains have explored the degree to which CTA-informed instruction has influenced learning outcomes. In another review, Clark (2014) noted that CTA results in nearly 30-45% learning performance increases as compared to instruction that is informed by traditional observation or task analysis. The author further points out that there is evidence pertaining to the front-end of training being advantageous for increasing learning and reducing the number of mistakes made by recently graduated students within the field of healthcare. Therefore, CTA-informed learners or employees may be considered better trained and perhaps more appealing to employers throughout the medical field. In Crandall and Getchell-Reiter's (1993) study, their findings were favorable in supporting CTA as the most effective method for capturing expertise. They discovered that the formal analysis helped generate more instances from experts, which captured the subtle nuances of what was considered highly subjective material. Other studies (see Schaafstal, Schraagen, & Marcel, 2000; Velmahos, et al.,

2004) associated to the task of troubleshooting were conducted in order to gain a sense of what analyses surfaced the most accuracies and abilities in solving problems. The CTA method proved to be effective because it gained expertise from both people who were from a theoretical background as well as practitioners. Overall, CTA has been shown to be effective in capturing expertise and informing instruction in a wide range of domains, including software development (Schaafstal et al., 2000), military (Fackler, et al., 2009; Flynn, 2011), business sector (Klein, 2004; Mayer, 2011), and medical fields (Clark, 2014).

Summary

Cognitive Task Analysis is an interview and observation methodology that is used to capture underlying cognitive procedures experts use to perform and solve complex problems. When experts are asked to describe how to perform complex tasks, they unconsciously omit up to 70% of action and decision steps novices need to successfully perform the complex task. Teaching solving quadratic equations is a complex task which expert knowledge and skills is required to meet the instructional needs of 8^{th} and 9^{th} grade students in K-12 who need to build mastery of Algebra One because Algebra One is a building block for higher level mathematics in high school. The level of mathematics students in high school attain in their senior year has a correlation to whether they will earn a bachelor's degree. As such, the task of solving quadratic equations in algebra may gain from doing a CTA to inform teaching. Therefore, the purpose of this study was to perform a CTA to examine the knowledge and skills that expert math teachers use to teach solving quadratic equations in Algebra and the extent at which expert math teachers omit the critical conceptual knowledge, action steps, and decision steps when describing instruction for solving quadratic equations.

CHAPTER THREE: METHODOLOGY

The purpose of this study was to investigate the expertise of mathematics teachers that teach algebra, specifically how to solve quadratic equations, to 8th and/or 9th grade students in K-12. The study used Cognitive Task Analysis (Clark, Feldon, van Merriënboer, Yates, & Early, 2008; Cooke, 1999; see also Cooke, 1994; Wei & Salvendy, 2004) methods to capture the knowledge of expert math teachers when they describe how they teach solving quadratic equations. Clark et al. (2008) suggested four steps that could be used for conducting a Cognitive Task Analysis procedure to elicit knowledge. These four steps were used for knowledge elicitation in this study and they included, (a) collecting preliminary domain-specific knowledge, (b) identifying the types of knowledge associated with the task, (c) applying the knowledge elicitation technique in a semi-structured interview, and finally (d) verifying and analyzing the results from the interviews.

The following were the research questions that guided this study:

1. What are action and decision steps that expert math teachers recall when they describe how to teach solving quadratic equations in Algebra?
2. What percentage of actions and/or decision steps, when compared to a gold standard, do expert math teachers omit when they describe how to solve quadratic equations in Algebra One?

Task

The task for this study was to elicit the action and decision steps Algebra One teachers can recall when describing how to teach solving quadratic equations to 8th and/or 9th grade students. Students taking algebra find understanding and solving quadratic equations very challenging yet quadratic equations are a major component of building mastery in algebra. To be proficient in solving quadratic equations, students require both declarative (what and why to perform the task) and procedural (when and how to perform the task) knowledge. Although solving quadratic equations is common in algebra, it involves complex procedural steps, which can result in getting an erroneous solution if the steps are inaccurately executed. As described in Chapter Two, expert math teachers' knowledge and skills may be automated and unconscious to the extent that when they teach mathematics they may be providing incomplete or inaccurate instruction that ultimately diminishes students' achievement of the task.

Population and Sample

Yates, Sullivan, and Clark (2012) suggest that 3 to 4 experts are needed for CTA to capture the optimal amount of significant information during a procedure. Therefore, this study's sample included four algebra teachers from two Southern California School Districts plus one senior expert to review the final gold standard protocol. The researcher collaborated with the two school districts superintendents and other district leaders to identify expert math teachers for Algebra One. The teachers were selected based upon their expertise in teaching algebra to 8^{th} and 9^{th} grade students. Feldon (2007) describes the characteristics of expertise to include extensive and highly structured knowledge of the domain, effective strategies for solving problems within the domain, and expanded working memory that utilizes elaborated schemas to organize information effectively for rapid storage, retrieval, and manipulation. According to Clark et al., (2008) a subject matter expert (SME) is a person with wide experience and is capable of performing a range of tasks fast and successfully. For this study, expertise was shown by the algebra teacher's years of experience, teaching expertise, and more so performance of their students in state standardized tests. The sample of expert math teachers was asked to voluntarily participate in CTA guided semi-structured interviews to capture their subject matter expertise in teaching solving quadratic equations for the purpose of aggregating a gold standard protocol.

Data Collection and Instrumentation

Using Clark et al. (2008) guidelines, CTA was conducted using the five common steps that entailed: (a) collecting preliminary knowledge, (b) identifying knowledge representations, (c) applying focused knowledge elicitation methods, (d) analyzing and verifying acquired data, and (e) formatting results for the intended application. The steps are described below, as they were used during each step of the CTA process.

Step 1: Collecting preliminary knowledge

In this preliminary stage, the researcher is a high school mathematics teacher and is familiar with the task of solving quadratic equations. As part of this research, the researcher identified the task sequences and procedures that became the focus of the CTA. While the researcher did not need to become an SME, the researcher was familiar with the procedures and steps of solving quadratic equations. This stage also included document analysis of books and other resources describing this task. This orientation prepared the researcher for subsequent task analysis activities. The information elicited during semi-structured interviews may be more robust when analysts are already familiar with experts' language.

Step 2: Identifying Knowledge Representations

The information collected in the first step was examined to identify and generate a preliminary list of subtasks and types of knowledge required to solve quadratic equations. This involved using flowcharts in order to provide a systematic way of organizing the information that was elicited from the SMEs.

Stage 3: Applying Focused Knowledge Elicitation Methods

At this stage, semi-structured interviews following the protocol attached as Appendix A were contacted to elicit information from the four SMEs. During the semi-structured CTA interview, the SMEs were asked a series of questions that focused on the major tasks and potential difficulties a student may encounter when solving quadratic equations. The action and decision steps are considered the critical information novices need to perform the task. Action steps begin with a verb and are statements about what a person should do, such as “When driving out of a garage, close the garage door.” Decision steps contain two or more alternatives to consider before taking an action, such as “When driving, IF you are backing up, THEN look back and the rear view mirrors; IF it is not clear and safe, THEN wait; IF it is clear and safe, THEN proceed to back up.”

These experts were asked to describe the action and decision steps they used to solve quadratic equations. The preliminary semi-structured interview began with a clear description of the CTA process by the researcher. The SMEs were asked to list or outline the performance sequence of all key subtasks necessary to successfully solve quadratic equations. SMEs were also asked to describe at least five problems that an expert should be able to solve if they have mastered how to solve quadratic equations. These problems ranged from routine to highly complex ones. The initial Round One interviews lasted approximately three hours, followed by Round Two interviews that allowed the SMEs to review the individual draft protocol to add or delete any unnecessary steps for this task. The Round Two interviews lasted approximately two hours.

Step 4: Analyzing and Verifying Data Acquired

The information generated from the SMEs through the semi-structured interviews was used to create a protocol for solving quadratic equations. However, before this protocol was prepared, the researcher coded the data using domain specific codes and formatted the results for verification, refinement, and revision by participating SMEs to ensure that the representations of procedures were complete and accurate.

CTA coding scheme. After the four interviews were transcribed, a CTA coding scheme was generated (Appendix B) that allowed the researcher to analyze the data from the semi-structured interviews. The coding categories used were: objective, conditions/cues, main procedures, action steps, decision steps, standards, equipment, reasons, new concepts and others that were determined as the coding was in progress.

Inter-rater reliability (IRR). The researcher and another trained coder for inter-rater reliability coded one of the coded transcripts. After the coding was completed, it was analyzed and an inter-rater reliability was calculated as a percentage of agreement between the two coders. Once there is an 85% or higher agreement in inter-rater reliability, then the coding process is considered as reliable among the different coders (Hoffman, Crandall, & Shadbolt, 1998). Crandall, et al., (2006) suggested that if the inter-rater reliability is less than 85%, then the coding scheme may need to be further revised and refined.

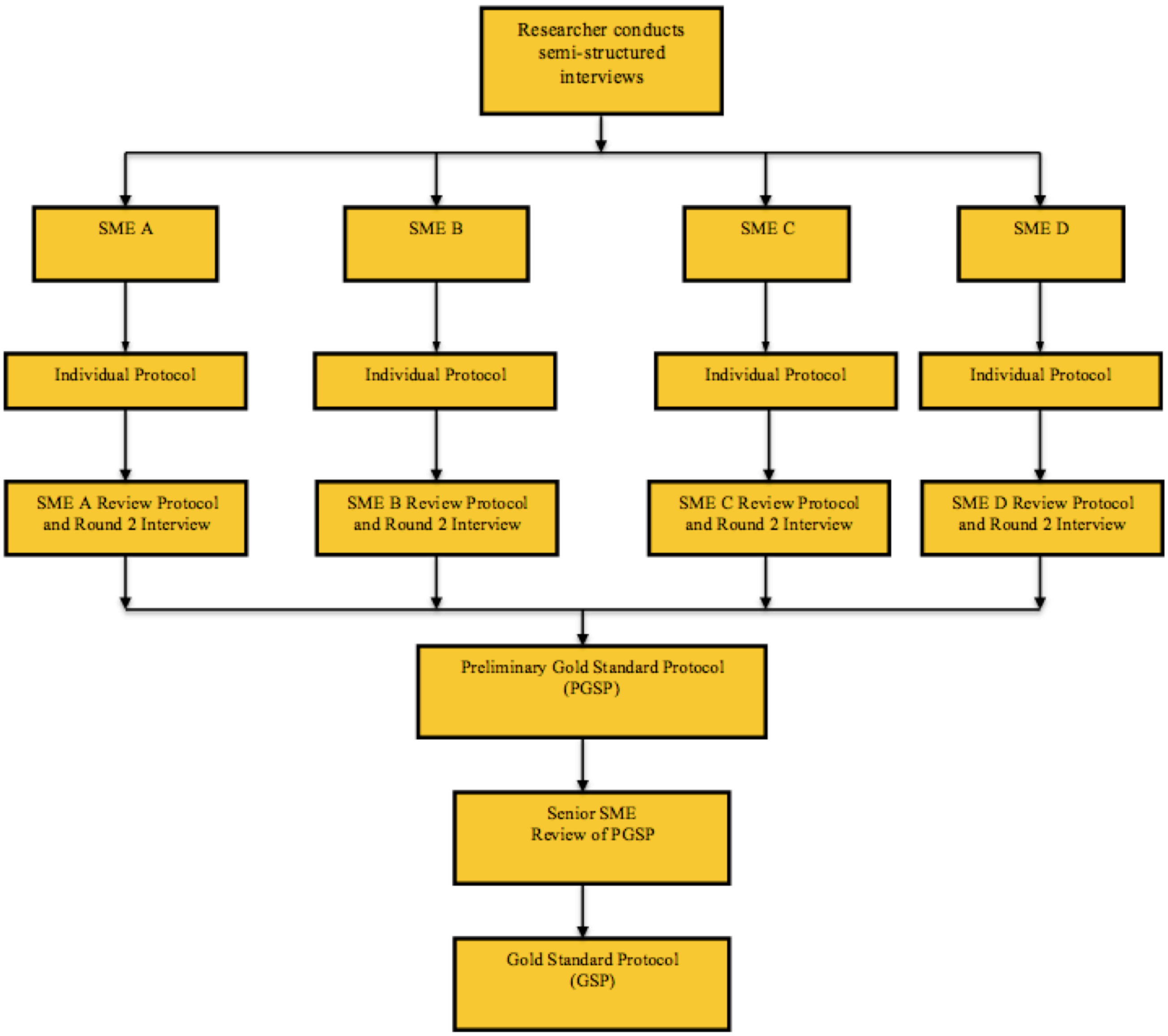
Step 5: Formatting Results for the Intended Application

Data Analysis

Research Question 1. What are action and decision steps that expert math teachers recall when they describe how to teach solving quadratic equations in Algebra?

Gold standard protocol (GSP). Each of the individual reviewed and corrected CTA protocols was aggregated to develop one draft protocol. This protocol served as a preliminary gold standard protocol (PGSP) and was given to another senior expert who was not part of the initial CTA interviews to review and edit for completeness and accuracy to produce a corrected and final GSP (Appendix D) that was used to generate the action and decision steps expert mathematics teachers make to teach solving quadratic equations in Algebra One. See Appendix C for a description of a complete procedure for creating a GSP. Figure 3.1 visually represents the five-stage process.

Figure 3.1: Process of conducting CTA semi-structured interviews and aggregating the GSP



Research Question 2. What percentage of action and/or decision steps, when compared to a gold standard, do expert math teachers omit when they describe how to solve quadratic equations in Algebra One?

Microsoft Excel spreadsheet analysis. The gold standard protocol that was generated from the four individual CTA protocols was transferred to a Microsoft Excel spreadsheet. This GSP included the action and decision steps that were recorded for teaching solving quadratic equations. Each of these steps that were included in the individual SME, a “1” was put in the column under that SME, otherwise if the step was missing, then a “0” was put in that column under the SME. Microsoft Excel spreadsheet formulas were used to add the number of “1’s” for each SME and the number of “0’s,” which indicated the number of omissions recorded for each SME. The analysis of this data allowed the researcher to calculate the percentage of omissions of actions and decision steps compared to the gold standard protocol. The mean percentage of omissions was then calculated and recorded to respond to Research Question 2.

CHAPTER FOUR: RESULTS

Overview of Results

This study examined the action and decision steps of five expert mathematics teachers using a CTA knowledge elicitation procedure to capture their expertise in teaching solving quadratic equations to 8^{th} and 9^{th} grade students in K-12. The data will be presented using frequency counts, percentages and graphs in order to answer the two research questions.

Research Questions

Question 1

What are action and decision steps that expert math teachers recall when they describe how to teach solving quadratic equations in Algebra One?

Inter-rater reliability. The researcher and one other knowledge analyst coded data from one of the four interview transcripts for the purpose of knowledge type coding of action and decision steps captured from the SME identified as SME C. After the researcher and the other knowledge analyst coded SME C's transcript independently, they got together and discussed their respective coding and attempted to reconcile any differences. Tallying all the coded items that were in agreement between the two coders and dividing this number with the total number of coded items determined inter-rater reliability. The inter-rater reliability showed a consistent interpretation of knowledge items for this SME with a 97.5% agreement across 16 distinct codes as shown in Appendix B. This researcher coded the remaining three SME interview transcripts before developing the initial individual protocol for solving quadratic equations for each SME.

Flowchart analysis. A flowchart was created from the interview data of SME C that is shown as Appendix D. After the flowchart was created, the researcher analyzed it to ensure that action and decision steps captured from the first interview were logical and could be executed when teaching solving quadratic equations. Also the flowchart showed instances where some decisions were being made without appropriate actions. These observations led the researcher to ask further questions to the SMEs during Round Two of the knowledge elicitation interviews that followed. On the agreed day for the review of the initial draft protocol and Round Two interviews, the researcher emailed the draft protocol to the SMEs to download into their computers. Each draft protocol steps where numbered and the transcript line number where that information was extracted from were indicated at the end of each sentence for ease during the review process. The SMEs were asked to use the Microsoft Word Track Changes feature on their computer to track any changes made on the draft protocol. Each SME was asked to read their draft protocol entirely before starting to make any changes to allow them to understand the document and have a general view of what was captured from their transcript. The researcher had at their disposal the original transcript that was used to generate the draft protocol. The researcher asked the SME to add any new steps that may make the execution of the procedures complete and/or delete any steps that may have been misleading and to give reasons.

Gold standard protocol. Once all the SMEs had reviewed their respective draft protocols, the researcher generated a preliminary gold standard protocol (PGSP). The researcher aggregated the data from the protocols generated from the four SMEs to solve quadratic equations. An example of the process of aggregating the gold standard protocol (GSP) is shown in Table 4.1.

Table 4.1

Example of aggregating action and decision steps for the preliminary gold standard protocol (PGSP)

SME C – Action step:

- Sing to students a song on the quadratic formula: *"x-equals negative b, plus or minus square root of b squared minus 4ac, all over 2a"* (P1)

SME D – Additions (in bold) to SME C's Action step:

- **Teach students a way to memorize** the quadratic formula: Sing, *"x-equals negative b, plus or minus square root of b squared minus 4ac, all over 2a"* Or say: "**A negative boy could not decide whether or not to go to a radical party. He decided to be square and he missed out on 4 awesome chicks. The party was all over at 2 am."** (P1, P2)

SME A – Additions to SMEs C and D Action step
SME A – said: "Play the quadratic formula song video (You Tube) so students can memorize the quadratic formula." Therefore the action step as aggregated from SMEs C and D remained as is.

- **Teach students a way to memorize** the quadratic formula: Sing: *"x-equals negative b, plus or minus square root of b squared minus 4ac, all over 2a"* Or say: "**A negative boy could not decide whether or not to go to a radical party. He decided to be square and he missed out on 4 awesome chicks. The party was all over at 2 am."** (P1, P2, P3)

SME B – Additions (underlined) to PGSP (as Action step reads in final GSP)
SME B – said: "Play the quadratic formula song and make students sing along.

GSP Step: 3.1: **Teach students a way to memorize** the quadratic formula: GSP Step: 3.1.1: Sing: *"x-equals negative b, plus or minus square root of b squared minus 4ac, all over 2a."* <u>Make students sing along</u> (P4) or teach students to memorize the quadratic formula using the phrase: "**A negative *b*oy could not decide whether or not to go to a radical party. He decided to be square and he missed out on 4 awesome chicks. The party was all over at 2 am."** (P1, P2, P3, P4)

Note: P1 – represents SME C's contribution to the GSP, P2 – represents SME D's contribution to the GSP, P3 – represents SME A's contribution to the GSP, and P4 – represents SME B's contribution to the GSP.

Thereafter, the researcher set an appointment with a fifth senior SME to review the initial gold standard protocol for accuracy, consistency, and completeness. The initial preliminary gold standard protocol was not send to this SME prior to the meeting because the researcher wanted the SME to read the protocol in his presence. The SME was asked to download the document into her computer and to turn on the Microsoft Word Track Changes feature and then review the protocol in its entirety. Any additions, deletions, modifications and re-arrangements of the main procedures, action and decision steps from the initial gold standard were captured. After the senior SME's review, a final gold standard protocol was created, which is the response to Research Question 1 and is attached as Appendix E. The gold standard protocol represents the action and decision steps that expert math teachers use to teach solving quadratic equations. Overall there were found to be seven main procedures for solving quadratic equations. These seven procedures are:

1. Review linear equations to activate prior knowledge
2. Teach solving quadratic equations by factoring
3. Teach solving quadratic equations by using the quadratic formula
4. Teach solving quadratic equations by graphing
5. Teach solving quadratic equations by completing the square
6. Teach solving quadratic equations by the square root method
7. Teach application of these methods of solving quadratic equations to solving real-life word problems

The disaggregated results are described in the following sections.

Recalled action and decision steps. In this study, action steps and decision steps indicate those steps that provide "how-to" procedural information for solving quadratic equations. Action steps refer to the steps that SMEs may be observed performing while decision steps refer to steps that are unobservable cognitive decisions that may inform an action. Table 4.2 and Table 4.3 show the data results for action steps and decision steps in frequency counts and percentages respectively.

Table 4.2

Counts of action and decision steps for each SME

	SME				Summary Statistics		
	A	B	C	D	Median	M	SD
Action steps	81	58	262	115	98	129	91.71
Decision steps	16	16	76	24	20	33	28.91
Total action and decision steps	97	74	338	139	106	156	122.49

There were a total of 402 action and decision steps on the GSP for solving quadratic equations. The SME with the highest count of action and decision steps was SME C with a total of 338 steps that accounted for 83.66% of the GSP action and decision steps for solving quadratic equations. SME B had the lowest recorded action and decision steps with a total count of 74 steps that represented 18.32% of the GSP action and decision steps. It should be noted that total tally of the action and decision steps shown in Table 4.2 do not add up to 402 because the action and decision steps elicited through CTA may not be distinctive to one SME, as a result the action and decision steps in Figure 4.1 when aggregated do not equal the total number of action and decision steps reported in the GSP because some action steps were described by more than one SME. Therefore, these data confirm that SMEs will tend to provide the same action or decision steps when describing how to perform a complex task through CTA techniques of knowledge elicitation. Of the combined 402 action and decision steps recalled by the four SMEs, there were 317 action steps and 85 decision steps for solving quadratic equations.

As shown in Table 4.3, individual SMEs recalled between 18.32% and 83.66% of action and decision steps, a range of 65.34% and median of 29.21%. The variation between the SMEs when describing the action and decision steps for solving quadratic equations was extremely high with a standard deviation (SD) of 29.79% and a mean (M) of 40.10% of action and decision steps described.

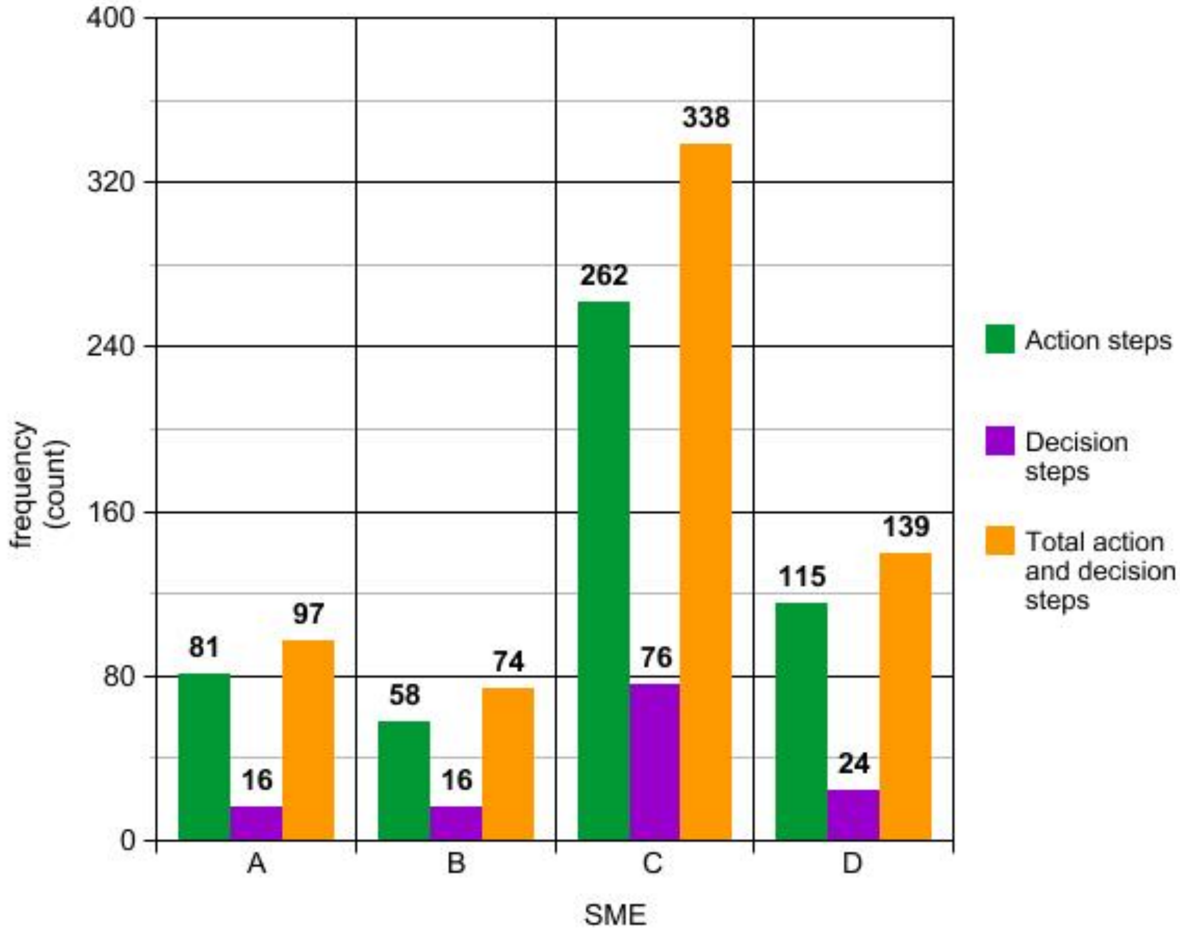
Table 4.3

Percentage of action steps and decision steps for each SME compared to the total number of action and decision steps.

	SME				Summary statistics			
	A	B	C	D	*Range*	*Median*	*M*	*SD*
Action steps	25.47%	18.24%	82.39%	36.16%	64.15%	30.82%	40.57%	28.84%
Decision steps	18.60%	18.60%	88.37%	27.91%	69.77%	23.26%	38.37%	33.62%
Total action and decision steps	24.01%	18.32%	83.66%	34.41%	65.34%	29.21%	40.10%	29.79%

SME C had the highest recall at 83.66% of the 402 action and decision steps captured in the GSP, which was more than the sum of the other three SMEs. SME B had the lowest recall of action and decision steps at 18.32%, which was a difference of 65.34%. All in all, the SMEs recalled more action steps than decision steps. Figure 4.3 shows a graphical representation of the number of action steps, decision steps and total action and decision steps captured from SMEs A, B, C and D. These results will be discussed further in Chapter Five: Discussions.

Figure 4.1: Number of action steps, decision steps, and total action and decision steps for SMEs A, B, C and D captured through CTA



Note. The graph represents the individual non-repeating SME action and decision steps captured from the four SMEs. There were a total of 317 action steps and 85 decision steps captured in the gold standard making a total of 402 steps for solving quadratic equations.

Action and decision steps captured in Round Two interviews. Following the initial interview with each SME, an individual protocol of action and decision steps for each SME was generated**.** The researcher conducted a round two interview with each SME to respond to questions that arose while preparing the individual CTA protocol. This exercise gave the SMEs an opportunity to make corrections to the individual protocol as much as they could recall the necessary action and decision steps required to solve quadratic equations. In addition to these

SMEs, a fifth senior SME reviewed the final combined draft gold standard protocol and was also given an opportunity to make changes that would make the protocol effective as a job-aid in teaching solving quadratic equations. Table 4.4 shows both round 1 (R1) and round 2 (R2) additional action steps and decision steps that were added during the Round Two interview process.

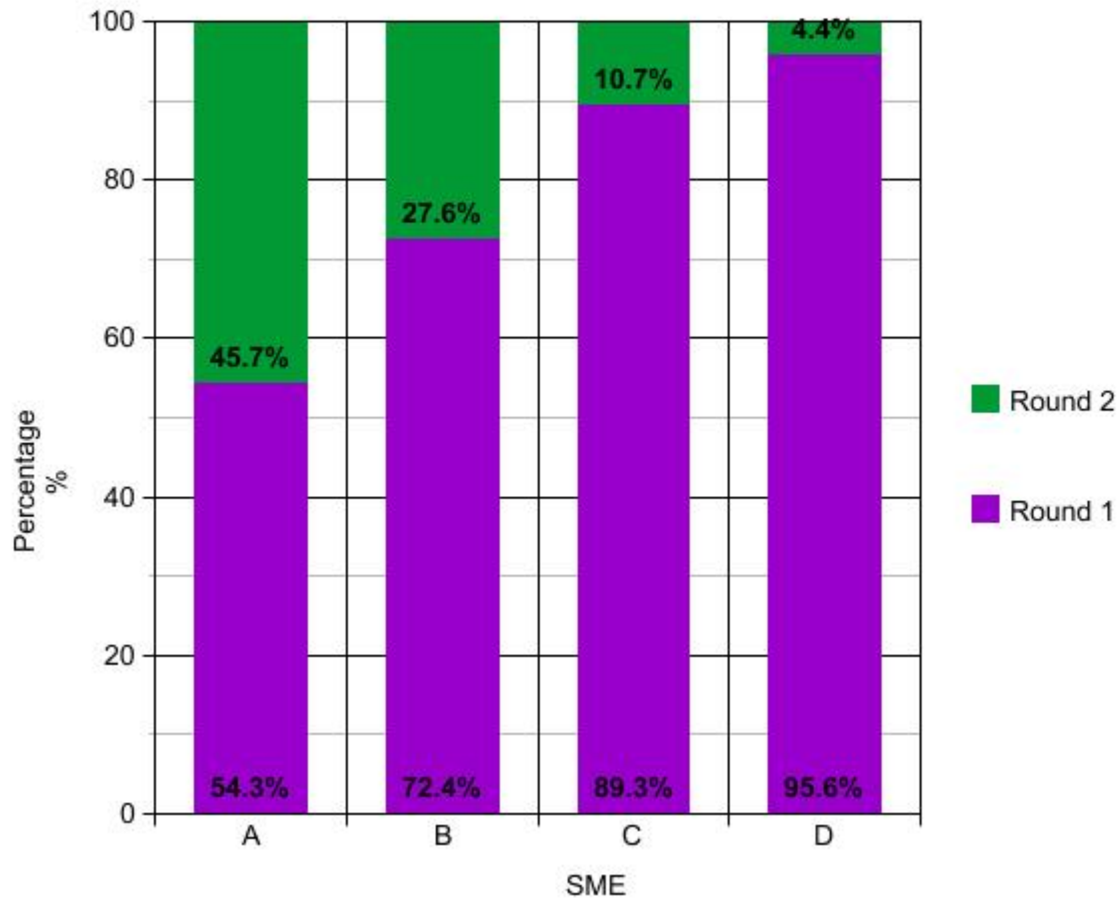
Table 4.4

Additional action and decision Steps during Round two interviews

	SME								
	A		B		C		D		Senior SME
	R1	R2	R1	R2	R1	R2	R1	R2	
Action steps	44	37	42	16	234	28	110	5	0
Decision steps	4	12	6	10	74	2	21	3	0
Total action and decision steps	48	49	48	26	308	30	131	8	0
Round Two Interviews		In person no prior email		In person no prior email		In person no prior email		In person no prior email	In person no prior email

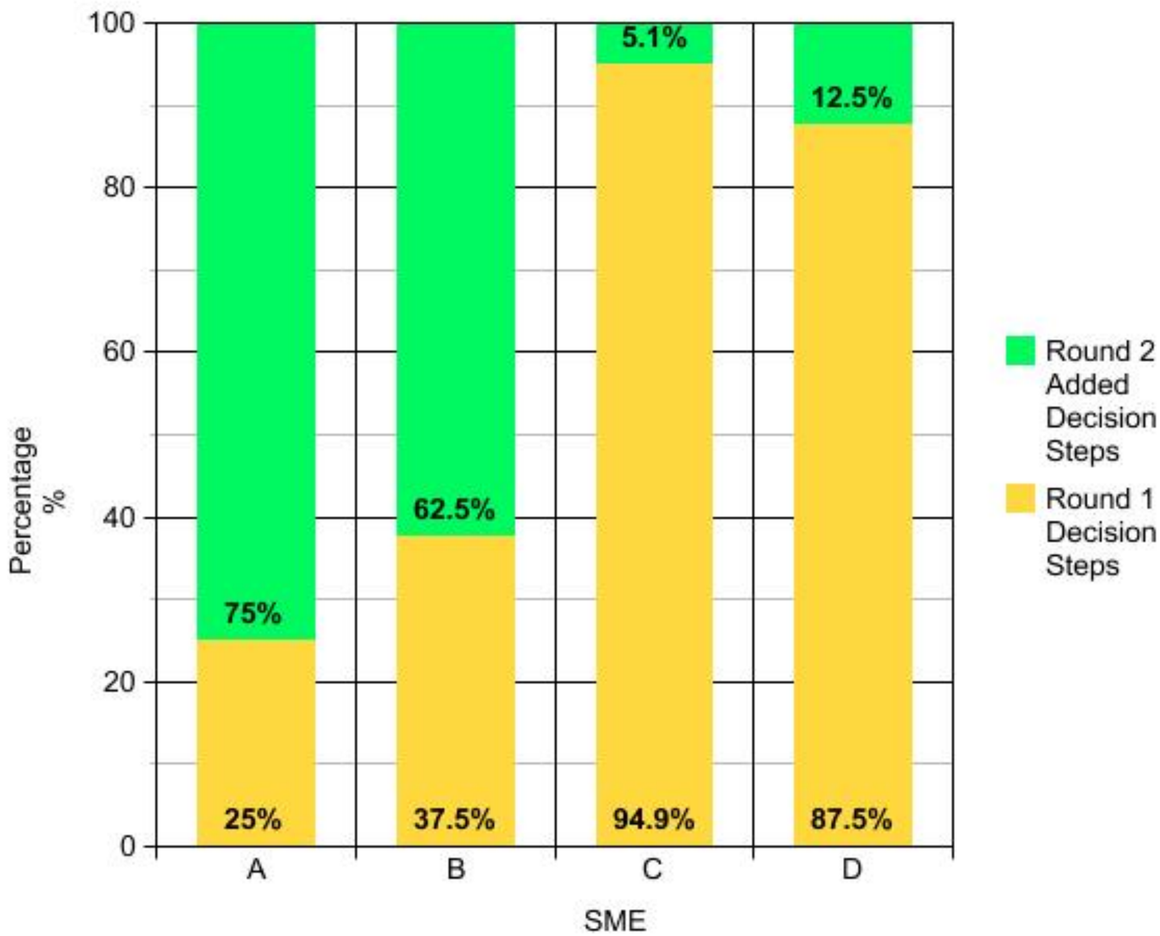
Note: The senior SME did not participate in initial CTA semi-structured interviews but reviewed the preliminary gold standard protocol for accuracy and completeness.

Figure 4.2: Percentage of action steps of SMEs for Round 1 and Round 2 interviews



All the four SMEs recalled extra action and decision steps at this stage. SME A and SME B recalled more decision steps compared to their initial interview. SME A recalled 12 (75%) new decision steps while SME B recalled an additional 10 (62.5%) decision steps as shown in Table 4.4. Equally important were the extra action steps recalled by both SME A and SME B, which were 37 (45.7%) and 16 (27.6%) respectively as shown in both Table 4.4 and Figure 4.2.

Figure 4.3: Percentage of decision steps of SMEs for Round 1 and Round 2 interviews



It appeared that SME C and SME D recalled more during Round One interviews with SME C adding only 28 (10.7%) new action steps and 2 (5.1%) new decision steps while SME D added up to 4.4% of new recalled action steps and 12.5% new recalled decision steps as shown in Figure 4.4 and Figure 4.5 respectively.

Alignment of SMEs in describing the same action and decision steps. The spreadsheet analysis was also used to determine the number and percentage of action and decision steps described by each SME that were fully aligned, substantially aligned, partially aligned, or not aligned. For each action and decision step, if the step was only added by one SME, it was identified as being “not aligned” then the number “1” was added in the alignment column. If an action or decision step was described by two of the SMEs, then the number “2” was added in the alignment column to indicate that the step was “partially aligned.” If an action or decision step was described by three of the four SMEs, then the number “3” was added in the alignment column indicating that the step was “substantially aligned” with the GSP. Finally, if an action or decision step was described by all four SMEs the number “4” was added in the alignment column to indicate the step was “fully aligned” with the GSP. Table 4.4 shows a summary of the results of this analysis.

Table 4.5

Count and percentage of action and decision steps that are fully, substantially, partially, or not aligned with the GSP

	Count	Percentage
Full Alignment	26	6.68%
Substantial Alignment	44	10.89%
Partial Alignment	81	20.05%
No Alignment	251	62.38%

Together the SMEs were “fully aligned” on 26 (6.68%), “substantially aligned” on 44 (10.89%), “partially aligned” on 81 (20.05%), and “not aligned” on 251 (62.38%) of the total action and decision steps (402) on the GSP. The implications of these findings are discussed in Chapter Five.

Question 2

What percentage of actions and/or decision steps, when compared to a gold standard, do expert math teachers omit when they describe how to solve quadratic equations in Algebra One? To answer Research Question 2, the number of steps omitted was aggregated using Microsoft Excel to determine the extent expert mathematics teachers omit critical action and decision steps required when describing instruction for solving quadratic equations. This data was analyzed and is shown in Tables 4.6, Table 4.7 and Figure 4.4 that shows a graphical representation of the same data. Action and decision steps that were included in the GSP but omitted by individual SMEs were marked "0." Using the Microsoft Excel, the omitted action and decision steps were aggregated and summarized in Table 4.4 and Table 4.5 that show the frequency and percentage omissions respectively for each SME.

Table 4.6

Total action and decision steps omitted by SMEs when compared to the GSP

	SME				Summary statistics		
	A	B	C	D	*Median*	*M*	*SD*
Action steps	237	260	56	203	220	189	91.706
Decision steps	70	70	10	62	66	53	28.914
Total action and decision steps	307	330	66	265	286	242	120.488

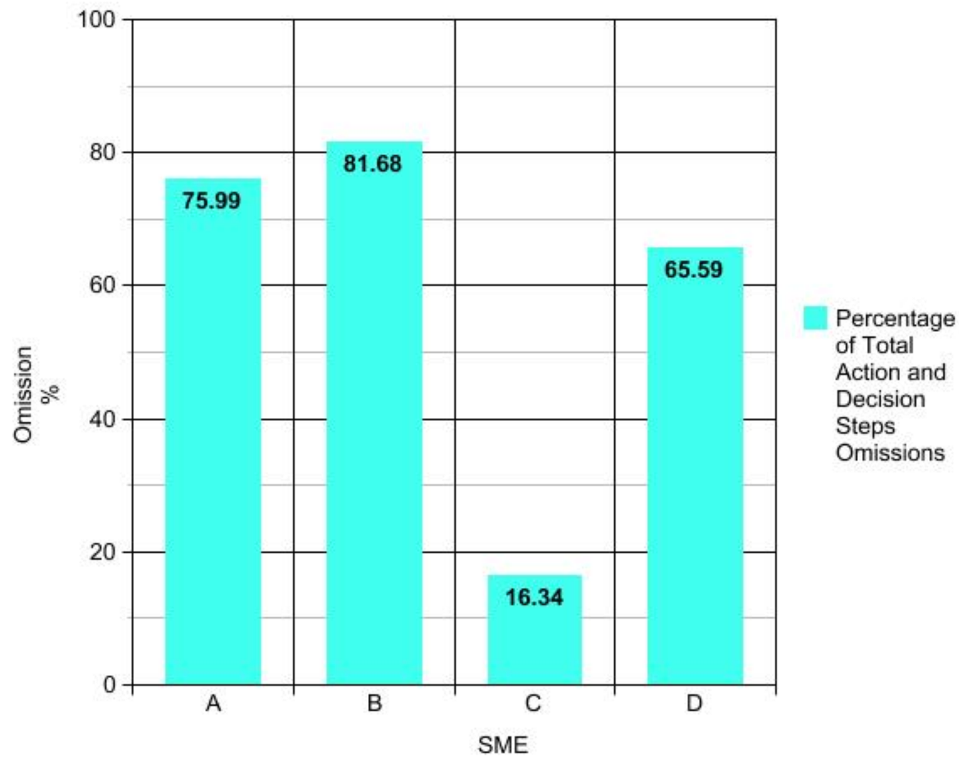
Table 4.7

Percentage of total action and decision steps omitted by SMEs when compared to the GSP

	SME				Summary statistics			
	A	B	C	D	*Range*	*Median*	*M*	*SD*
Action steps	74.53%	81.76%	17.61%	63.84%	64.15%	69.19%	59.44%	28.84%
Decision steps	81.40%	81.40%	11.63%	72.09%	69.77%	76.75%	61.63%	33.62%
Total action and decision steps	75.99%	81.68%	16.34%	65.59%	65.34%	70.79%	59.90%	29.79%

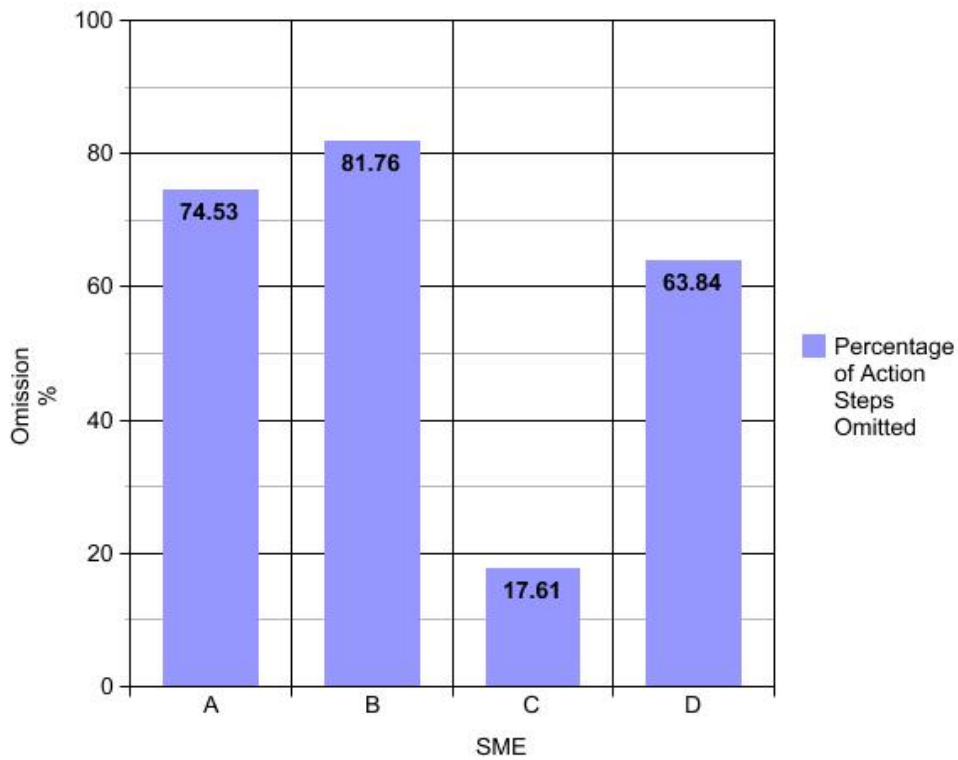
The GSP had a total of 402 action and decision steps for solving quadratic equations. The lowest percentage of action and decision steps omissions was 16.34% while the highest percentage of omissions was 81.68% after Round Two interviews as shown in Figure 4.4. When the SMEs described how to solve quadratic equations, the mean percentage of omissions of actions steps was 59.44% with a standard deviation of 28.84% of 338 action steps recorded in the GSP.

Figure 4.4: Percentage of total action and decision steps omitted by SMEs when compared to the GSP



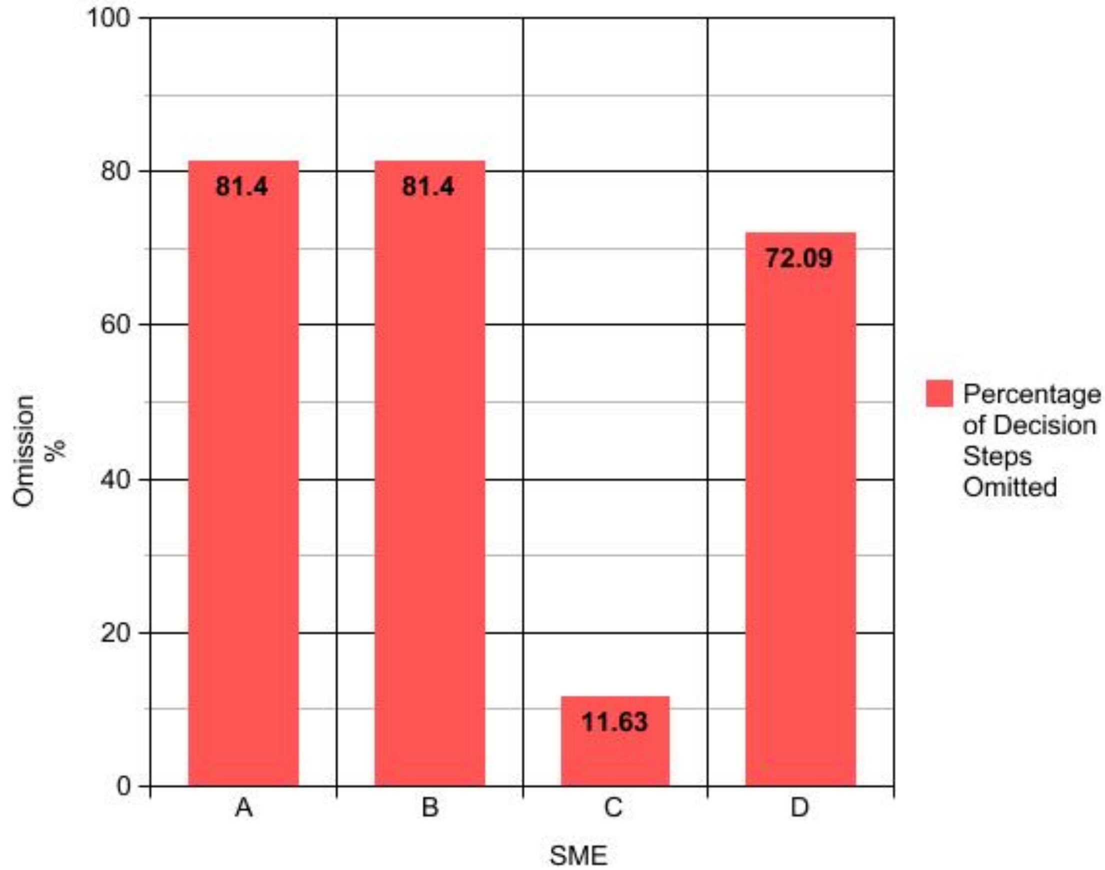
As shown in Figure 4.5, SME A and SME B omitted 74.53% and 81.76% respectively of action steps required to successfully solve quadratic equations while SME D omitted 63.84% of action steps.

Figure 4.5: Percentage of action steps omitted by the SMEs compared to the GSP



There were a total of 402 action and decision steps of which SME A and SME B omitted 75.99% and 81.68% respectively as indicated in Figure 4.4. SME D omitted up to 65.59% of the 402 action and decision steps shown in the GSP.

Figure 4.6: Percentage of decision steps omitted by the SMEs compared to the GSP



The mean percentage of omissions of decision steps was 61.63% with a standard deviation of 33.62% of 86 decision steps recorded in the GSP. Overall, it was established that expert mathematics teachers that participated in this study omitted up to an average of 59.90% with a standard deviation of 29.79% of the total action and decision steps captured in the GSP (Table 4.7).

Chapter Five will include an overview of the study, a discussion of the findings, limitations of the study, implications, and future research.

CHAPTER FIVE: DISCUSSION

Overview of the Study

The main purpose of this study was to use cognitive task analysis to capture the knowledge and skills expert mathematics teachers of Algebra One use to teach solving quadratic equations to 8^{th} and 9^{th} grade students in the K-12 education system. Also this study sought to establish the percentage of critical information omitted when describing solving quadratic equation procedures. While no formal hypotheses were stated, these two research questions guided and informed this study. Research has shown that experts in an instructional role may unintentionally leave out information that students must master when learning procedural skills. Recent research (Clark, Pugh, Yates, Inaba, Green, and Sullivan, 2011) has shown that when experts describe how they perform a difficult task, they may unintentionally omit up to 70 percent of the critical information novices need to learn to successfully perform the task. According to Clark (2008) this is a serious problem because it forces novices to "fill in the blanks" using less efficient and error-prone trial-and-error methods.

As new knowledge becomes automated and unconscious, experts are often unable to completely and accurately recall the knowledge and skills that comprise their expertise, negatively impacting instructional efficacy and leading to subsequent difficulties for learners (Chi, 2006; Feldon, 2007). Feldon (2007) observes that automated procedures are deeply rooted and not easy to change and therefore automaticity impairs the development of expertise. Experts regularly cannot articulate their knowledge because much of their knowledge is implied and their overt intuitions can be flawed (Chi, 2006). Therefore, in K-12 consultants who are experts in their field and may be hampered by the effects of expertise and automaticity when recalling the critical knowledge and skills often do professional development for teachers. While CTA has been used successfully to capture explicit and implicit knowledge of experts in various domains to use for training (Hoffman & Militello, 2009), this is the first study to use CTA to capture expert mathematics teachers' knowledge and skills to solve quadratic equations. This study may form the basis for more research in K-12 education to provide effective professional development modules to teachers and training institutions. Furthermore, the expert knowledge and skills elicited through this study may provide clear guidelines in designing effective training programs for mathematics teachers than those currently in use. Darling-Hammond et al. (2009) notes that improving professional development and collaborative learning opportunities for educators is a crucial step in transforming schools and improving academic achievement for all students. Therefore, capturing expertise in teaching solving quadratic equations, can lead to offering targeted professional development that would assist in improving instruction.

As such, this chapter discusses the process of conducting CTA, discussion of findings, limitations of the study, its implications and suggestions for future research.

Process of Conducting Cognitive Task Analysis

Selection of Experts

Yates, Sullivan, and Clark (2012) and Crispen (2010) recommend that 3 to 4 experts are needed for CTA to capture the optimal amount of significant information during a procedure. Merriam (2009) notes, "If the purpose is to maximize information, the sampling is terminated when no new information is forthcoming from new sampled units" (p. 80). And as Crispen (2010) confirmed, the investment of resources into three to four experts yields a reliable amount of expert knowledge to create a gold standard protocol. Further Merriam (2009) asserts that selecting information-rich cases is important.

The selection of the participants to this study was a challenge to the researcher. This researcher identified one unified school district in Southern California and approached its superintendent for clearance to allow three of its 8th or 9th grade Algebra One teachers who have shown high student achievement in state standardized tests over a period of five or more years. Unfortunately it took longer than anticipated for these experts to be identified. Meanwhile, this researcher approached a second unified school district that within a short time identified three subject matter experts and the process of Round One interviews began. No sooner had the researcher began interviewing experts from the second school district than the first school district identified three experts for this study. That meant that this CTA study had six identified experts though according to Crispen (2010), three to four SMEs provide the optimal level of action and decision steps in a CTA gold standard protocol.

Since this study involved a small sample of mathematics teachers, selecting the sample purposefully was necessary (Merriam, 2009; Patton, 1990) to allow the researcher to select teachers that were highly qualified as determined by their training, student achievement scores and experience. Feldon (2007) describes the characteristics of expertise to include extensive and highly structured knowledge of the domain, effective strategies for solving problems within the domain, and expanded working memory that utilizes elaborated schemas to organize information effectively for rapid storage, retrieval, and manipulation. According to Clark et al., (2008) a subject matter expert (SME) is a person with wide experience and is capable of performing a range of tasks fast and successfully. For this study, expertise was shown by the algebra teachers' years of experience, teaching expertise, and moreover, performance of their students in state standardized tests. This was a critical criterion of selecting the experts.

Of the six SMEs identified, five were interviewed but only four experts' interviews were aggregated to generate the GSP, as the fifth expert was not able to verbally describe the action and decision steps required to successfully solve quadratic equations. It may be that this expert's procedural knowledge was so automated that delineating the individual actions and decisions became impossible, similar to the difficulty a person would have in describing how to drive a car. The sixth SME was used to review the aggregated gold standard protocol for accuracy and completeness.

Collection of Data

Multiple CTA methods. During the process of collecting data, the semi-structured interview protocol described in Chapter Three was used for the first two experts, SME A and SME B. This method of eliciting knowledge was followed strictly, however, the experts found it difficult to recall critical action and decision steps for solving quadratic equations. When the third and fourth SMEs (C and D) were interviewed, the researcher followed the same semi-structured interview protocol but when requested, permitted the SMEs to write out the action and decision steps as they verbalized them. This change in tact seemed to elicit significantly more critical action and decision steps from these two SMEs (C and D) compared to those elicited from SMEs A and B. In fact, SME C and SME D recalled more than double the number of action and decision steps required to solve quadratic equations compared to both SME A and SME B after they were allowed to think aloud as they wrote these steps. A possible explanation of these results follows.

According to Yates (2007), there are over 100 types of CTA methods that have been identified and classified. Yates suggests that since the current classification schemes organize CTA methods by process rather than the desired outcome or application, practitioners may find it difficult to select an optimal method for their specific purpose (or knowledge outcome). Crandall et al. (2006) note that many knowledge elicitation methods have analytical processes and representational formats embedded within the method. Therefore, Yates (2007) identified the most frequently used CTA methods and the knowledge types associated with the respective methods and outcomes (product approach versus the existing process approach) supporting Chao and Salvendy's (1994) conclusion that the percentage of accurately recalled decisions and procedures varied by task type and by elicitation method used.

Considering the literature, it may be that the researcher inadvertently changed the methods of elicitation for SME C and D. The method used in SME C and D process of collecting data has been described in the literature as the Think Aloud protocol. Ericsson and Simon (1984) describe this method as verbalizing a description of task performance while actually performing the task or visualizing performing the task. As such, it may be that SME C and D recalled more action and decision steps than the other SMEs because the Think Aloud method was more appropriate for the task than the intended semi-structured interview method described in Chapter Three. Further research is needed to examine these results.

Length of interviews. The CTA interviews took more time than it had been anticipated, however, the additional time may have increased the number of action and decision steps recalled by the experts. The initial round one interviews SMEs A and B took approximately two hours, while SMEs C and D took on average three hours. These interviews were expected to take about 90 minutes but the SMEs seemed to "get into it" and continued to give information that sometimes may not have been relevant to the process of solving quadratic equations. SMEs C and D interviews may have taken about one hour longer because these SMEs were given the opportunity to freely talk and write out the steps without interruption from the interviewer. This, in turn, may have helped to elicit more action and decision steps from SME C and SME D than were elicited from SMEs A and B.

In sum, the length of the interviews in this study depended on the process of collecting data. The process that involved the CTA semi-structured interview protocol took about two hours while the process in which the SMEs were given the opportunity to write freely as they described the action and decision steps took approximately one-hour more time. Further research is needed in this area to examine the length of the interviews and their effect on the knowledge elicited.

Confirmation bias. The researcher was a high school mathematics teacher with 15 years experience. To avoid confirmation bias, the researcher recorded and took notes on what the SMEs were describing as the critical action and decision steps they make while solving quadratic equations. According to Nickerson (1998) and Plous (1993), confirmation bias is the tendency to search for, gather and/or interpret information to confirm one's beliefs or hypothesis. The researcher made sure by constantly reviewing the interview recording and transcript that the protocol and the flowchart were based on what the SMEs said *they do* while teaching 8th and/or 9th grade students how to solve quadratic equations and not what the researcher thought *should have been done*.

Discussion of Findings

While no formal hypotheses were developed for this study, the study was guided by two research questions. The results from the data collection are discussed here below.

Research Question 1: *What are action and decision steps that expert math teachers recall when they describe how to teach solving quadratic equations in Algebra?*

To answer this question, four subject matter experts described the action and decision steps used to solve quadratic equations in algebra for 8th and 9th grade students in K-12 education system. Out of all the procedural steps that were recalled, 78.7% were action steps, a significantly higher number than decision steps, which were 21.3%.

Differences in SMEs

In knowledge elicitation, experts are more likely to recall more action than decision steps because decision steps are unobservable cognitions and are often more difficult to recall when performing a task (Clark, 2014; Clark & Estes, 1996) as a consequence of automated expert knowledge (see also Ericsson, 2004; Ericsson et al., 1993). In this study, the SMEs had significance differences in the number of action steps and the number decision steps they recalled (SME A = 97; SME B = 74; SME C = 338; SME D = 139). An examination of the biographical differences among these SMEs may provide insight into these findings.

Biographical data. The five SMEs that participated in this study had varied experiences and educational backgrounds. SME A had six years of experience teaching mathematics the least experience of all the SMEs. SME B, on the other hand, was an expert with 16 years experience teaching Algebra One to 8th grade students. SME B became a teacher after spending 23 years in an unrelated industry. The third expert, SME C had 11 years experience teaching Algebra One and the fourth SME, SME D, had 11 years experience teaching Algebra One to 8th grade students.

The expert with most experience, SME B recalled the least action and decision steps which was in step with literature which indicates that as SMEs gain expertise, their skills become automated and the steps of the procedure blend together. Experts perform tasks largely without conscious knowledge as a result of years of practice and experience. This causes experts to omit critical steps when describing a procedure because this information is no longer accessible to conscious processes (Clark & Elen, 2006). Also SME B with six-year experience recalled less action and decision steps compared to SME C and SME D who had 11 years of teaching Algebra One. Based on previous studies (Canillas, 2010; Clark & Elen, 2006) and the concept of automaticity in experts (Feldon, 2007), the SME with the least experience should recall the most action and decision steps compared to the SMEs with more experience in the subject matter. However, SME B with six years experience compared to SMEs C and D who had 11 years of experience teaching algebra one recalled less action and decision steps. Therefore, these results are inconclusive and do not support previous studies. It may also be that the knowledge elicitation method, which changed from semi-structured interview for SMEs A and B to a think aloud method for SMEs C and D and discussed in the next section might have influenced the results.

Interview methods. It was intended that the semi-structured CTA interview protocol described in Chapter Three be used to elicit expert knowledge from the four experts. During the semi-structured CTA interview process, the SMEs were asked a series of questions that focused on the major tasks when solving quadratic equations. The first two SMEs (A and B) interview protocol was adhered to without deviation, however, SMEs C and D had more difficulty responding to the semi-structured interview questions and thus were allowed and encouraged to think aloud and even write what they were thinking as they responded to the semi-structured CTA interview questions. The results show that SME C and D recalled significantly more action and decision steps compared to the first two experts interviewed after the method of eliciting knowledge was changed. These data support the conclusions of Yates (2007) and Chao and Salvendy (1994) that different knowledge elicitation methods may be more appropriate for specific tasks and knowledge types than other methods. The differences in the number of action and decision steps recalled by the experts can be further examined by comparing the number of action steps recalled versus the number of decision steps recalled by the experts.

Action Steps Verses Decision Steps

On average the four SMEs recalled more action steps than decision steps. These SMEs recalled an average of 317 (78.7%) action steps compared to an average of 85 (21.3%) decision steps. These findings confirm Canillas' (2010) findings that SMEs are consistently able to describe more knowledge steps on "how" to do a performance task, than knowledge of "when" to do the task in a decision step. Canillas (2010) found that experts described 75.8% action steps compared to 24.2% decision steps in describing the critical information required for the placement of a central venous catheter (CVC). Due to automaticity, experts perform tasks largely without conscious knowledge as a result of years of practice and experience. This phenomenon causes experts to omit critical steps when describing a procedure because this information is no longer accessible to conscious processes (Clark & Elen, 2006). Secondly, many SMEs are not able to share the complex thought processes of behavioral execution of skills. Even experts who make an attempt to "think aloud" during the process of complex problems often omit essential information because their knowledge is automated (Clark & Elen, 2006; Clark & Estes, 1996). Consequently, it is difficult to identify points during a procedure where an expert makes decisions (Clark & Elen, 2006) and as such they are not able to describe these decision steps and procedures.

SME C was the most efficient expert in recalling 83.66% of action and decision steps aggregated in the GSP while the other three SMEs, A, B and D recalled an average of 24.58% action and decision steps as enumerated in the GSP. The reason SME C may have recalled more action and decision steps than the other SMEs may have been because of how the interview was done. This researcher did not use the CTA semi-structured interview protocol alone like it was done for SMEs A and B. Several elicitation methods were used together to maximize the knowledge recall for both SME C and D. These elicitation methods were semi-structured interview paired with both Diagram Drawing and Think Aloud (Clark & Estes, 1996; Yates, 2007). According to Yates and Feldon (2011), these methods result in both declarative and procedural knowledge. Therefore SME C made more contributions to the GSP than three other SMEs combined. During the Round One interview, SME D found it very hard to describe the action and decision steps required to solve quadratic equations. This researcher then asked SME D to "think aloud" and write these steps on paper as he verbalized what he was writing so that the recording device could pick his voice to transcribe later. When this option was offered to SME D, the SME was able to articulate and describe action and decision steps for solving quadratic equations with ease. This essence of performing the task of solving quadratic equations instead of describing the task allowed SME D to recall more action and decision procedural steps required to solve quadratic equations though it was only 34.41% of the steps in the GSP. Clark and Estes (1996) notes that differences between the various CTA approaches tend to be based more on the specific nature of the types of tasks being analyzed and the eventual use of the information being collected. The first two SMEs (A and B) found it challenging to describe these steps because their knowledge was automated and they were not given the option to Think Aloud during the initial interview. In fact, SME B kept on stating "let the kids play around" but never articulated what "playing around" was in the context of solving quadratic equations.

Follow up interviews. During the Round Two interviews, all the SMEs recalled more action and decision steps as shown in Table 4.4 and this may have been attributed to two things: (1) the SMEs were able to recall more action and decision steps having known what the task was about after the first interview, and/or (2) this researcher gained CTA interview skills to ask the right probing questions and allowed all the SMEs to Think Aloud and utilize the Diagram

Drawing method of eliciting critical knowledge. At this stage SME A and SME B added significantly more action and decision steps during Round Two interviews after they were encouraged to verbalize their thinking and also to write the processes step by step as they remembered them. The implication of this process was that the knowledge analyst must have a variety of tools for interview, to match up the methods with the kinds of knowledge being sought (Chao & Salvendy, 1994).

Expert Review of Draft Gold Standard Protocol

After this researcher aggregated the four individual protocols generated from the SMEs, a fifth senior SME was asked to review the preliminary gold standard protocol for accuracy and completeness. The fifth SME did not add any action or decision steps to the existing preliminary gold standard protocol. The fifth SME noted that the preliminary gold standard protocol represented the complete process for teaching how to solve quadratic equations and therefore did not contribute to the PGSP. During the review of the preliminary gold standard protocol by the fifth senior SME, this researcher noted that the senior SME was distracted because at the same time the SME was supervising students doing club activities. What this researcher found very useful was that during the Round Two interviews each of the four SMEs read their individual protocols and clarified all the questions this researcher generated during the process of preparing the individual protocols. This was important to capture all the knowledge and skills these experts could recall in response to Research Question 1 as shown in the GSP (Appendix E).

Research Question 2: *What percentage of actions and/or decision steps, when compared to a gold standard, do expert math teachers omit when they describe how to solve quadratic equations in Algebra?*

Expert knowledge omissions. Recent research (Clark et al., 2011) has shown that when experts describe how they perform a difficult task, they may unintentionally omit up to 70 percent of the critical information novices need to learn to successfully perform the task. As new knowledge becomes automated and unconscious, experts are often unable to completely and accurately recall the knowledge and skills that comprise their expertise, negatively impacting instructional efficacy and leading to subsequent difficulties for learners (Chi, 2006; Feldon, 2007). As such, in this study when compared to the GSP, on average 59.90% of action and decision steps were omitted by SMEs with a standard deviation of 29.79% when describing the overall steps needed to successfully solve quadratic equations. Individually, SME A and SME B had the most omissions of 75.99% and 81.68%, which was much higher than the literature suggests that expert may omit up to 70% of the critical information novices need to successfully perform the task. SME D omitted 65.59% compared to the gold standard protocol while SME C had the least omissions at 16.34% of action and decision steps enumerated in the GSP for solving quadratic equations.

On further analysis of these data, with n = 4 participants, *sample mean* of omissions = 59.90%, and standard deviation (*SD*) = 29.79% and with the assumption that the population mean of omissions of experts is 70% based on literature, the calculated two-sided statistic is t = -0.6780 with a p = 0.5464 which is substantial evidence against an alpha level, $\alpha = 0.05$, showing this could not have happened by chance and therefore provides further evidence that experts may omit up to 70% of critical information novices need to successfully perform the task. Therefore, these results confirm that experts do omit critical information when describing how to solve quadratic equations.

Limitations

Though the study's findings are consistent with results from other CTA studies related to capturing expert knowledge in the form of action and decision steps and expert knowledge omissions, there were several study limitations.

Confirmation Bias

The first limitation of this study is that the researcher is a high school mathematics teacher with extensive knowledge and experience in teaching solving quadratic equations to high school students for 15 years. This background and experience put the researcher to be attentive of biases that may crop up while conducting the CTA interviews. The researcher had to constantly control his facial outlook and emotions while listening to responses by the SMEs. According to Clark (2014), when a knowledge analyst has experience in a performance task, the analyst tends to change the information captured from SMEs to suit the analyst's knowledge and expectations. This knowledge analyst did not need to participate in any bootstrapping procedures, where the analyst should read materials to gain a general familiarity with job or task and knowledge of the specialized vocabulary (Crandall et al., 2006; Schraagen et al., 2000) because this analyst had experience and knowledge of solving quadratic equations. Therefore, extra effort was needed by the researcher to not put his preexisting knowledge and experiences in solving quadratic equations onto the data collected. The potential for bias can never be completely eliminated.

Internal Validity

The second limitation of this study is the validation of these results against what the SMEs do in practice as they teach solving quadratic equations to 8th and 9th grade students in the K-12 education system. According to Merriam (2009) "internal validity deals with the question of how research findings match reality" (p. 213). In other words, validity must be assessed in terms of something other than reality itself. In order to validate the gold standard protocol developed from the CTA interviews, there would need to be a study on the effectiveness of CTA based instruction. The validity test will be to observe the teachers whose data produced the GSP and see how many of the actions and decisions steps s/he actually performs in reality.

External Validity

The final limitation of the present study is it may not be generalizable because of the small sample size (*n* = 4) and that the participants were limited to two neighboring school districts in Southern California. Merriam (2009) observes that the question of generalizability has plagued qualitative researchers for some time. In other words, Merriam believes that part of the difficulty lies in thinking of generalizability in the same way as do researchers using experimental designs. However, the generalizability of the use of CTA for this task could be measured as more teachers use it with successful results in student achievement. Also, future research may replicate this study that will result in increasing the sample size and therefore reduce external validity.

Implications

Darling-Hammond et al. (2009) assert that improving professional development and collaborative learning opportunities for educators is a crucial step in transforming schools and improving academic achievement for all students. Therefore, the declarative and procedural knowledge captured from the CTA study when applied to training and instruction may increase novice performance and decreases the amount of time and resources needed for training. In fact according various studies (Embry, 2010; Zepeda-McZeal, 2014; Canillas, 2010), CTA has been shown to be an effective methodology of capturing expert knowledge needed for the performance of complex tasks. As such, in most schools and school districts expert consultants may omit up to 70% of the critical action and decision steps when they conduct professional development; training based on the results of CTA may prove to be advantageous (Clark et al., 2011; Clark et al., 2008).

The current study supports the use of CTA research to capture expert knowledge and skills in complex instructional tasks, such as solving quadratic equations in Algebra One not just for training and instruction but to a major extend improve student achievement in algebra which is a gateway to success in career and college (Gamoran & Hannigan, 2000; Moses & Cobb, 2001; Smith, 1996). Most importantly, although none of the SMEs admitted it, they did not seem to know how to describe how to solve real-life word problems that involve quadratic equations. As this is a requirement for the new Common Core State Standards' (CCSS) performance tasks in mathematics, omissions such as these may have an implication in student performance. CTA may be an efficient method of capturing these skills for Common Core professional development.

Clark (2011) notes that the use of CTA in instruction and training has been proven to be positively related to cost savings due to reduced training times with comparable learning outcomes. Further Clark (2011) maintains that CTA training results in 50% learning gains and with reduced training times and cost savings, the implication is that school districts will spend less resources in training and achieve more in well prepared teachers which will be expected to translate to improved performance for students in the classroom.

Overall, CTA training and instruction has been shown to significantly improve performance, including patent examiners finish 75% faster, six months vs. two years (Clark, 2011), surgical residents finish 25% faster, learn 40% more and important mistakes are reduced by 50% (Velmahos et al., 2004), a meta-analysis of 34 studies averaged 47% performance increase (Lee, 2004) and another meta-analysis of more than 100 studies averaged 25% learning increase (Tofel-Grehl & Feldon, 2013). Therefore, K-12 professional development that includes CTA training and instruction would benefit teachers by offering a targeted professional development that would aid in learning that in turn may lead to improved instruction.

Future Research

A search of studies in this field of research did not result in any studies in the area of solving quadratic equations in algebra using cognitive task analysis. Therefore as a result of this current study, future studies may consider using the gold standard protocol generated by the research and implement a randomized experimental study with mathematics teachers teaching solving quadratic equations to 8^{th} and 9^{th} grade students. This study would involve a control group of teachers who would teach solving quadratic equations using the current traditional method and an experimental group in which the teachers would use the gold standard protocol. These two groups would be compared using a two-sample t-test of differences to see whether there is a significant difference in performance between the control group and the experimental group of students taught using the GSP. Longitudinal research may also benefit this body of research to determine short- and long-term learning gains in solving quadratic equations. During this research study, it was also established that the experts did not articulate clearly how real-life word problems are introduced while teaching quadratic equations and this would be one area that may warrant future research using CTA methods.

Conclusion

The purpose of this study was to add to the body of knowledge on the benefits of CTA for capturing critical information experts use when solving challenging problems and performing complex tasks and the omissions experts make when describing their knowledge and skills. The complex task of capturing the expert information that mathematics teachers use when teaching 8^{th} and 9^{th} grade students how to solve quadratic equations in Algebra is the first of its kind in K-12, however there are other similar studies that explore the knowledge and skills captured and omitted by experts through CTA methods. Expert mathematics teachers in this study omitted up to an average of 60%, which was statistically not different from 70% and therefore this study gives evidence in support of earlier studies that experts omit up to 70% of the critical action and decision steps needed to successfully solve complex tasks when describing how to solve quadratic equations. CTA methods were shown in this research to be effective in capturing the unconscious, automated knowledge of expert mathematics teachers when they perform the complex task of solving quadratic equations. The expert knowledge captured and aggregated into a gold standard protocol in this study may be used to train teachers in teacher education programs and in professional development in schools and school districts to assist in cutting down costs and improving student achievement in Algebra One which is a building block for upper level mathematics courses in high school. Darling-Hammond (1999) observed that the effects of well-prepared teachers on student achievement are stronger than the influences of poverty, language barriers, and minority status. Therefore capturing expertise of teaching solving quadratic equations can improve Algebra One instruction that may lead to higher student achievement.

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Appendix A

Cognitive Task Analysis Interview Protocol

Begin the interview: Meet the Subject Matter Expert (SME) and explain the purpose of the interview. Ask the SME for permission to record the interview. Explain to the SME the recording will be only used to ensure that you do not miss any of the information the SME provides.

Name of task(s):

Performance Objective:
Ask: *"What is the objective of solving quadratic equations? What action verbs should be used?"*
Step 1:
Objective: Capture a complete list of student learning outcomes for teaching solving quadratic equations.

(a) Ask the SME to list student outcomes when these tasks are complete. Ask them to make the list as complete as possible.
(b) How are the students assessed on these outcomes?

Step 2:
Objective: Provide practice exercises that are authentic to the teaching context in which the tasks are performed.

(a) Ask the SME to list all the contexts in which these tasks are performed (e.g. using the quadratic formula, completing the square, graphing, factorization, or real-life problem type)
(b) Ask the SME how the tasks would change for each method of solving quadratic equations.

Step 3:
Objective: Identify main steps or stages to accomplish the task.

(a) Ask SME the key steps or stages required to accomplish the task.
(b) Ask SME to arrange the list of main steps (procedures) in the order they are performed, or if there is no order, from easiest to difficult.

Step 4:
Objective: Capture a list of "step-by-step" actions and decisions for each task.

(a) Ask the SME to list the sequence of actions and decisions necessary to complete the task and/or solve the problem
(b) Ask: "Please describe how you would accomplish this task step-by-step, so a novice teacher could perform it."

(c) For each step the SME gives you, ask yourself, “Is there a decision being made by the SME here?” If there is a possible decision, ask the SME.

(d) If the SME indicates that a decision must be made …

Ask: “Please describe the most common alternatives (up to a maximum of three) that must be considered to make the decision and the criteria novice teachers should use to decide between the alternatives.”

Step 5:

Objective: Identify prior knowledge and information required to perform the task.

(a) Ask SME about the prerequisite knowledge and other information required to perform the task.

i) Ask the SME about Cues and Conditions

Ask: “For this task, what must happen before someone starts the task? What prior knowledge, order, or other initiating event must happen? Who decides?”

ii) Ask the SME about New Concepts and Processes

Ask: “Are there any concepts or terms required of this task that may be new to the novice teacher?

Concepts – terms mentioned by the SME that may be new to the novice

Ask for a definition and at least one example

Processes – How something works

If the trainee is teaching solving quadratic equations, then ask the SME to “Please describe how and at what stage quadratic equations fit in teaching algebra – in words that novices will understand what procedures are taught first? Think of it as a flowchart.”

Ask: “Must novices know these procedures to do the task?” “Will they have to use it to change the task in unexpected ways?”

If the answer is NO, do NOT collect information about the process.

a) Ask the SME about Equipment and Materials

Ask: “What equipment if any and materials are required to succeed at this task in routine situations? Where are they located? How are they accessed?”

b) Performance Standard

Ask: “How do we know the objective has been met? What are the criteria, such as time, efficiency, quality indicators (if any)?”

c) Sensory experiences required for task

Ask: "Must novices see, hear, or touch something in order to learn any part of the task? For example, are there any parts of this task they could not perform unless they could touch something (such as a calculator)?"

Step 6:
Objective: Identify problems that can be solved by using the procedure

(a) Ask the SME to describe at least one routine problem and two or three complex problems that the novice should be able to solve if they can perform each of the tasks on the list you just made.

Ask: "Of the tasks we just discussed, describe at least one simple or routine problem and two to three complex problems that the novice should be able to solve IF they learn to perform the task."

Appendix B

Inter-rater Reliability Code Sheet

Acquillahs MUTETI
Main Coder

Douglas WIELAND
Secondary Coder

Code	Tally	Frequency	Agree	Disagree	% Agreemen (IRR)
O (Objective)	HHT X	5	5	Ø	100%
C (Conditions/cues)	HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT III	58 ~~47~~	53	5	91.4%
M (Main Procedures)	HHT I	6	6	Ø	100%
A (Action Step) 200 191	HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT	191. ~~201~~	190	1	99.5%
D (Decision Step)	HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT HHT II	55 ~~57~~	55	2	96.5%
S (Standards)	HHT HHT HHT HHT HHT IIII	24. ~~29~~	24	2	92.3%
E (Equipment)	HHT HHT	~~8~~ 10	10	0	100%
P (Pre-req know/skl)	HHT I	6.	6	0	100%
NCONC (New Concept)	IIII	4	4	Ø	100%
NPROS (New Process)					
NPRIN (New Principle)	III	~~3~~ 2	2	0	100%
SENSE (Sensory Info)					
REASON (Reasons)	HHT HHT HHT HHT HHT HHT HHT HHT IIII	46 ~~43~~	42	4	91.3%
PROB (Problems)					
SAFE (Safety Factors)					
REF (References)					
		~~408~~ 407	397	10	97.5%

Appendix C

Job Aid for Developing a Gold Standard Protocol Richard Clark and Kenneth Yates (2010, Proprietary)

The **goals** of this task are to 1) aggregate CTA protocols from multiple experts to create a "gold standard protocol" and 2) create a "best sequence" for each of the tasks and steps you have collected and the best description of each step for the design of training.

Trigger: After having completed interviews with all experts and capturing all goals, settings, triggers, and all action and decision steps from each expert – and after all experts have edited their own protocol.

Create a gold standard protocol

STEPS Actions and Decisions

1. For each CTA protocol you are aggregating, ensure that the transcript line number is present for each action and decision step.

a) If the number is not present, add it before going to Step 2.

2. Compare all the SME's corrected CTA protocols side-by-side and select one protocol (marked as P1) that meets all the following criteria:

a) The protocol represents the most complete list of action and decision steps.
b) The action and decision steps are written clearly and succinctly.
c) The action and decision steps are the most accurate language and terminology.

3. Rank and mark the remaining CTA protocols as P2, P3, and so forth, according to the same criteria.

4. Starting with the first step, compare the action and decision steps of P2 with P1 and revise P1 as follows:

a) IF the step in P2 has the same meaning as the step in P1, THEN add "(P2)" at the end of the step.
b) IF the step in P2 is a more accurate or complete statement of the step in P1, THEN revise the step in P1 and add "(P1, P2)" at the end of the step.
c) IF the step in P2 is missing from P1, THEN review the list of steps by adding the step to P1 and add "(P2N)"* at the end of the step.

5. Repeat Step 4 by comparing P3 with P1, and so forth for each protocol.

6. Repeat Steps 4 and 5 for the remaining components of the CTA report such as triggers, main procedures, equipment, standards, and concepts to create a "preliminary gold standard

protocol" (PGSP).

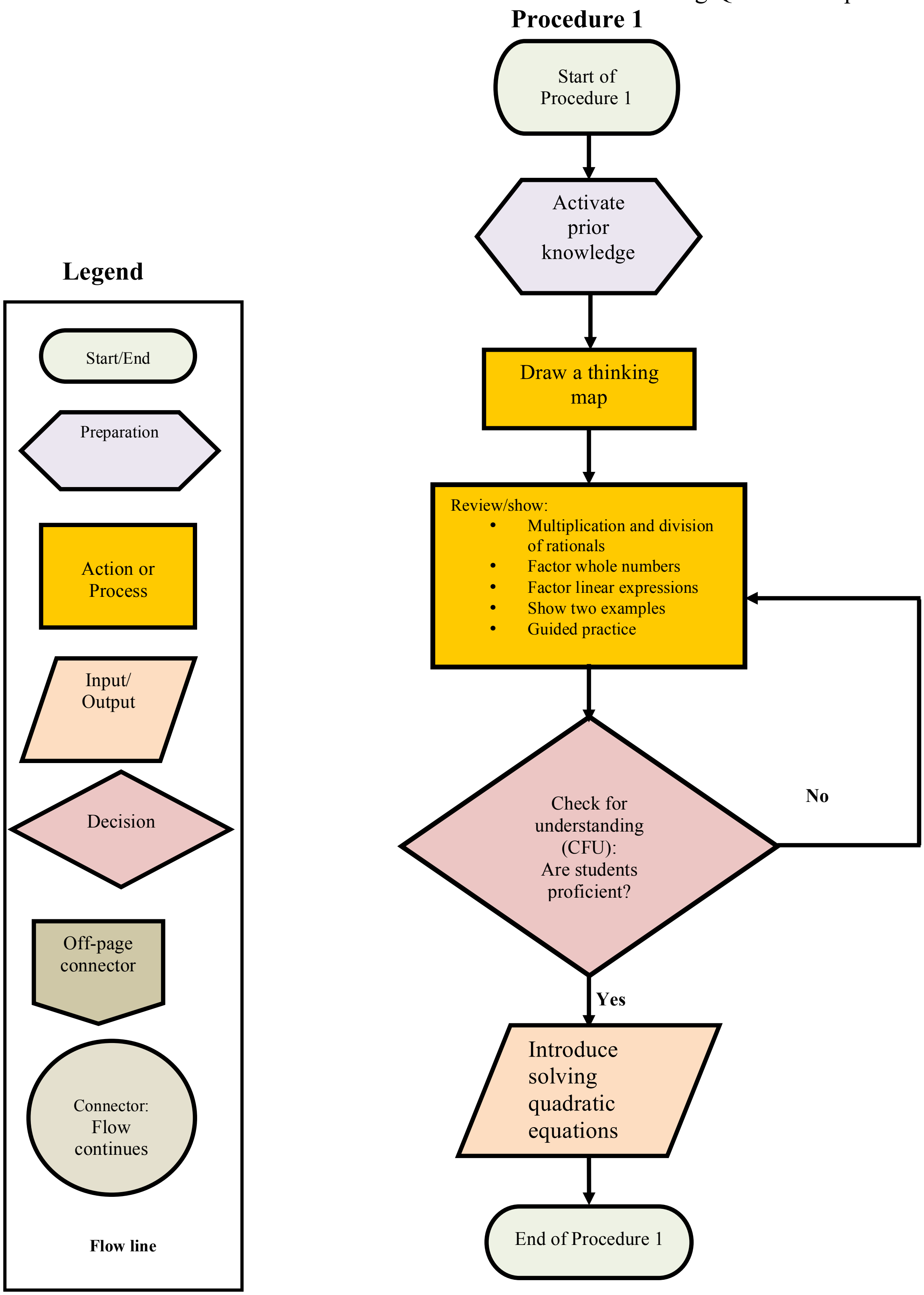
7. Verify the PGSP by either:

a) Asking a senior SME, who has not been interviewed for a CTA, to review the PGSP and note any additions, deletions, revisions, and comments.
b) Asking each participating SME to review the PGSP, and either by hand or using MS Word Track Changes, note any additions, deletions, revisions, or comments.
    (i) IF there is disagreement among the SMEs, THEN either:
        a. 1. Attempt to resolve the differences by communicating with the SMEs, or
        b. Ask a senior SME, who has not been interviewed for a CTA, to review and resolve the differences.

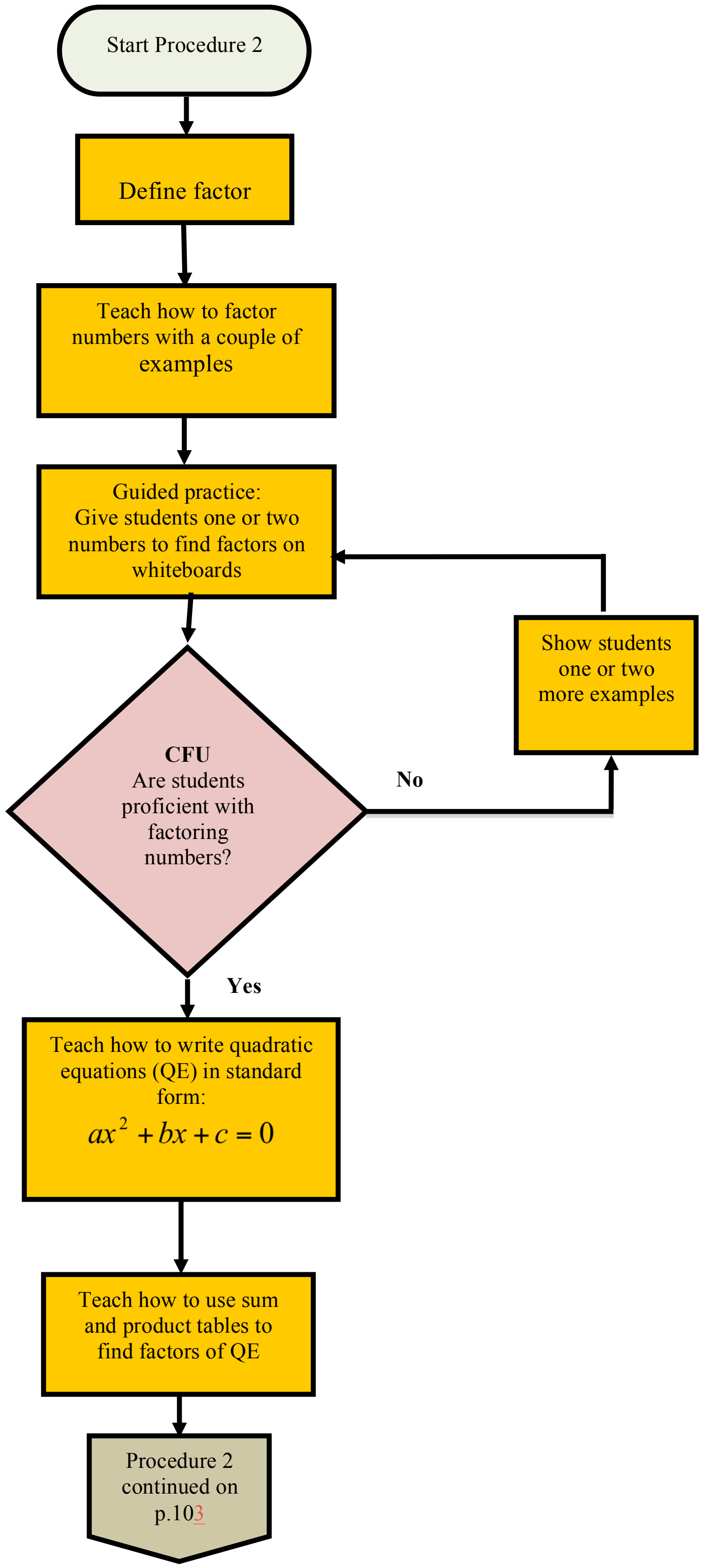
8. Incorporate the final revisions in the previous Step to create the "gold standard protocol" (GSP).

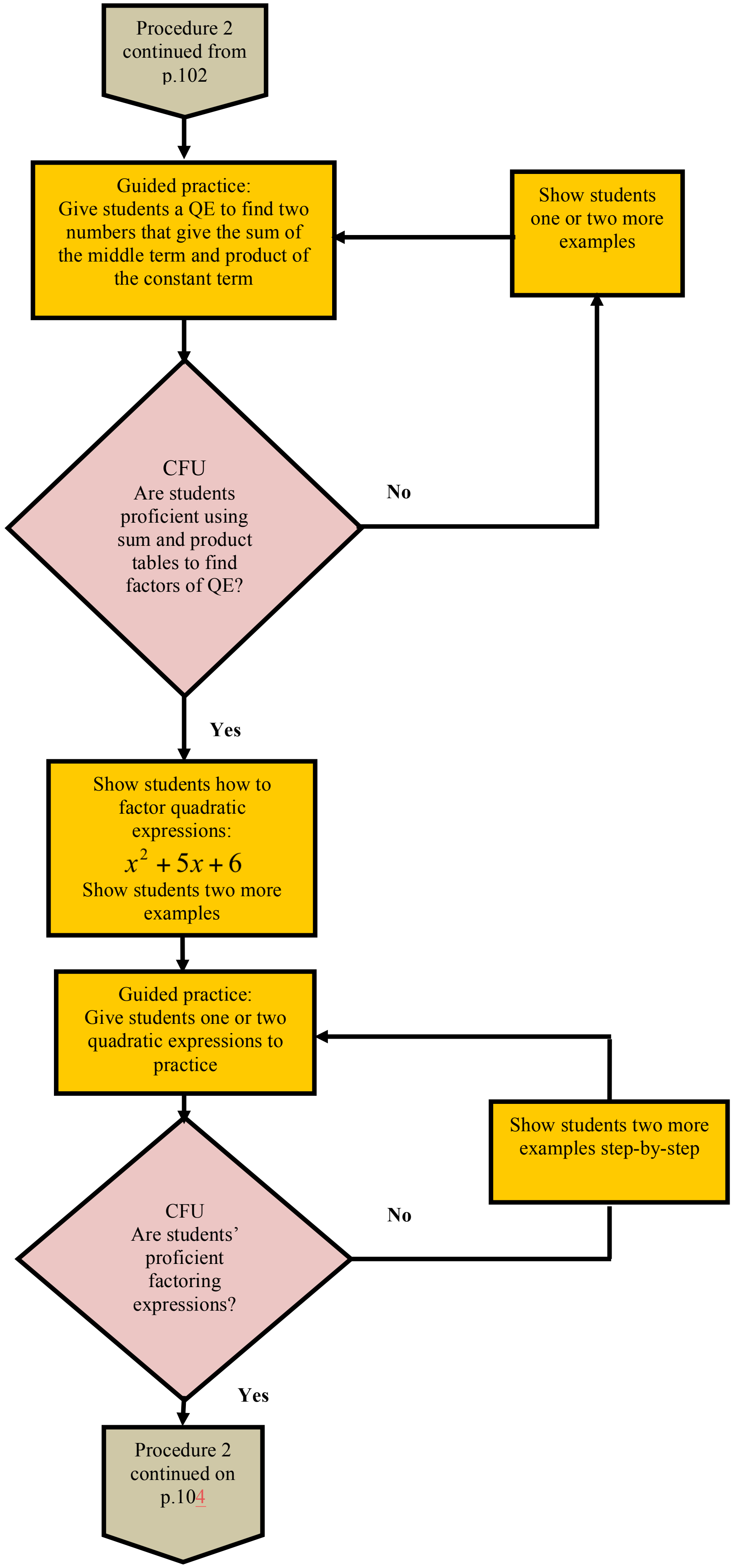
Appendix D

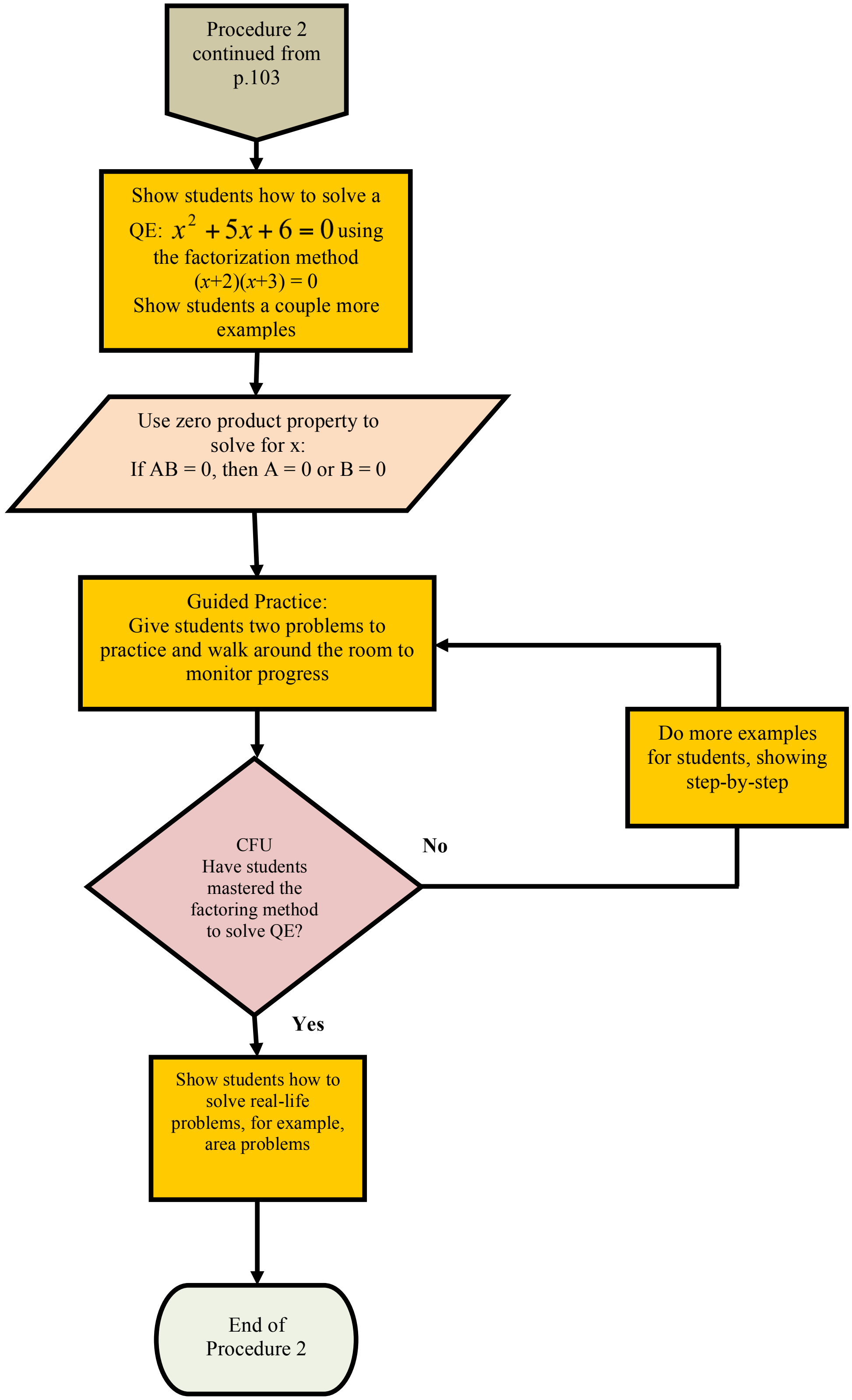
SME C Initial Interview Flowchart for solving Quadratic Equations



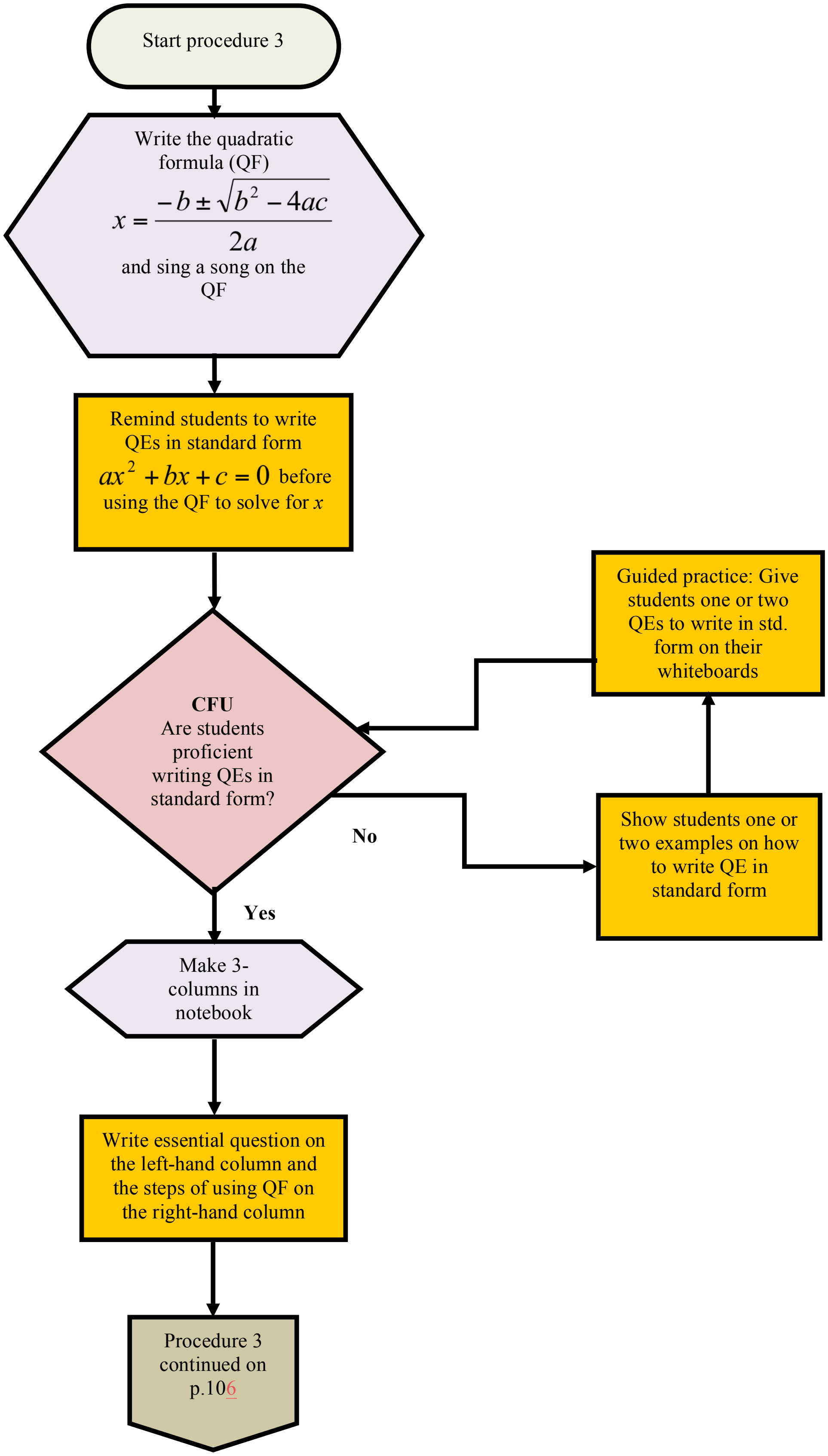
Procedure 2

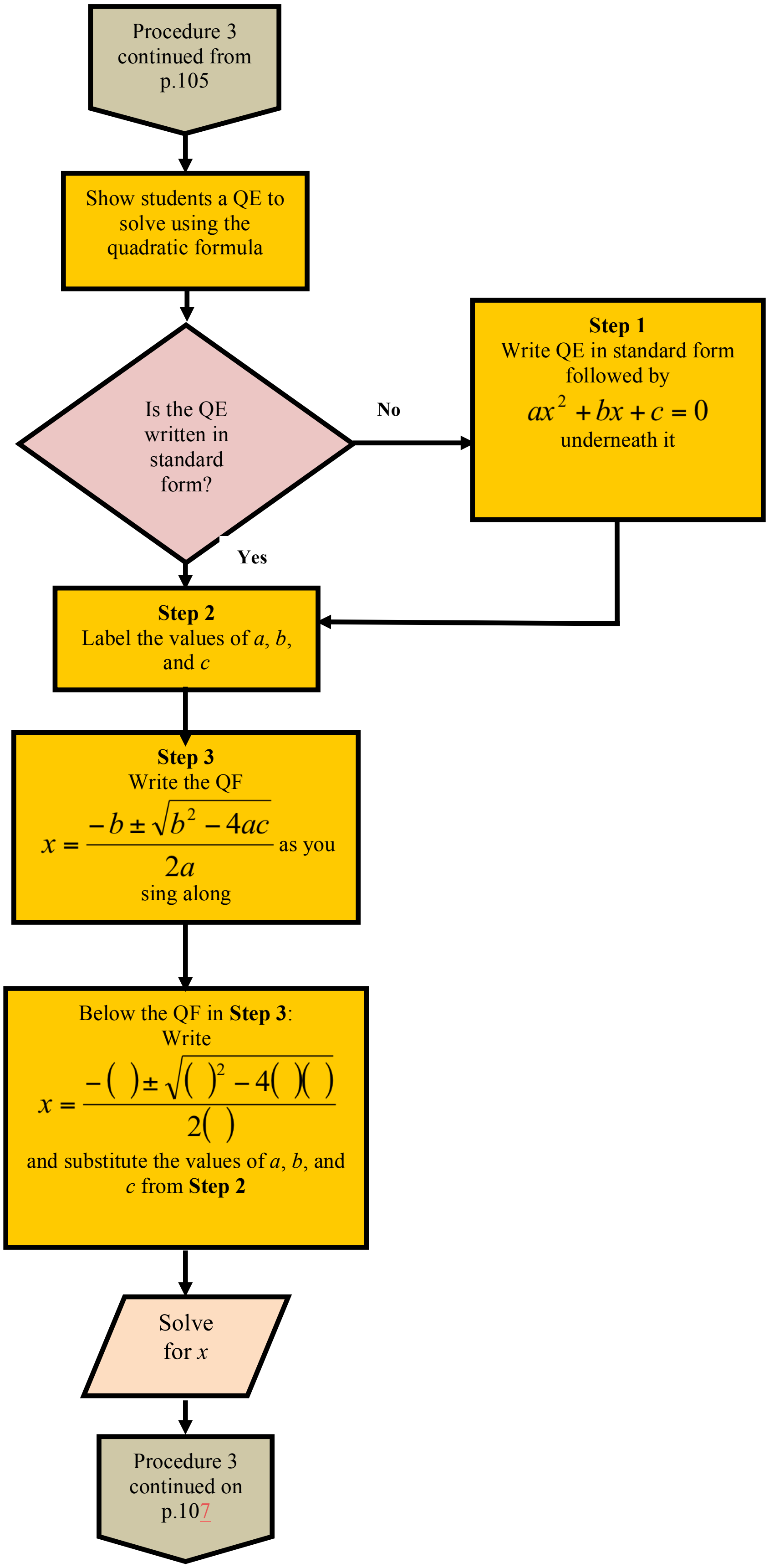

Start Procedure 2
Define factor
Teach how to factor numbers with a couple of examples
Guided practice: Give students one or two numbers to find factors on whiteboards
CFU
Are students proficient with factoring numbers?
No
Show students one or two more examples
Yes
Teach how to write quadratic equations (QE) in standard form:
$ax^2 + bx + c = 0$
Teach how to use sum and product tables to find factors of QE
Procedure 2 continued on p.103

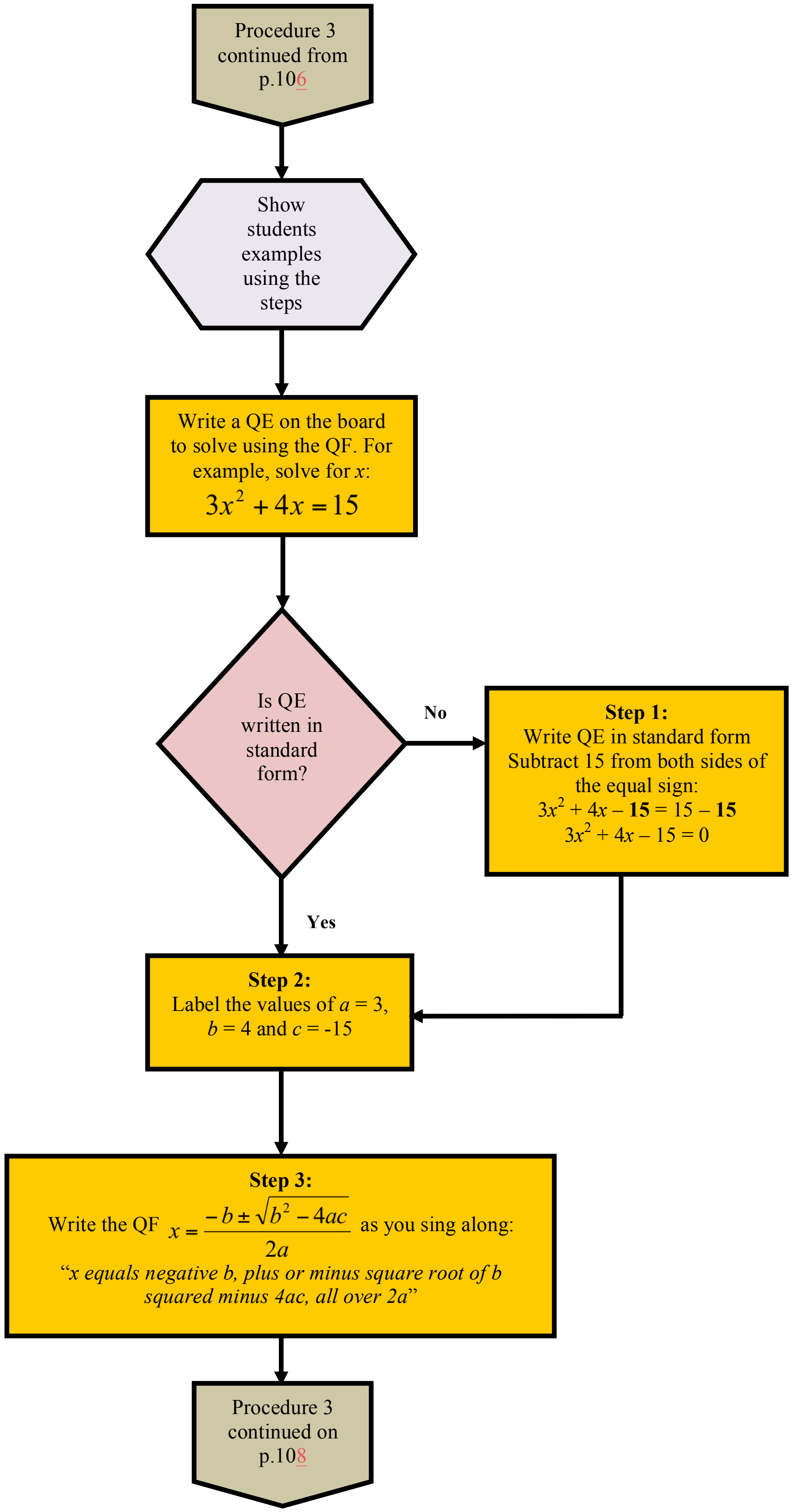

Procedure 2 continued from p.102
Guided practice: Give students a QE to find two numbers that give the sum of the middle term and product of the constant term
Show students one or two more examples
CFU Are students proficient using sum and product tables to find factors of QE?
No
Yes
Show students how to factor quadratic expressions:
$x^2 + 5x + 6$
Show students two more examples
Guided practice: Give students one or two quadratic expressions to practice
Show students two more examples step-by-step
CFU Are students' proficient factoring expressions?
No
Yes
Procedure 2 continued on p.104

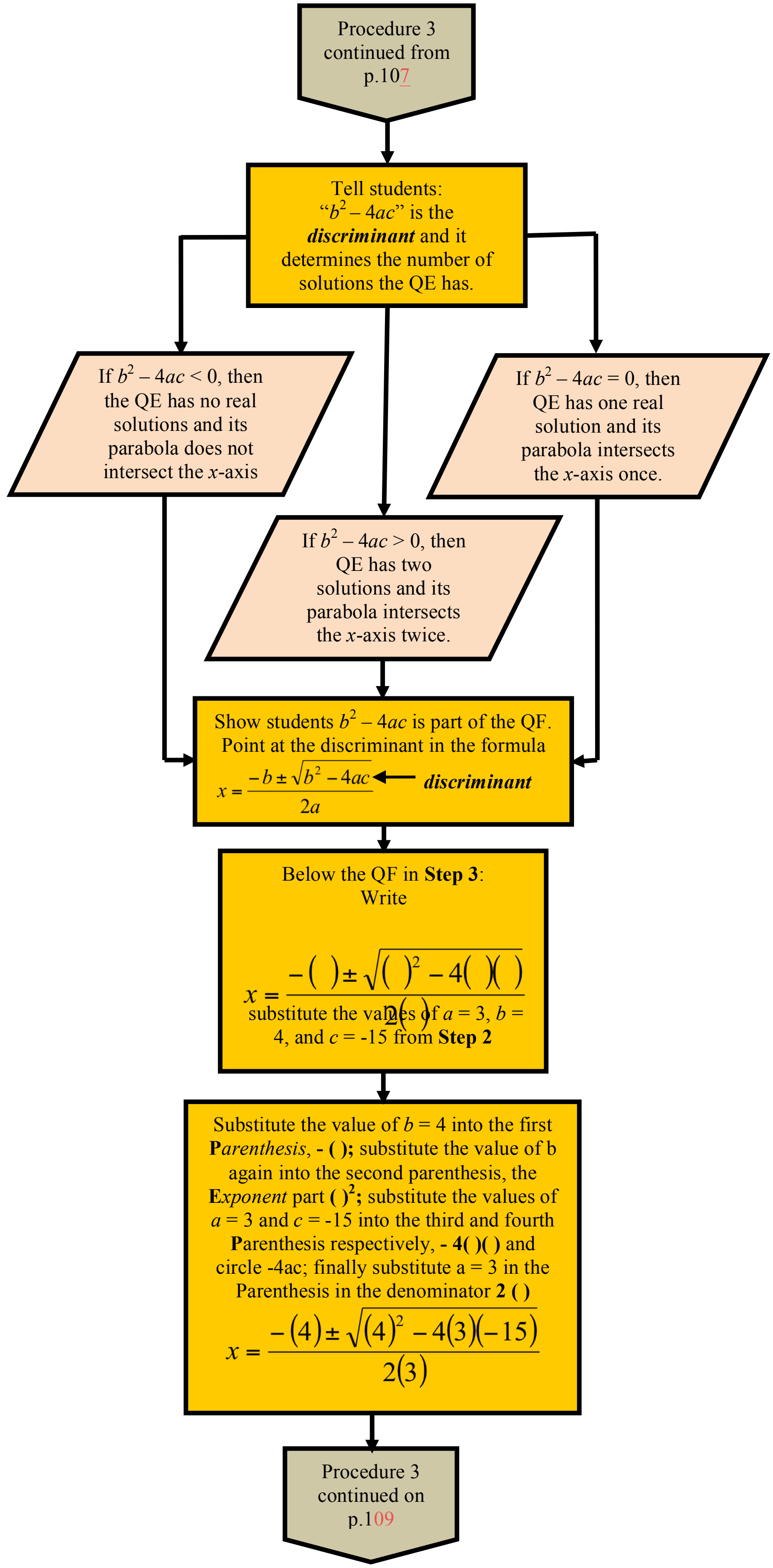

Procedure 2 continued from p.103
Show students how to solve a QE: $x^2 + 5x + 6 = 0$ using the factorization method $(x+2)(x+3) = 0$
Show students a couple more examples
Use zero product property to solve for x:
If AB = 0, then A = 0 or B = 0
Guided Practice:
Give students two problems to practice and walk around the room to monitor progress
CFU
Have students mastered the factoring method to solve QE?
No
Do more examples for students, showing step-by-step
Yes
Show students how to solve real-life problems, for example, area problems
End of Procedure 2

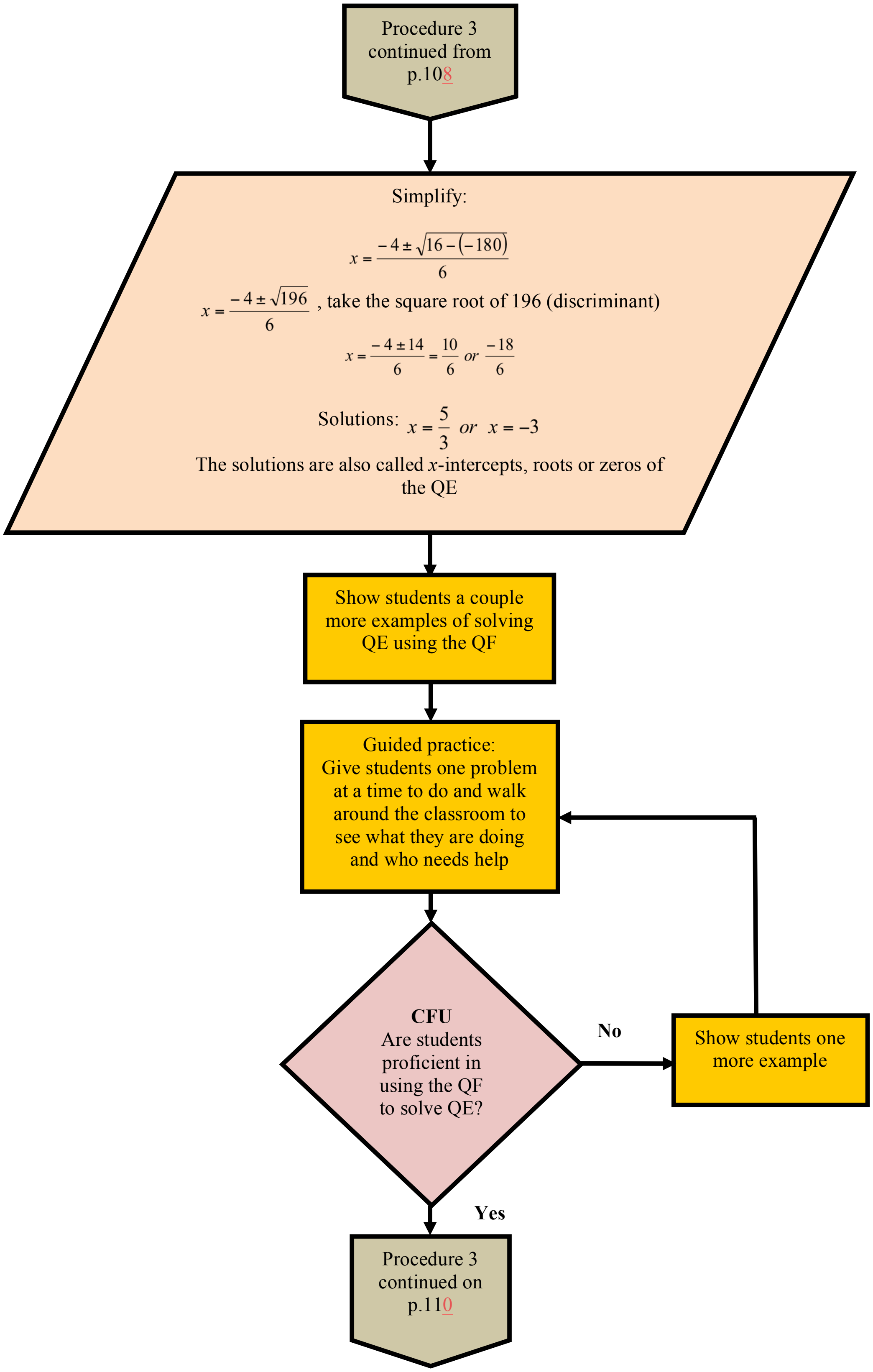
Procedure 3

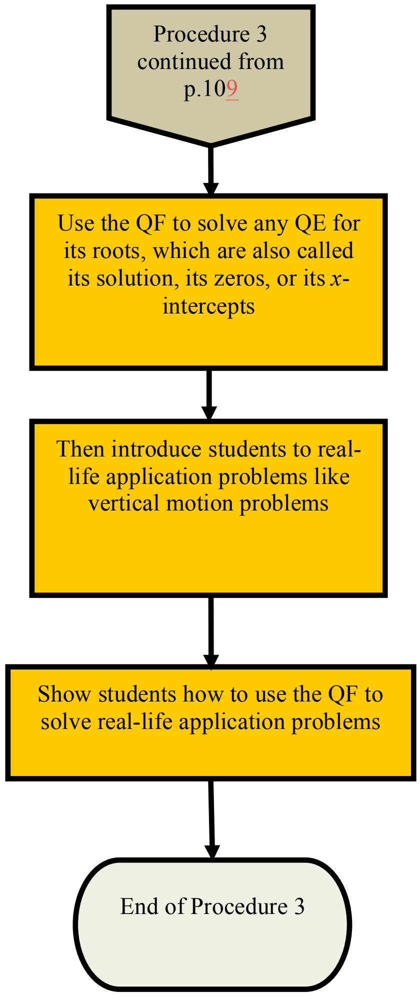

Start procedure 3
Write the quadratic formula (QF)
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
and sing a song on the QF
Remind students to write QEs in standard form $ax^2 + bx + c = 0$ before using the QF to solve for x
CFU
Are students proficient writing QEs in standard form?
No
Show students one or two examples on how to write QE in standard form
Guided practice: Give students one or two QEs to write in std. form on their whiteboards
Yes
Make 3-columns in notebook
Write essential question on the left-hand column and the steps of using QF on the right-hand column
Procedure 3 continued on p.106


Procedure 3 continued from p.105
Show students a QE to solve using the quadratic formula
Is the QE written in standard form?
No
Step 1
Write QE in standard form followed by
$ax^2 + bx + c = 0$
underneath it
Yes
Step 2
Label the values of a, b, and c
Step 3
Write the QF
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ as you sing along
Below the QF in Step 3: Write
$x = \frac{-( ) \pm \sqrt{( )^2 - 4( )( )}}{2( )}$
and substitute the values of a, b, and c from Step 2
Solve for x
Procedure 3 continued on p.107

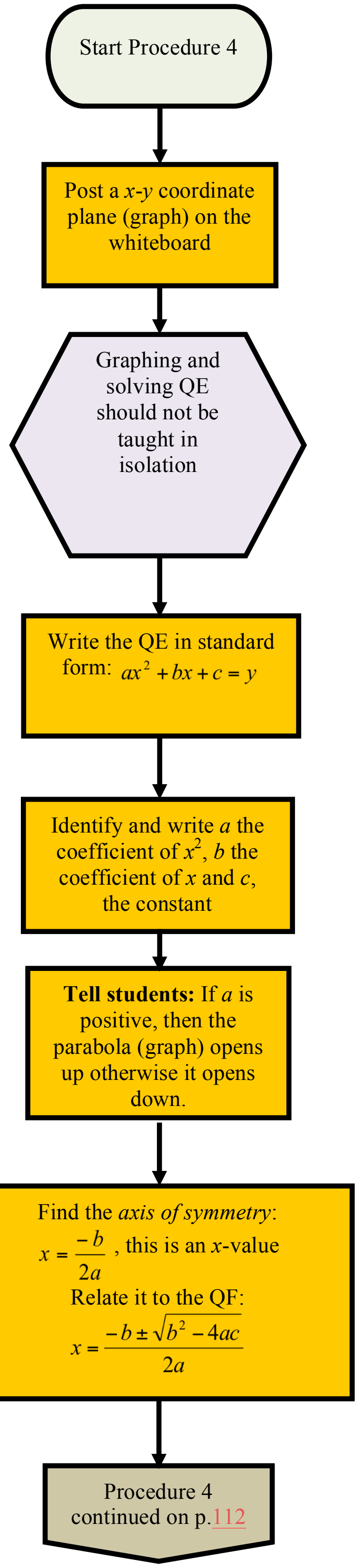

Procedure 3 continued from p.106
Show students examples using the steps
Write a QE on the board to solve using the QF. For example, solve for x:
$3x^2 + 4x = 15$
Is QE written in standard form?
No
Step 1:
Write QE in standard form
Subtract 15 from both sides of the equal sign:
$3x^2 + 4x - \mathbf{15} = 15 - \mathbf{15}$
$3x^2 + 4x - 15 = 0$
Yes
Step 2:
Label the values of a = 3, b = 4 and c = -15
Step 3:
Write the QF $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ as you sing along:
"x equals negative b, plus or minus square root of b squared minus 4ac, all over 2a"
Procedure 3 continued on p.108


Procedure 3 continued from p.107
Tell students: "$b^2 - 4ac$" is the ***discriminant*** and it determines the number of solutions the QE has.
If $b^2 - 4ac < 0$, then the QE has no real solutions and its parabola does not intersect the *x*-axis
If $b^2 - 4ac = 0$, then QE has one real solution and its parabola intersects the *x*-axis once.
If $b^2 - 4ac > 0$, then QE has two solutions and its parabola intersects the *x*-axis twice.
Show students $b^2 - 4ac$ is part of the QF. Point at the discriminant in the formula
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
discriminant
Below the QF in **Step 3**: Write
$x = \frac{-(\ ) \pm \sqrt{(\ )^2 - 4(\ )(\ )}}{2(\ )}$
substitute the values of $a = 3$, $b = 4$, and $c = -15$ from **Step 2**
Substitute the value of $b = 4$ into the first **P***arenthesis*, **- ( );** substitute the value of b again into the second parenthesis, the **E***xponent* part **( )^2;** substitute the values of $a = 3$ and $c = -15$ into the third and fourth **P**arenthesis respectively, **- 4( )( )** and circle -4ac; finally substitute a = 3 in the Parenthesis in the denominator **2 ( )**
$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(3)(-15)}}{2(3)}$
Procedure 3 continued on p.109


Procedure 3 continued from p.108
Simplify:
$x = \frac{-4 \pm \sqrt{16-(-180)}}{6}$
$x = \frac{-4 \pm \sqrt{196}}{6}$, take the square root of 196 (discriminant)
$x = \frac{-4 \pm 14}{6} = \frac{10}{6}$ or $\frac{-18}{6}$
Solutions: $x = \frac{5}{3}$ or $x = -3$
The solutions are also called x-intercepts, roots or zeros of the QE
Show students a couple more examples of solving QE using the QF
Guided practice: Give students one problem at a time to do and walk around the classroom to see what they are doing and who needs help
CFU
Are students proficient in using the QF to solve QE?
No
Show students one more example
Yes
Procedure 3 continued on p.110


Procedure 3 continued from p.109
Use the QF to solve any QE for its roots, which are also called its solution, its zeros, or its x-intercepts
Then introduce students to real-life application problems like vertical motion problems
Show students how to use the QF to solve real-life application problems
End of Procedure 3

Procedure 4


Start Procedure 4
Post a x-y coordinate plane (graph) on the whiteboard
Graphing and solving QE should not be taught in isolation
Write the QE in standard form: $ax^2 + bx + c = y$
Identify and write a the coefficient of x^2, b the coefficient of x and c, the constant
Tell students: If a is positive, then the parabola (graph) opens up otherwise it opens down.
Find the *axis of symmetry*:
$x = \frac{-b}{2a}$, this is an x-value
Relate it to the QF:
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Procedure 4 continued on p.112

Procedure 4 continued from p.111

Substitute the values of *a* and *b* into the *axis of symmetry* equation $x = \frac{-b}{2a}$ to find the *axis of symmetry*

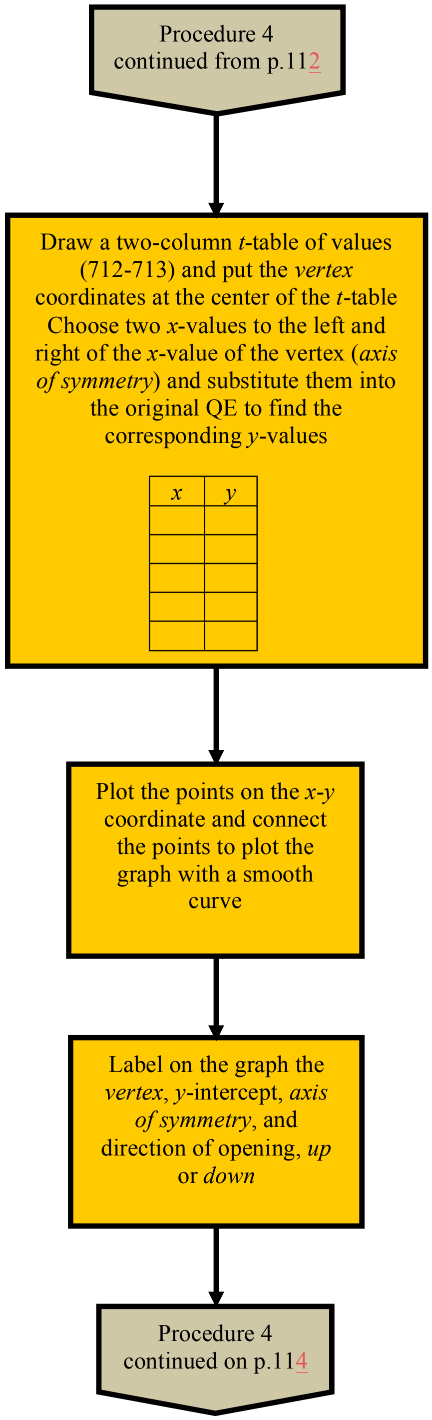
Draw a dotted line through the *axis of symmetry* on the x-y coordinate plane on the board.

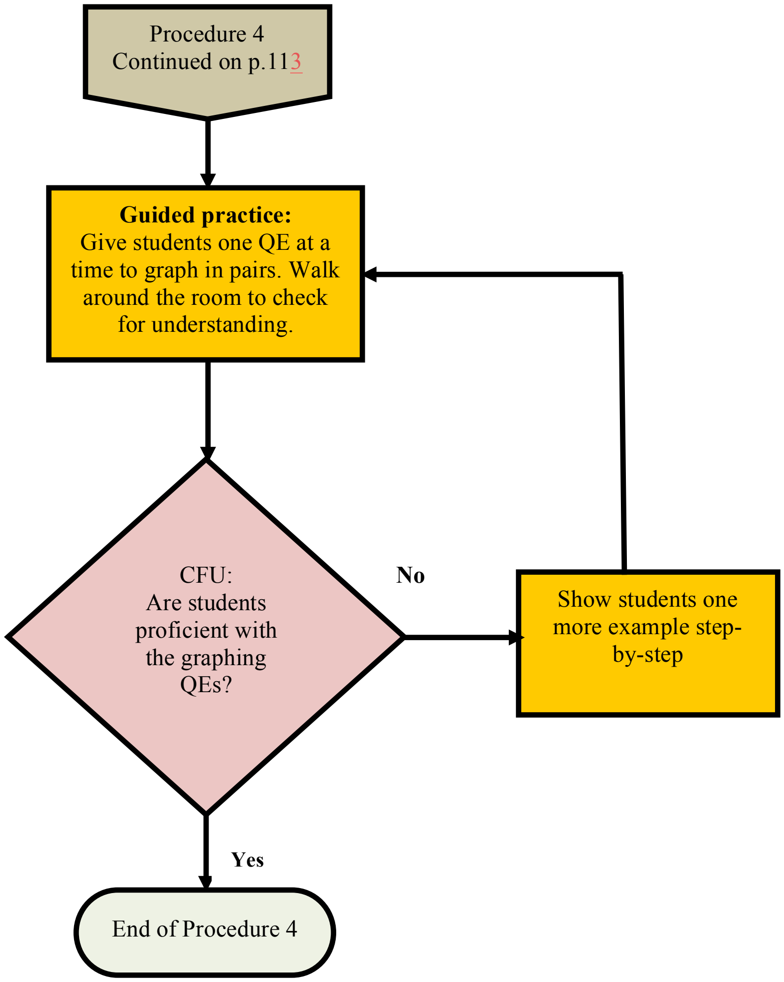
Find the *vertex* by substituting the *x*-value of the *axis of symmetry* into the original equation: $ax^2 + bx + c = y$ to find the *y*-value of the *vertex*

Write the *vertex* in the form (*x*, *y*) and **plot** the point on the *x-y* coordinate plane on the board

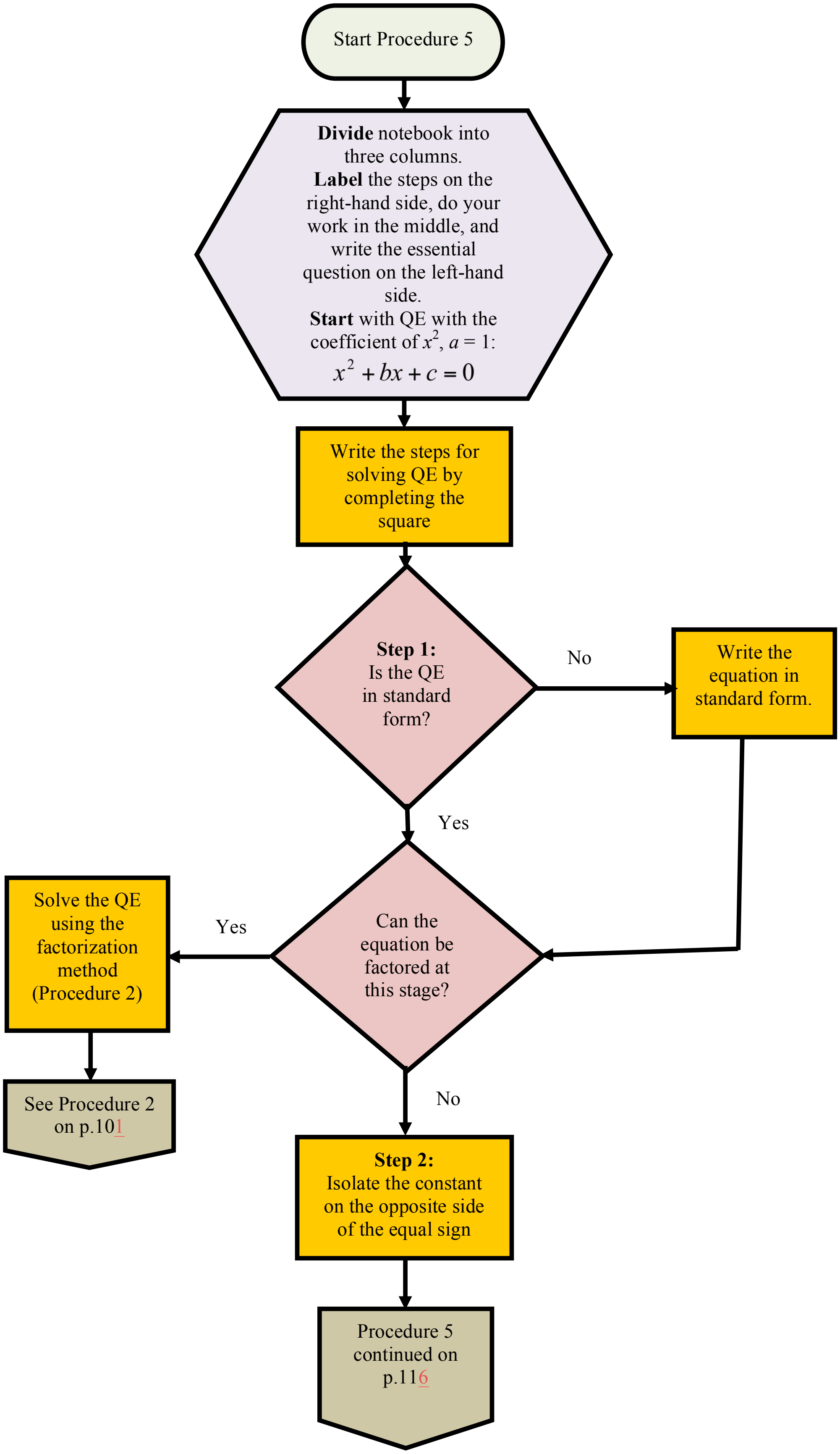
Tell students: "The *vertex* is where the *parabola* opens from"

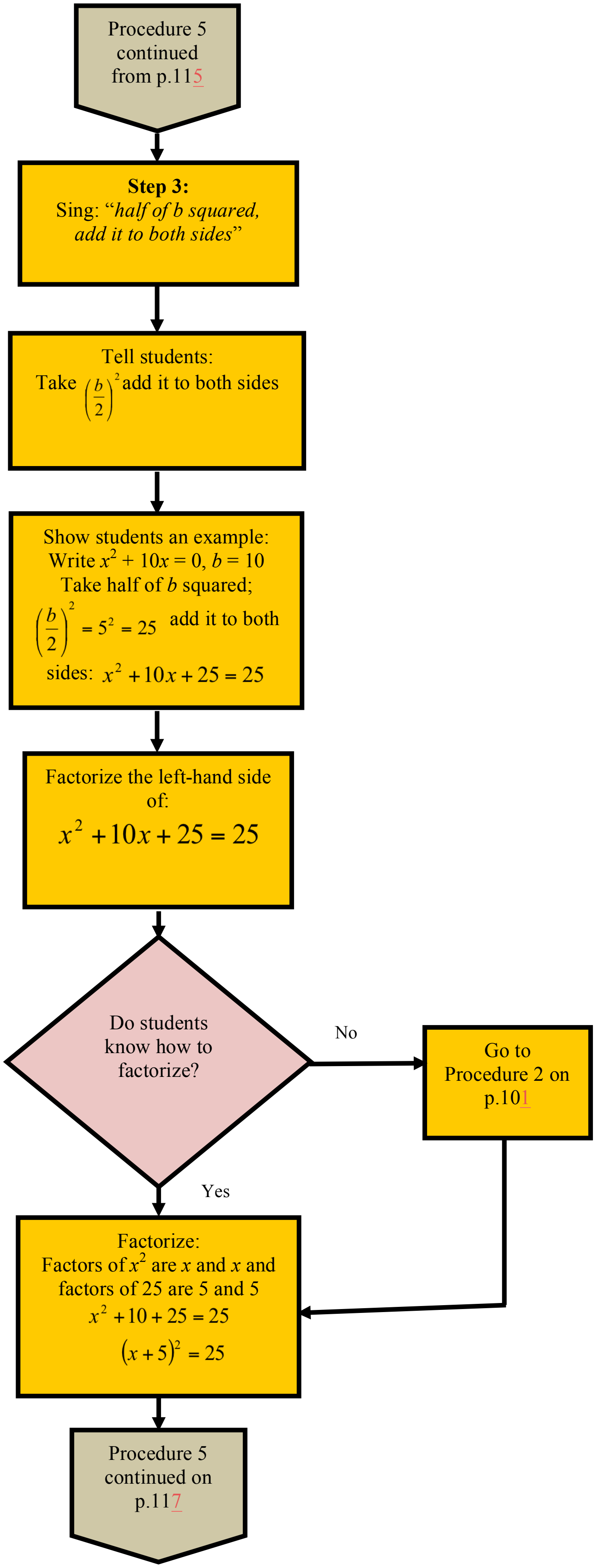
Procedure 4 continued on p.113

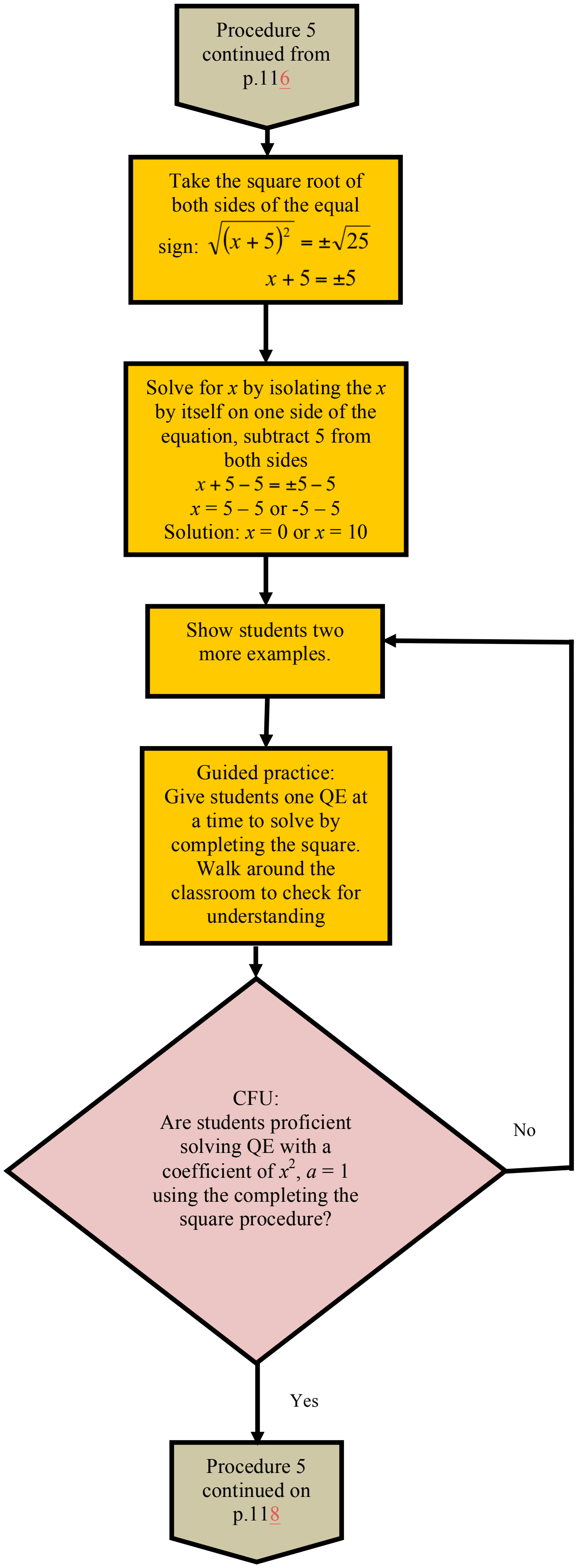

Procedure 4
continued from p.112
Draw a two-column t-table of values (712-713) and put the vertex coordinates at the center of the t-table Choose two x-values to the left and right of the x-value of the vertex (axis of symmetry) and substitute them into the original QE to find the corresponding y-values
x
y
Plot the points on the x-y coordinate and connect the points to plot the graph with a smooth curve
Label on the graph the vertex, y-intercept, axis of symmetry, and direction of opening, up or down
Procedure 4
continued on p.114

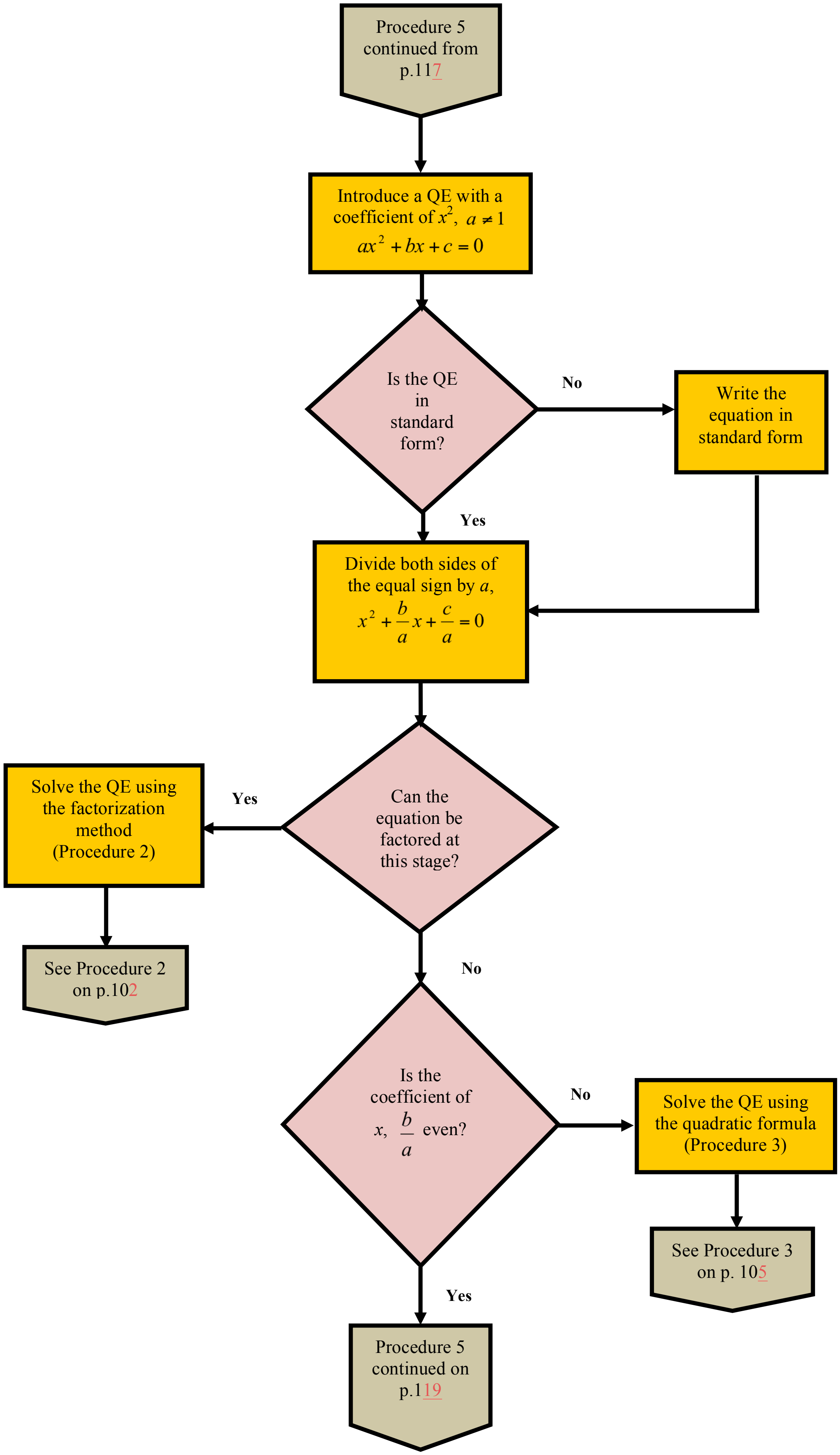

Procedure 4
Continued on p.113
Guided practice:
Give students one QE at a time to graph in pairs. Walk around the room to check for understanding.
CFU:
Are students proficient with the graphing QEs?
No
Show students one more example step-by-step
Yes
End of Procedure 4

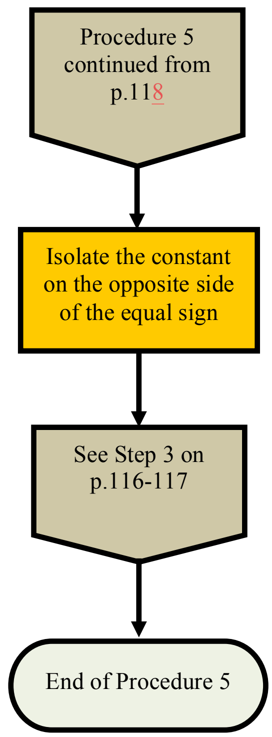
Procedure 5


Start Procedure 5
Divide notebook into three columns. **Label** the steps on the right-hand side, do your work in the middle, and write the essential question on the left-hand side. **Start** with QE with the coefficient of x^2, $a = 1$:
$x^2 + bx + c = 0$
Write the steps for solving QE by completing the square
Step 1: Is the QE in standard form?
No
Write the equation in standard form.
Yes
Can the equation be factored at this stage?
Yes
Solve the QE using the factorization method (Procedure 2)
See Procedure 2 on p.101
No
Step 2: Isolate the constant on the opposite side of the equal sign
Procedure 5 continued on p.116

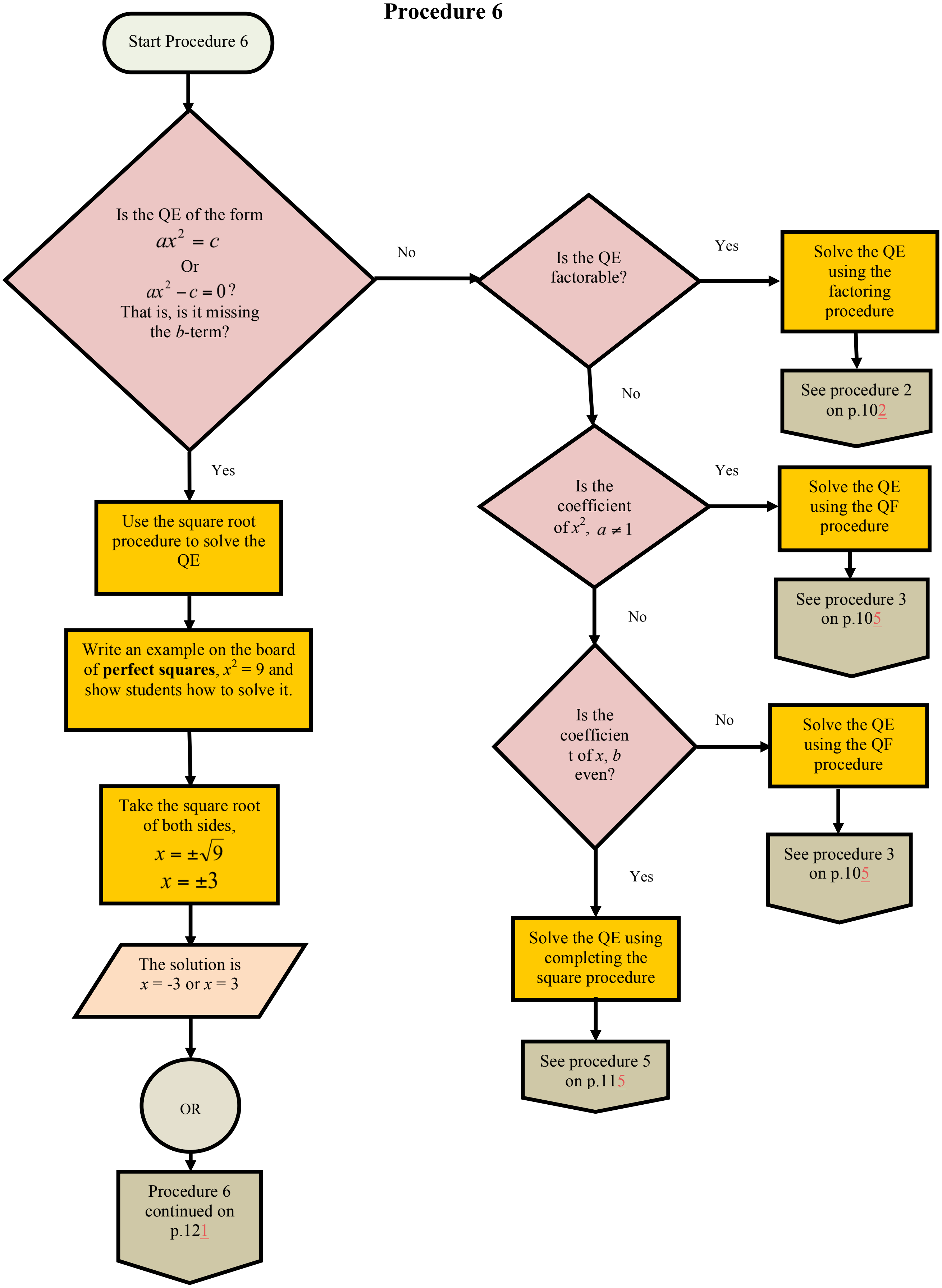

Procedure 5 continued from p.115
Step 3:
Sing: "*half of b squared, add it to both sides*"
Tell students:
Take $\left(\frac{b}{2}\right)^2$ add it to both sides
Show students an example:
Write $x^2 + 10x = 0$, $b = 10$
Take half of b squared;
$\left(\frac{b}{2}\right)^2 = 5^2 = 25$ add it to both
sides: $x^2 + 10x + 25 = 25$
Factorize the left-hand side of:
$x^2 + 10x + 25 = 25$
Do students know how to factorize?
No
Go to Procedure 2 on p.101
Yes
Factorize:
Factors of x^2 are x and x and factors of 25 are 5 and 5
$x^2 + 10 + 25 = 25$
$(x+5)^2 = 25$
Procedure 5 continued on p.117

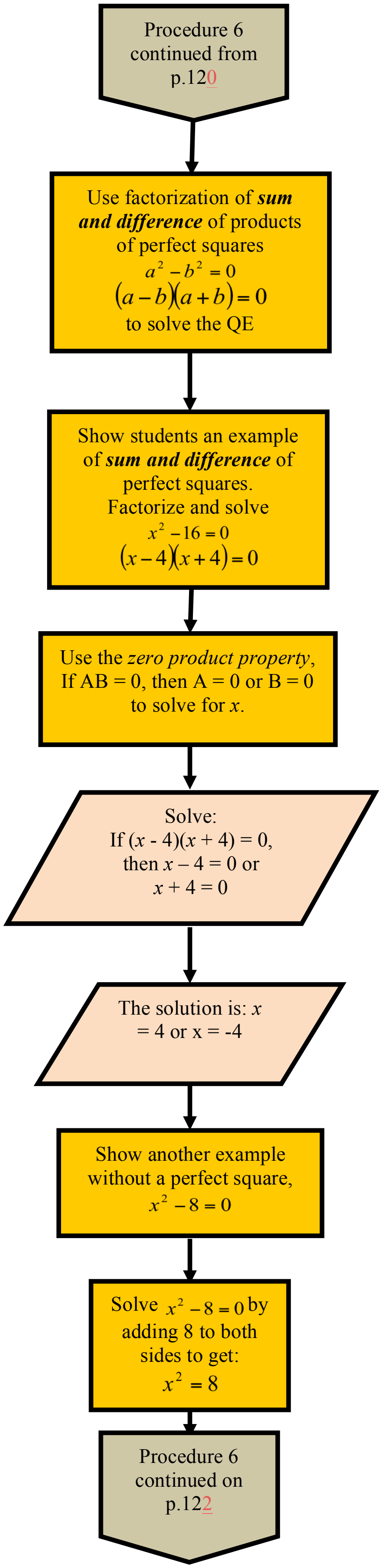

Procedure 5 continued from p.116
Take the square root of both sides of the equal sign: $\sqrt{(x+5)^2} = \pm\sqrt{25}$
$x + 5 = \pm 5$
Solve for x by isolating the x by itself on one side of the equation, subtract 5 from both sides
$x + 5 - 5 = \pm 5 - 5$
$x = 5 - 5$ or $-5 - 5$
Solution: $x = 0$ or $x = 10$
Show students two more examples.
Guided practice: Give students one QE at a time to solve by completing the square. Walk around the classroom to check for understanding
CFU: Are students proficient solving QE with a coefficient of x^2, $a = 1$ using the completing the square procedure?
No
Yes
Procedure 5 continued on p.118

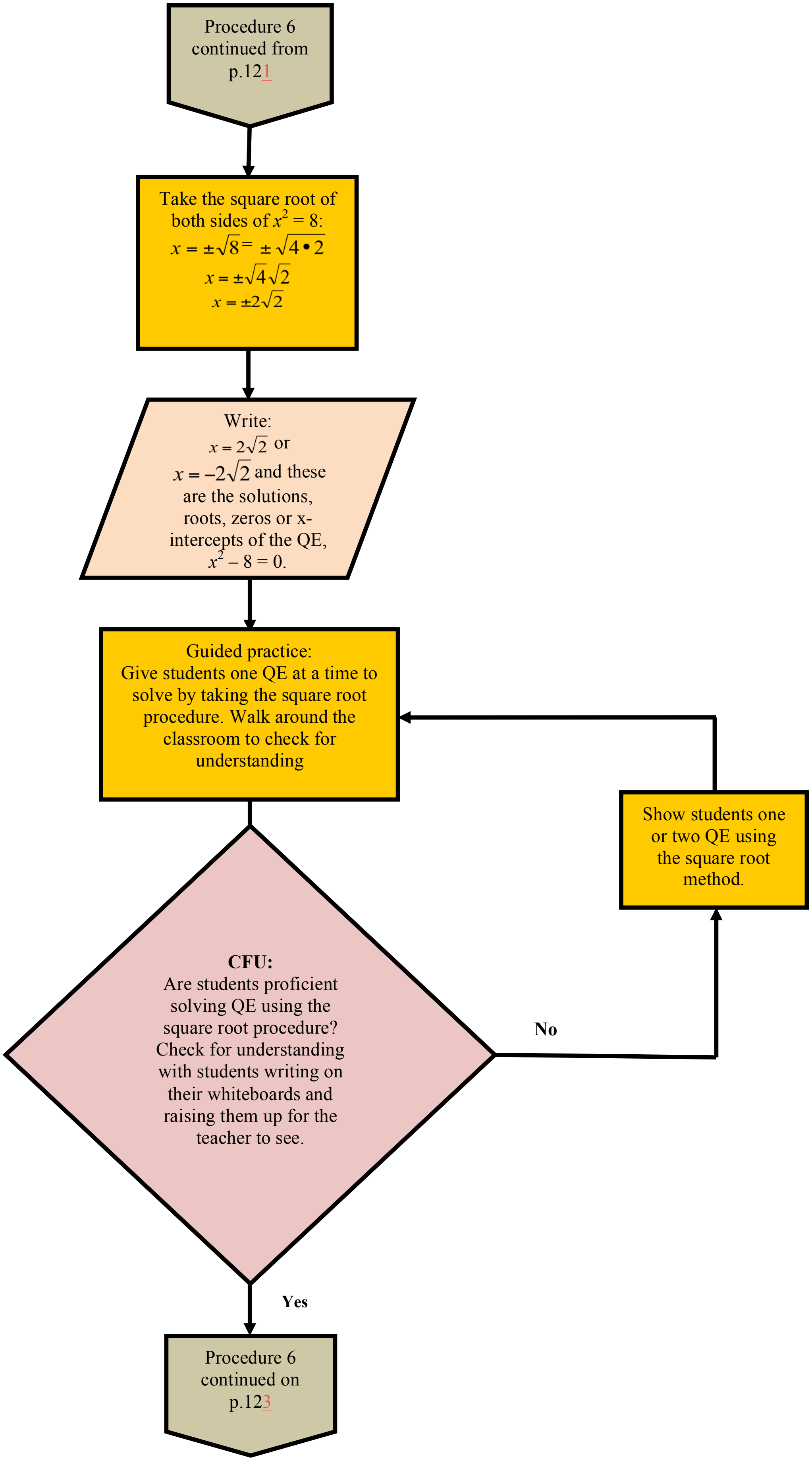

Procedure 5 continued from p.117
Introduce a QE with a coefficient of x^2, $a \neq 1$
$ax^2 + bx + c = 0$
Is the QE in standard form?
No
Write the equation in standard form
Yes
Divide both sides of the equal sign by a,
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
Can the equation be factored at this stage?
Yes
Solve the QE using the factorization method (Procedure 2)
See Procedure 2 on p.102
No
Is the coefficient of x, $\frac{b}{a}$ even?
No
Solve the QE using the quadratic formula (Procedure 3)
See Procedure 3 on p. 105
Yes
Procedure 5 continued on p.119

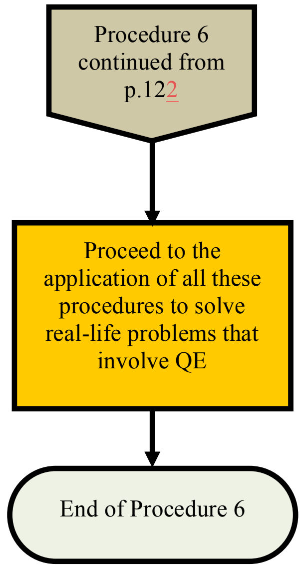

Procedure 5 continued from p.118
Isolate the constant on the opposite side of the equal sign
See Step 3 on p.116-117
End of Procedure 5

Procedure 6

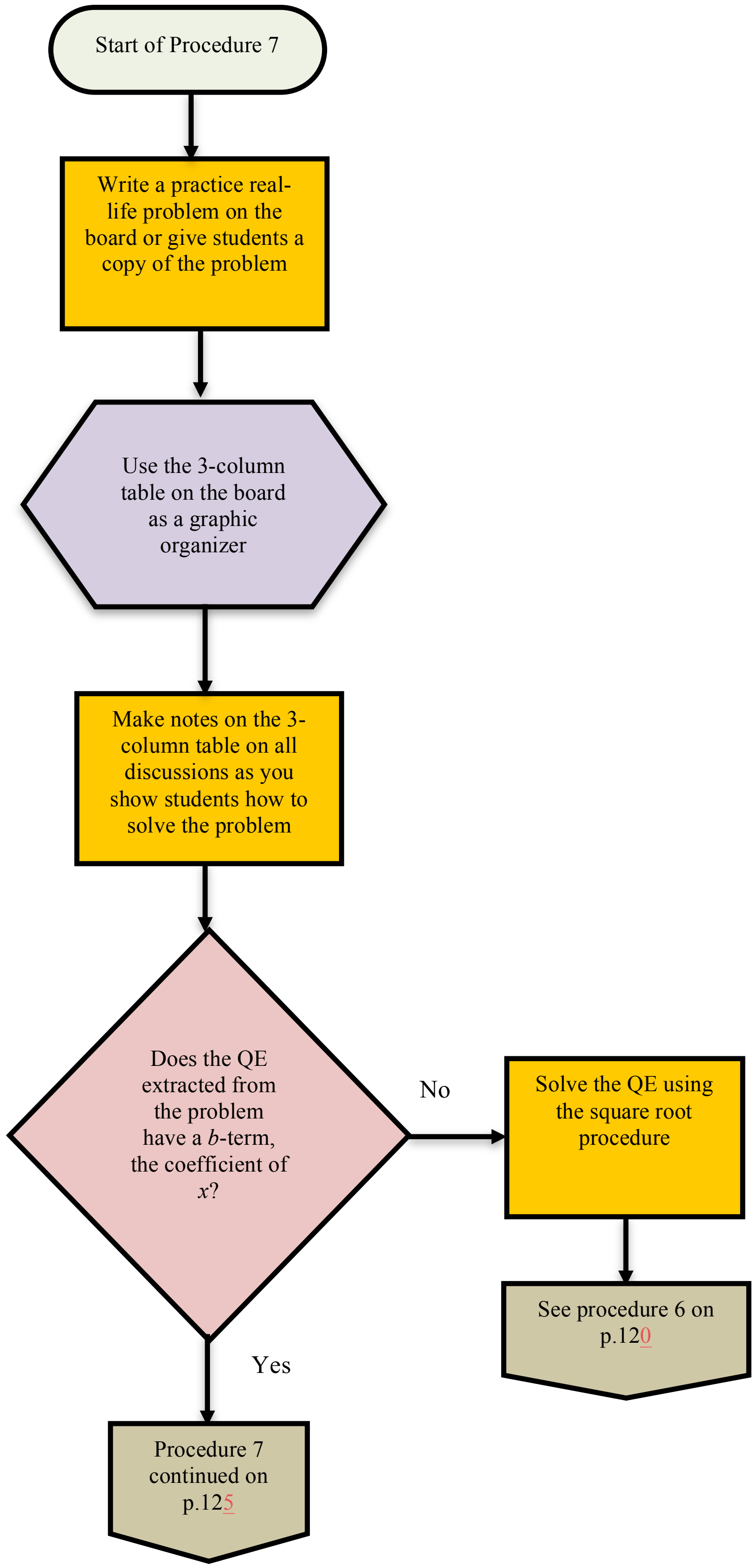

Start Procedure 6
Is the QE of the form $ax^2 = c$ Or $ax^2 - c = 0$? That is, is it missing the *b*-term?
Yes
Use the square root procedure to solve the QE
Write an example on the board of **perfect squares**, $x^2 = 9$ and show students how to solve it.
Take the square root of both sides, $x = \pm\sqrt{9}$ $x = \pm 3$
The solution is $x = -3$ or $x = 3$
OR
Procedure 6 continued on p.121
No
Is the QE factorable?
Yes
Solve the QE using the factoring procedure
See procedure 2 on p.102
No
Is the coefficient of x^2, $a \neq 1$
Yes
Solve the QE using the QF procedure
See procedure 3 on p.105
No
Is the coefficient of *x*, *b* even?
No
Solve the QE using the QF procedure
See procedure 3 on p.105
Yes
Solve the QE using completing the square procedure
See procedure 5 on p.115

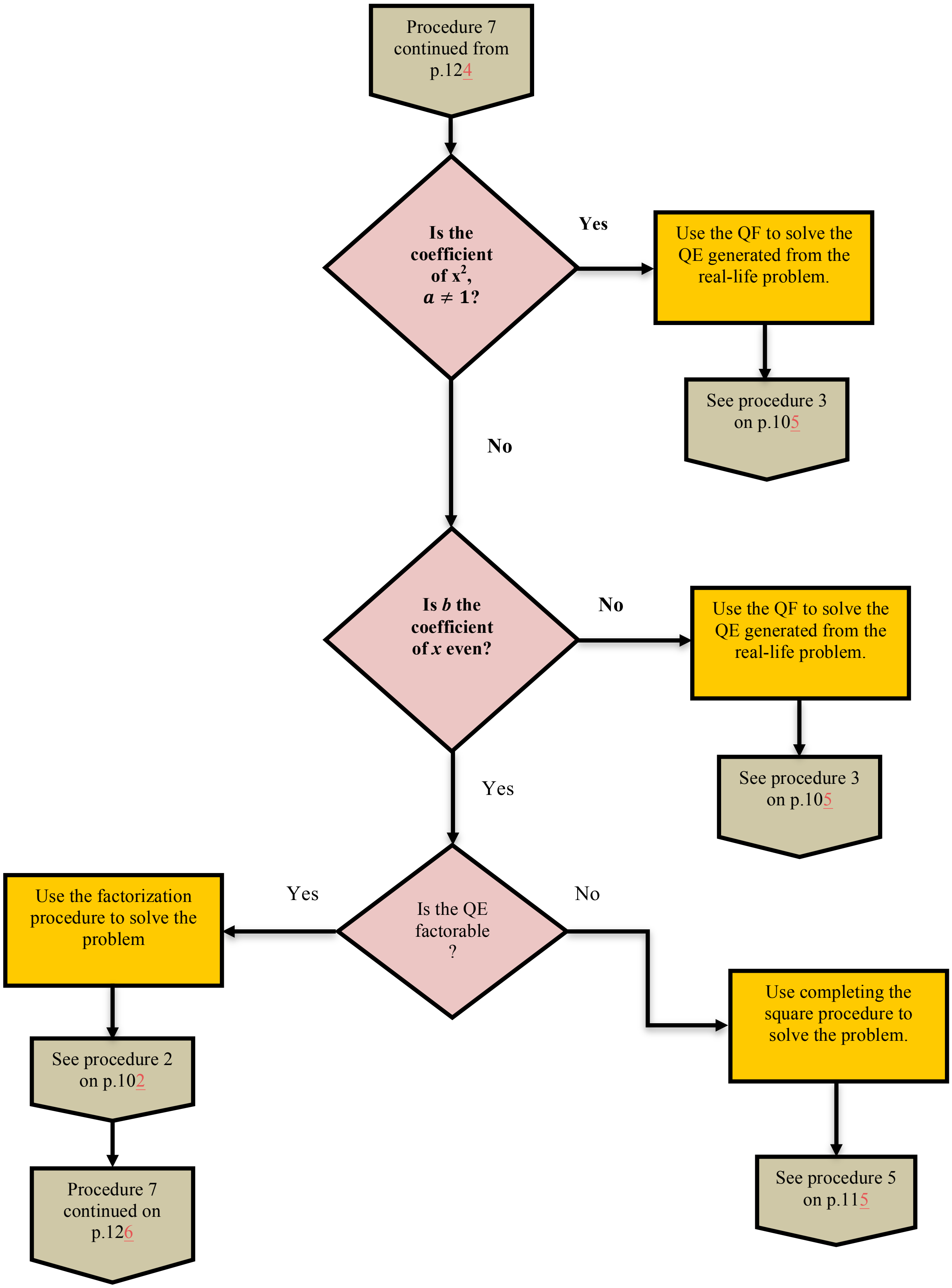

Procedure 6 continued from p.120
Use factorization of ***sum and difference*** of products of perfect squares
$a^2 - b^2 = 0$
$(a-b)(a+b) = 0$
to solve the QE
Show students an example of ***sum and difference*** of perfect squares.
Factorize and solve
$x^2 - 16 = 0$
$(x-4)(x+4) = 0$
Use the *zero product property*, If AB = 0, then A = 0 or B = 0 to solve for *x*.
Solve:
If $(x - 4)(x + 4) = 0$, then $x - 4 = 0$ or $x + 4 = 0$
The solution is: $x = 4$ or $x = -4$
Show another example without a perfect square,
$x^2 - 8 = 0$
Solve $x^2 - 8 = 0$ by adding 8 to both sides to get:
$x^2 = 8$
Procedure 6 continued on p.122

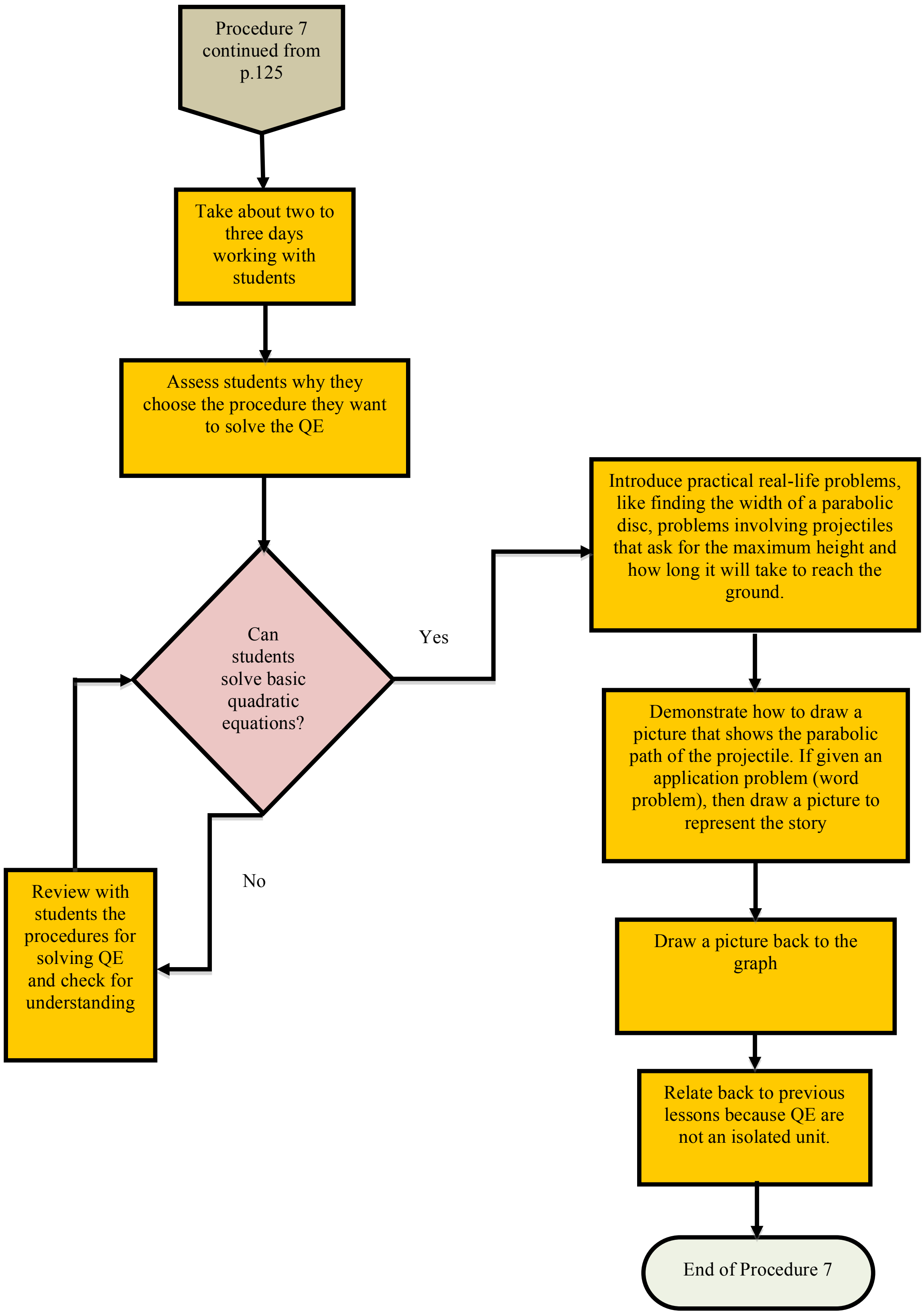

Procedure 6 continued from p.121
Take the square root of both sides of $x^2 = 8$:
$x = \pm\sqrt{8} = \pm\sqrt{4 \bullet 2}$
$x = \pm\sqrt{4}\sqrt{2}$
$x = \pm 2\sqrt{2}$
Write: $x = 2\sqrt{2}$ or $x = -2\sqrt{2}$ and these are the solutions, roots, zeros or x-intercepts of the QE, $x^2 - 8 = 0$.
Guided practice: Give students one QE at a time to solve by taking the square root procedure. Walk around the classroom to check for understanding
Show students one or two QE using the square root method.
CFU:
Are students proficient solving QE using the square root procedure? Check for understanding with students writing on their whiteboards and raising them up for the teacher to see.
No
Yes
Procedure 6 continued on p.123


Procedure 6 continued from p.122
Proceed to the application of all these procedures to solve real-life problems that involve QE
End of Procedure 6

Procedure 7


Start of Procedure 7
Write a practice real-life problem on the board or give students a copy of the problem
Use the 3-column table on the board as a graphic organizer
Make notes on the 3-column table on all discussions as you show students how to solve the problem
Does the QE extracted from the problem have a *b*-term, the coefficient of *x*?
No
Solve the QE using the square root procedure
See procedure 6 on p.120
Yes
Procedure 7 continued on p.125


Procedure 7 continued from p.124
Is the coefficient of x^2, $a \neq 1$?
Yes
Use the QF to solve the QE generated from the real-life problem.
See procedure 3 on p.105
No
Is b the coefficient of x even?
No
Use the QF to solve the QE generated from the real-life problem.
See procedure 3 on p.105
Yes
Is the QE factorable ?
Yes
Use the factorization procedure to solve the problem
See procedure 2 on p.102
Procedure 7 continued on p.126
No
Use completing the square procedure to solve the problem.
See procedure 5 on p.115


Procedure 7 continued from p.125
Take about two to three days working with students
Assess students why they choose the procedure they want to solve the QE
Can students solve basic quadratic equations?
Yes
No
Review with students the procedures for solving QE and check for understanding
Introduce practical real-life problems, like finding the width of a parabolic disc, problems involving projectiles that ask for the maximum height and how long it will take to reach the ground.
Demonstrate how to draw a picture that shows the parabolic path of the projectile. If given an application problem (word problem), then draw a picture to represent the story
Draw a picture back to the graph
Relate back to previous lessons because QE are not an isolated unit.
End of Procedure 7

Appendix E

Solving Quadratic Equations – Gold Standard Protocol

Task: To Teach Solving Quadratic Equations in Algebra 1

Objective: Students understand how quadratic equations connect to real-life situations and that there are multiple ways to solve a quadratic equation

Main Procedures:

1. Review Linear Equations to Activate prior knowledge
2. Teach solving quadratic equations by Factoring.
3. Teach solving quadratic equations by using the quadratic formula.
4. Teach solving quadratic equations by graphing
5. Teach solving quadratic equations by completing the square.
6. Teach solving quadratic equations by the Square root method.
7. Teach Application of these methods of solving quadratic equations to solving real-life problems.

Procedure 1: Review linear equation to Activate prior knowledge

1.1 Give students an overview of the unit of solving quadratic equations.

1.1.1 Tell students: "I'm going to teach you how to solve quadratic equations"

1.1.2 Tell students: "There are multiple methods for solving quadratic equations: factorization, graphing, the quadratic formula, the square root method, and completing the square

1.1.3 Tell students: "Some of these methods will work better for some of the quadratic equations. Some quadratic equations are perfect square binomials and therefore will be easy to recognize that it can be factored and solved, while other quadratic equations may be missing a "*b*" value and so it will be useful to use the square root method. Other quadratic equations may be factorable but have many factors to try and so quadratic formula will be quickest".

1.2 Draw a full page size Tree map – see Figure E1

1.2.1 Relate solving quadratic equations to solving linear equations

1.2.1.1 Reason: Get students to see there are many ways to solve a quadratic equation by always referring back to the tree map and these procedures are related to solving linear equations

Figure E1: Tree map

Procedures for Solving Quadratic Equations

Factoring (And use zero product property (zpp))

QE in standard form must be factorable

Example:
$x^2 + 7x + 10 = 0$

sum *product*

(x + 5)(x + 2) = 0

x+5=0 x+2=0
zpp

x = -5 and x = -2
these are zeros, roots, solutions of the QE

Non-example:
$x^2 + 1x + 5 = 0$

sum *product*

There are no factors of 5 that have a sum of 1

Quadratic formula

Use with QE in standard form:
$ax^2 + bx + c = 0$

The quadratic formula will work to solve any quadratic equation.

Graphing

Standard form (Brace Map)

- Opening up or down depends on *a*.
- *c*-value is the y-intercept
- Axis of symmetry, $x = \frac{-b}{2a}$
- Axis of symmetry cuts parabola into 2 symmetrical pieces
- Vertex $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ substitute the x-value of the axis of symmetry into the function to find the y-value of the vertex.
- The vertex is a minimum or a maximum
- Other points: select and substitute an x to the left and right of the vertex

Completing the Square

- Isolate c
- Song/dance
- Use when *b*-value is even

Square Root

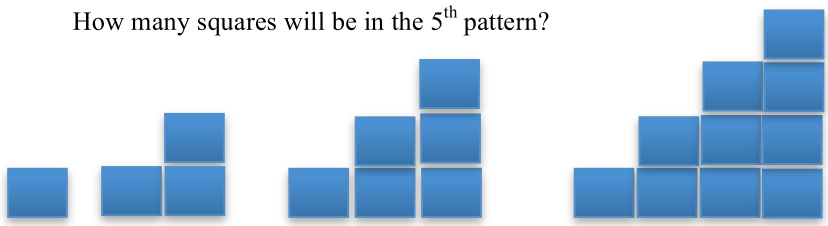
- Use only when *b*-value is 0 ($b = 0$)
- $ax^2 = c$

1.3 Put patterns on the screen over the computer or write them on the board, see Figure E2

Figure E2: Patterns

Display Patterns (P3)

How many squares will be in the 5th pattern?



1st 2nd 3rd 4th 5th

Position of squares	Number of squares	1st diff	2nd diff
0	0		
1	1	1	
2	3	2	1
3	6	3	1
4	10	4	1
5	?		

Therefore, there are:

y = ½ (5^2) + ½ (5)

y = 12.5 + 2.5

y = 15

There are 15 squares in the 5th pattern.

Show students how to find the constants *a*, *b*, and *c*:
The second difference is a constant, which indicates the pattern is quadratic.
Find *a* by dividing the common difference of 1 by 2: $a = {}^{1}/_{2}$, *b* = ? and *c* = 0, *y*-intercept

Use the 4th pattern with 10 squares: (4, 10) ; $y = ax^2 + bx + c$
Substitute for x and y into the quadratic equation

$$y = ax^2 + bx + c$$

10 = ½ (4^2) + *b*(4) + 0

10 = 8 + 4*b*

2 = 4*b*, b = ½

Therefore the quadratic equation is y = ½ x^2 + ½ *x*

Check: Use the 3rd pattern, (3, 6)

6 = ½ (3^2) + ½ (3)

6 = 4.5 + 1.5

6 = 6, therefore this quadratic equation checks for this pattern.

1.3.1 Show students how to solve linear equations step-by-step
1.3.2 Draw a two-column table on the board
1.3.3 Show students how to generate a table of values from patterns or linear equations (see Fig. 2)
1.3.4 Plot the points on an *x*-*y* coordinate plane drawn on the white board
1.3.5 Examine and compare linear equations by changing the slope (*m*) and/or changing the *y*-intercept (*b*)
1.3.6 IF students have access to graphing calculators or graphing software, THEN give opportunities to check the graph they have made the linear equations from the table of values
1.3.7 IF you show students a problem or two, THEN give them a few to try individually to check for understanding
1.3.8 IF students are not proficient in solving linear equations, THEN reteach the concept
1.3.9 Ask students randomly to come to the board to show that they have successfully completed the problem
1.3.10 Assess students by walking around the room to get a visual of what they are doing and that they are communicating using math language in their groups
1.3.11 IF you see that students are showing that they can do it on the board and you are walking around the room and making sure that students are understanding from what you can see, THEN continue to progress with the lesson of activating prior knowledge

1.4 Review with students how to multiply and divide rational numbers
1.4.1 Give examples: $3*4=12$; $-2*5=-10$; $2*-5=-10$; $-3*-2=6$.
1.4.1.1 Reason: The intention is to remind students about the rules for multiplying integers since factoring quadratics assumes students can factor constant values
1.4.2 Define a factor to students: "Factors are numbers you can multiply together to get another number"

1.5 Review with students how to factor a whole number
1.5.1 Use factor trees (see Figure E3)
1.5.2 Show students the factors of 6: 1 and 6, and 2 and 3 are factors because the product of each pair is 6.
1.5.3 Give students a number to factorize, for example: factorize 18
1.5.3.1 Show students the factors of 18 are: 1 and 18, 2 and 9, 3 and 6, -1 and -18, -2 and -9, and -3 and -6
1.5.3.2 Show students that the product of these factors is 18
1.5.4 Remind students the rules for multiplying integers
1.5.4.1 Multiply a positive number by a positive number the product is another positive number; multiply a negative number by another negative number the product is positive while the product of a positive number by a negative number is a negative number
1.5.5 IF you are factorizing a positive number, THEN get two positive factors or two negative factors
1.5.6 IF you are factorizing a negative number, THEN get one positive factor and one negative factor. Be sure that the sign of the greater factor matches the sign of the middle term

1.6 Review with students how to multiply polynomials

1.6.1 Teach students exponent rules

1.6.1.1 Remind students, for example that x times x equals x^2

1.6.1.2 Factorize x^3 or x^2 into factors: $x^3 = (x)(x)(x)$ and $x^2 = (x)(x)$

1.6.1.3 Show students that $x + x = 2x$, *and* $x \bullet x = x^2$ on the whiteboard

1.6.1.3.1 Reason: To know the difference between addition and multiplication when factorizing (breaking down) polynomials into factors

1.7 Review with students how to factor linear expressions

1.7.1 Show students how to factor an expression like $3x + 12$. Tell students: “The terms $3x$ and 12 have a common factor of 3 because 3 can divide both $3x$ and 12. Factorize 3 and the new expression is $3(x + 4)$”

1.7.2 Give students another linear expression to practice factorizing

1.7.3 IF students are not proficient factoring linear expressions, THEN show more examples like in step 1.7.1

1.7.4 IF students are proficient factoring linear expressions, THEN introduce solving quadratic equations by factoring

Procedure 2: Teach solving quadratic equations by factoring

2.1 Remind students what a “factor” is

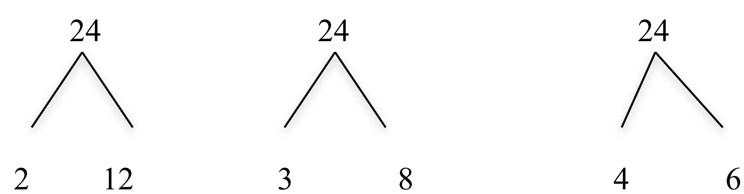
2.1.1 Define a factor to students again (line 1.4.2): “Factors are numbers you can multiply together to get another number”

2.1.2 Explain (step 1.5) what you will do when you factor a certain problem. When “factoring” we are showing students another way to write a product—as a multiplication problem. Sometimes the factored form will look like an expanded version of the original problem.

2.1.3 Give students a few factor tree problems to practice, for example: find the factors 24 (Fig. 3)

2.1.3.1 Factor 24: 24 = (2)(12) or (3)(8) or (4)(6). At this point, label (2)(12) as the “factored form” of 24, Figure E3.

Figure E3: Factor trees



2.1.3.2 Give students a number to factor individually

2.1.3.3 IF some students are not proficient in multiplication, then assign multiplication flash cards for homework practice, and encourage those students to use multiplication charts when factorizing

2.1.3.4 IF students are not proficient in factorizing, THEN the teacher does one more as students follow along

2.1.3.5 IF students are proficient with factoring numbers, THEN teach students how to write a quadratic expression in standard form, like $x^2 + 5x + 6$

2.2 Give an example of the standard form: $x^2 + 5x + 6 = 0$ and a non-example: $x^2 + 6 = -5x$. Tell students: "All the terms should be on one side of the equal sign

2.3 Show students coefficients of the quadratic equation. For example, the coefficients of $x^2 + 5x + 6 = 0$ are 1, 5, and 6"

2.4 Teach students how to use distributive property

2.4.1 Write an example of two binomials on the board, for example (x + 2) and (x + 1)

2.4.2 Show students how to use algebra tiles (see Figure E5)

2.4.3 Find the area of the product of these binomials: $x^2 + 2x + x + 2$

2.4.4 Write area $x^2 + 3x + 2$ by looking at tiles

2.4.5 Tell students: "Factoring is how we undo the distributive property (product of binomials) for example, getting the binomials that gave the product $3x^2 + 4x - 15$"

2.5 Show students how to factorize $x^2 + 5x + 6$

2.6 Show students a sum and product table

2.6.1 Show students how to use a sum and product table to find factors

2.6.1.1 Tell students: "Draw a sum and product table on your whiteboard"

2.6.1.2 Tell students: "Two numbers have a sum of 5 and a product of 6. With your partner, figure out which numbers they are"

2.6.1.3 Tell students to do the problem on their whiteboards and hold them up

2.6.1.4 Scan across the room as students raise their whiteboards checking for understanding

2.6.1.5 IF students have not mastered the use of sum and product tables to factorize, THEN show students another example. Like two numbers have a sum of 8 and a product of 15, show them how to find these two numbers

2.6.1.6 Repeat this procedure with different problems until students are proficient

2.6.1.7 IF students have mastered the use of sum and product tables, THEN show students how to factor the original problem $x^2 + 5x + 6$

2.6.1.8 Circle the term 6 which is the constant in $x^2 + 5x + 6$ and write the word constant above the 6. IF you circle the constant, THEN write the word product underneath it

2.6.1.8.1 Show students some numbers that give a product of 6, write these numbers in the sum and product table. Some pairs are 1 and 6, and 2 and 3

2.6.1.9 Circle the x-term, $5x$ and then write sum underneath it

2.6.1.9.1 Use the pairs of numbers in the sum and product table to determine which pair, 1 and 6 or 2 and 3 adds up to 5

2.6.1.9.2 Choose 2 and 3

2.6.1.10 Circle the x^2-term. Tell students: " x times x is x^2 and therefore x and x are the factors of x^2"

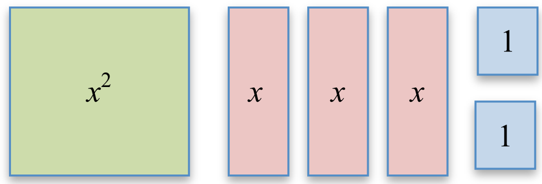
2.7 Teach students how to use algebra tiles, the X-BOX (Diamond Method), and the Parenthesis methods to factorize

2.7.1 Use algebra tiles to factor quadratic equations (see Figure E5)

2.7.2 Introduce quadratic equations that can be factorized using algebra tiles

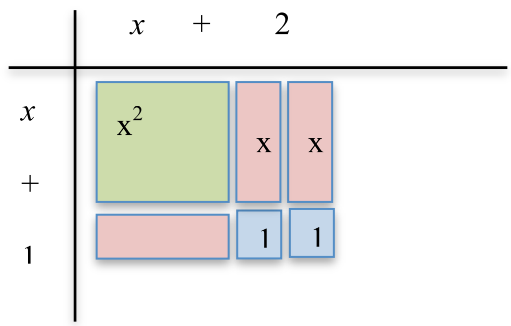
2.7.2.1 Ask students: IF I have $x^2 + 3x + 2$, THEN what tiles would I need? (Figure E4)
2.7.2.2 Ask students to gather tiles

Figure E4: Algebra tiles



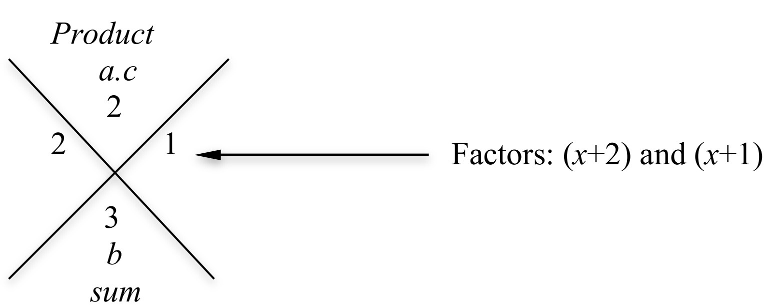
2.7.2.3 Tell students that $x^2 + 3x + 2$ represents an area
2.7.2.4 Arrange tiles in a rectangle, to find factors by looking at length and width of the rectangle

Figure E5: Factorization using tiles



2.7.2.5 Tell students: "The factors of $x^2 + 3x + 2$ are $(x + 2)$ and $(x + 1)$" (step 2.7.2.4)
2.7.2.6 IF students are not yet comfortable with factoring using algebra tiles, THEN reteach the concept as in step 2.7.2.4

2.7.3 If students are comfortable with factoring using algebra tiles, then introduce the X-BOX (Diamond) method for factoring so that students may have an alternative way of factorizing. For example, factorize $x^2 + 3x + 2$ using the X-BOX method, see below



2.7.4 IF students are not proficient factorizing using the X-BOX method (step 2.7.3), THEN reteach

2.7.5 IF students are proficient factorizing using the X-BOX method (step 2.7.3), THEN proceed to teach students the Parenthesis Method

2.8 Show students how to factorize a quadratic expression $x^2 + 5x + 6$ using the Parenthesis Method

2.8.1 Put two sets of parentheses, $(\ )(\ )$ at the top for the binomials that we are trying to factor this problem into Then, at the bottom of the parentheses, write the problem: $x^2 + 5x + 6$

2.8.2 Tell students: "We are going to find the factors of this problem, $x^2 + 5x + 6$"

2.8.3 Tell students: "Since factors of the first term x^2 are x and x, put each on a different parenthesis $(x + \ )(x + \ )$, since x times x is x^2, followed by 2 in the first parenthesis and 3 on the second parenthesis $(x+2)(x+3)$"

2.8.4 IF we factor $x^2 + 5x + 6$, THEN the product of these parts in the parentheses $(x+2)(x+3)$ have to match with the original expression, $x^2 + 5x + 6$

2.8.5 Give students another problem $x^2 + 11x + 30$ to factorize

2.8.6 IF students are not proficient with factoring $x^2 + 11x + 30$, THEN show students how to do it

2.8.6.1 Use the sum and product table. Two numbers have a sum of 11 and a product of 30.

2.8.6.2 Repeat the same process as in 2.3.1.1 through 2.3.1.4 to factorize $x^2 + 11x + 30$

2.8.7 IF students are proficient with factoring expressions, THEN introduce them to solving quadratic equations using the factorization method

2.8.7.1 Show students how to solve $x^2 + 5x + 6 = 0$ by finding the value of x that satisfies this equation.

2.8.7.2 Factorize the left hand side of the equation $x^2 + 5x + 6 = 0$

2.8.7.3 Put $(x+2)(x+3) = 0$ below the original quadratic equation

2.8.7.4 Use the zero product property, IF AB = 0, THEN A = 0 or B = 0

2.8.7.5 IF $(x+2)(x+3) = 0$, THEN $(x + 2) = 0$ or $(x + 3) = 0$

2.8.7.6 Solve: $x + 2 = 0$, subtract 2 from both sides of the equation: $x+2-2 = 0-2$ therefore $x = -2$. And $x + 3 = 0$, subtract 3 from both sides of the equation: $x+3-3=0-3$ therefore $x = -3$.

2.8.7.7 Solution is $x = -2$ or -3. Solution of a quadratic equation is also called the x-intercepts, zeros, and roots.

2.8.8 Give students another example to practice: $x^2 + 11x + 30 = 0$

2.8.9 Use the sum and product table to factorize the left hand side of the quadratic equation

2.8.9.1 Factorize $x^2 + 11x + 30 = 0$ to $(x +5)(x + 6) = 0$

2.8.9.2 Use the zero product property, IF AB = 0, THEN A=0 or B=0

2.8.9.3 Solve: IF $(x + 5)(x +6) = 0$, THEN $(x + 5) = 0$ or $(x + 6) = 0$

2.8.9.4 IF $x + 5 = 0$, THEN $x = -5$ and IF $x + 6 = 0$, THEN $x = -6$

2.8.9.5 Solution is $x = -5$ or -6. Tell students: "The solution of a quadratic equation is also called x-intercepts, zeros, and roots.

2.9 Give students 3-5 question assessment (open ended)

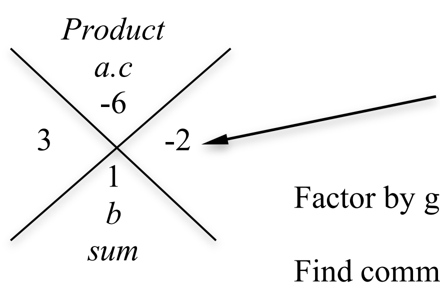
2.9.1 IF students have mastered factoring when the leading coefficient is 1, THEN move on to factoring where the leading coefficient is other than 1. IF not, THEN reteach.

2.10 Write a quadratic equation like $6x^2 + x - 1 = 0$ on the board

2.10.1 Identify $a = 6$, $b = 1$, and $c = -1$

2.10.2 Use the X-Box (Diamond) method, for example (Step 2.10.3)

2.10.3 Make a big cross (**X**) underneath the equation. Inside the top of the **X**, write the product of a and c $(a.c = -6)$ and inside the bottom of the **X**, write b (the sum) – (see below).



$6x^2 + 3x - 2x - 1 = 0$

a *b* *b* *c*

Factor by grouping: $3x(2x+1) -1(2x+1) = 0$

Find common factors: $(2x + 1)(3x - 1) = 0$

2.10.4 Find factors with a product of -6 and a sum of 1, as shown

2.10.5 Tell students: "There are many pairs that will give a product of -6 but they are not all going to give a sum of the middle coefficient, 1"

2.10.6 Show students the factors of -6 that give a sum of 1 are: 3 and -2

2.10.7 Write the expanded form of the quadratic equation: $6x^2 + 3x - 2x - 1 = 0$

2.10.8 Factor by grouping $3x(2x + 1) - 1(2x + 1) = 0$

2.10.9 Find common factors: $(2x + 1)(3x - 1)$

2.10.10 Check by multiplying using the box shown below

Multiply	2x	+1
3x	$6x^2$	3x
-1	-2x	-1

2.10.11 Write out $6x^2 + 3x - 2x - 1 = 0$ to confirm the product is the original quadratic equation $6x^2 + x - 1 = 0$

2.10.12 Solve the quadratic equation for x using the zero product property (ZPP): $(2x+1)(3x-1)=0$. If $(2x + 1)(3x – 1) = 0$, THEN $2x + 1 = 0$ or $3x – 1 = 0$

2.10.13 Solve $2x + 1 = 0$ to get $x = -1/2$ and $3x – 1 = 0$ to get $x = 1/3$

2.11 IF teacher does one example on the board, THEN give students one problem to try. Walk around the room to monitor what students are doing to check for understanding

2.12 IF students have not mastered factoring, THEN reteach the concept

2.13 IF students have mastered factoring, THEN give proceed to give assessment

2.14 Give students 5-10 question assessment (open ended) on factoring.

2.14.1 IF students have mastered factoring, THEN move on to solving quadratic equations using the quadratic formula. IF NOT, THEN reteach.

Procedure 3: Teach solving quadratic equations by using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3.1 Teach students a way to memorize the quadratic formula

3.1.1 Sing: "*x equals negative b, plus or minus square root of b squared minus 4ac, all over 2a.*" Make students sing along, or teach students to memorize the quadratic formula using this phrase: "A ***negative b***oy could not decide ***whether or not*** to go to a ***radical*** party. He decided to ***b***e ***square*** and he ***missed*** out on ***4 a***wesome *c*hicks. The party was ***all over*** at ***2 a***m."

3.1.2 Tell students: "The quadratic formula is a "catchall" for solving quadratic equations, it works every time"

3.1.3 Tell students: "When in doubt while solving quadratic equations, revert back to the quadratic formula, that is the reason for singing the song every day, multiple times during the period while using the quadratic formula"

3.2 Write the quadratic equation in standard form, $ax^2 + bx + c = 0$ before using the quadratic formula

3.2.1 IF the equation is not in standard form, THEN the equation may be misleading because either the value of *a*, *b*, or *c* may not be correct

3.3 Show students how to write a quadratic equation in standard form

3.3.1 Write an example on the board that has the *x*-term on the other side, $2x^2 + 5 = 3x$

3.3.2 Show students that the standard form would be $2x^2 - 3x + 5 = 0$ or $0 = -2x^2 + 3x - 5$

3.3.3 Choose which side of the equal sign to take all the terms to get the equation to standard form and pay attention to the sign change

3.3.4 IF there are terms on both sides of the equal sign, THEN the signs will be different when all the terms a collected on the same side

3.4 Tell students: 'Make three-columns in your notebook' see Table 1

Table E1: Three-column table

Essential Question (EQ): How is the quadratic formula used to solve a quadratic equation?

		Steps
Solve: $2x^2 + 3x - 5 = 0$	$2x^2 + 3x - 5 = 0$	Step 1: Standard form $ax^2 + bx + c = 0$
	$a = 2$ (opening-up), $b = 3$; $c = -5$ (y-intercept)	Step 2: Label a, b, and c
	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Step 3: Write quadratic formula
	$x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$	Step 4: Substitute values of a, b, and c into the quadratic formula
	$x = \dfrac{3 \pm \sqrt{9 + 40}}{4}$	Step 5: PEMM (4 steps in one)
	$x = \dfrac{3 \pm \sqrt{49}}{4}$	Step 6: Add and simplify the discriminant
	$x = \dfrac{3 \pm 7}{4}$	Step 7: Take the square root the discriminant
	$x = 10/4$ or $x = -4/4$	Step 8: Simplify and write: zeros, roots, solutions or x-intercept

3.4.1 Write the essential question (EQ): "How is the quadratic formula used to solve a quadratic equation" on the left-hand side

3.4.2 Write the steps of using the quadratic formula on the right-hand side [see Table E1)

3.4.3 Do the steps along with the students

3.4.3.1 **Step 1**: Write: Standard form

3.4.3.2 Write: $ax^2 + bx + c = 0$ or $ax^2 + bx + c = y$ underneath "Standard form"

3.4.3.3 **Step 2**: Label *a*, *b*, and *c*. To the left of *a*, write "opening"

3.4.3.3.1 Reason: So students know when they graph it, *a* is going tell them the direction the graph will open, either "up" or "down"

3.4.3.4 IF *a* equals a *negative* number, THEN the graph opens *down*

3.4.3.5 IF *a* equals a *positive* number, THEN graph opens *up*

3.4.3.6 Write *y*-intercept next to *c*

3.4.3.6.1 Reason: So that students know this is not in isolation

3.4.3.7 **Step 3**: Write the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, as you sing along with students the quadratic formula song: "*x equals negative b, plus or minus square root of b squared minus 4ac, all over 2a*"

3.4.3.7.1 Do not substitute the values of *a*, *b*, and *c* into the formula before writing the formula out

3.4.3.7.2 Write $x = \frac{-() \pm \sqrt{()^2 - 4()()}}{2()}$ below the quadratic formula. Sing the remix together with students: "*x equals negative parenthesis plus or minus square root of parenthesis squared minus four parenthesis parenthesis all over two parenthesis*" as you write this:

$$x = \frac{-(\ ) \pm \sqrt{(\ )^2 - 4(\ )(\ )}}{2(\ )}$$

3.4.3.7.3 Get students in the habit of writing the formula every time

3.4.3.7.4 IF students substitute directly without writing the formula first, THEN they do not get points for it

3.4.3.7.4.1 Reason: Because students must use parenthesis when substituting into the formula

3.4.3.8 Tell students: "To substitute is to replace"

3.4.4 Show students how to do the four steps in one (see Table E1)

3.4.4.1 Do order of operations, do PEMM

3.4.4.2 Substitute the value of *b* into the first **P***arenthesis*, negative ( )

3.4.4.3 Substitute the value of *b* again into the **E***xponent* part $(\ )^2$ and circle the *b*-squared part

3.4.4.4 Substitute the values of *a* and *c* into third and fourth parenthesis respectively, *negative* 4( )( ) and circle -4ac followed by putting **M** over it to indicate **M***ultiplication* will take place

3.4.4.5 Circle the denominator, 2*a*, substitute the value of *a* into the parenthesis 2( ) and put **M** over it to indicate **M***ultiplication* will take place

3.4.4.6 Circle what is under the square root sign

3.4.4.7 IF you are going to add the numbers b^2 and 4*ac*, THEN put **A** for addition above it and add the quantities

3.4.4.8 IF you are going to subtract the quantities b^2 and 4*ac*, THEN put **S** for subtraction above it and subtract the quantities

3.4.4.9 Write **D** for **D***ivision* but wait on D

3.4.4.10 Take the square root of the quantity $b^2 - 4ac$

3.4.4.11 **D**ivide the numerator by the denominator and then write *x* equals the result of the division

3.4.4.11.1 Reason: Because we are going through PEMMDAS, the order of operations

3.4.4.12 Circle using a red marker on the board and write the step being done

3.4.4.13 Draw a box around *x* equals, see below

$x =$

3.4.4.14 IF x has two solutions, THEN the graph intercepts the x-axis twice
3.4.4.15 IF x has only one solution, THEN the graph touches the x-axis once
3.4.4.16 IF x has no solution, THEN the graph does not touch the x-axis

3.4.5 Show an example: $2x^2 - 3x - 5 = 0$

3.4.6 Substitute the values of a, b, and c from the standard quadratic equation, $ax^2 + bx + c = 0$ into the formula with the parenthesis in place of a, b, and c:

$$x = \frac{-(\ ) \pm \sqrt{(\ )^2 - 4(\ )(\ )}}{2(\ )}$$

3.4.6.1 Substitute the value of b from the standard quadratic equation, $2x^2 - 3x - 5 = 0$ into the first and second parenthesis,

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(\ )(\ )}}{2(\ )}$$ the negative sign is still outside the parenthesis

3.4.6.2 IF b was negative, THEN substitute it together with its sign
3.4.6.3 Tell students: "Pay attention to the $(\ )^2$"
3.4.6.3.1 IF b is a *negative* number, THEN *negative* times *negative* is *positive*.
3.4.6.3.2 IF b is a positive number, THEN positive times positive is positive
3.4.6.4 IF any number is squared, THEN the product is always positive (P1)
3.4.6.5 Substitute values of a, the c and a again into: $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$
3.4.6.6 Simplify: $x = \frac{3 \pm \sqrt{9+40}}{4}$ = $x = \frac{3 \pm \sqrt{49}}{4}$ which becomes $x = \frac{3 \pm 7}{4}$
3.4.6.7 Tell students: "The plus or minus 7 means we have two solutions for this quadratic equation
3.4.6.8 Solution: $x = 10/4$ or $x = -4/4$ which are simplified to $x = 2.5$ or -1. These are also called roots, x-intercepts or zeros of the quadratic equation

3.4.7 See Table 1 (step 3.4)
3.4.7.1 Tell students: "$b^2 - 4ac$ is the *discriminant* and it helps determine the number of solutions of a quadratic equation"
3.4.7.2 IF the *discriminant* is positive, THEN the parabola intercepts the x-axis twice
3.4.7.2.1 Draw the graph to show students the parabola intercepts the x-axis twice
3.4.7.3 IF the *discriminant* is zero, THEN the parabola touches the x-axis once and turns around
3.4.7.3.1 Draw the graph to show students the parabola touches the x-axis once
3.4.7.4 IF the *discriminant* is negative, THEN the parabola does not touch the x-axis
3.4.7.4.1 Draw the graph to show students the parabola does not intersect the x-axis
3.4.7.5 IF students get the solution, THEN they have to write all the names every time: x-intercept(s), solution(s), zero(s) and root(s) of the quadratic equation
3.4.7.6 Tell students: "Write *x-intercepts*, *solutions*, *zeros* and *roots* under the answer on every quadratic equation problem you solve"

3.5 Show students $b^2 - 4ac$ is the discriminant and is part of the quadratic formula

3.5.1 Point at the discriminant: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ← Discriminant: 2 roots, 1 root or no root

3.6 Use the quadratic formula to solve any quadratic equation for its roots, its solution, and its zeros

3.7 IF students are able to solve a quadratic equation for its roots, solutions, or its zeros, THEN they can solve real-life application problems like vertical motion problems (Appendix D)

3.8 IF students are not comfortable using the quadratic formula, THEN show them two more examples and give them a few problems to practice (guided practice)

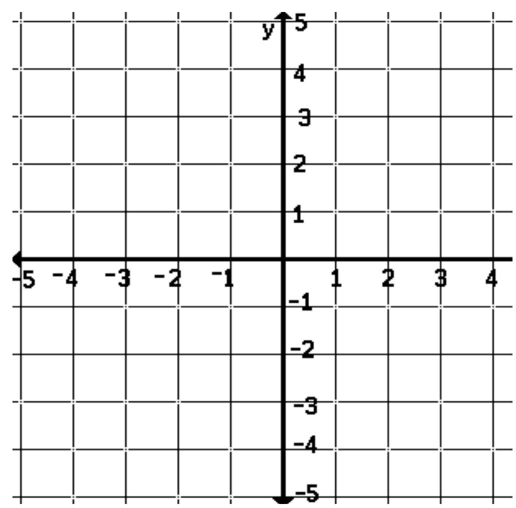
3.9 Give students 3-5 problems to solve using the quadratic formula.

3.9.1 IF students have mastered solving using the quadratic formula, THEN begin teaching graphing. IF not, THEN reteach.

3.9.2 IF students are proficient using the quadratic formula, THEN proceed with the lesson to show students the next procedure for solving quadratic equations

Procedure 4: Teach solving quadratic equations by graphing

4.1 Post an x-y coordinate plane on the whiteboard throughout the unit of quadratic equations (see below)



4.2 Solve all quadratic equations next to the graph so that students make connections and also see multiple representations

4.3 Solve while relating back to the graph because students have a hard time connecting different representations

4.4 Do not teach graphing of quadratic equations and solving quadratic equations using other procedures in isolation

4.4.1 Go back and forth between various methods of solving quadratic equations and their graphs

4.4.2 Relate the x-intercepts of the graph of a quadratic equation to its solutions after solving using any of the other procedures

4.5 Start with a quadratic equation in standard form $ax^2 + bx + c = y$ and reflects on the y-axis

4.6 Identify the parts that are obvious based on the equation

- 4.6.1 Identify a the coefficient of x^2, b the coefficient of x and c, the constant
- 4.6.2 IF a is positive, THEN the graph (parabola) will open up
- 4.6.3 IF a is negative, THEN the graph (parabola) will open down
- 4.6.4 Write c, the y-intercept

4.7 Teach how to find the axis of symmetry

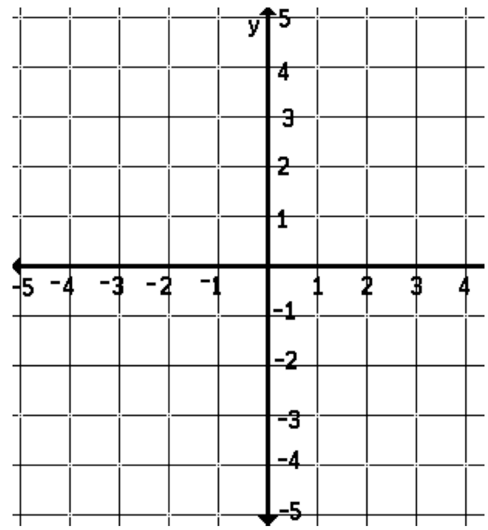
- 4.7.1 Find axis of symmetry and relate to the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- 4.7.2 Break apart the quadratic equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and show students that $x = \frac{-b}{2a}$ is the axis of symmetry
- 4.7.3 Write $x = \frac{-b}{2a}$ and carefully explain to students that this does not represent x-intercepts
- 4.7.4 Tell students: "This is an x-value and it is where on the x-axis the axis of symmetry cuts through"
- 4.7.5 IF the quadratic equation has a middle term bx, THEN the parabola will not reflect over the y-axis because $x = \frac{-b}{2a} \neq 0$. The axis of symmetry will not be on the y-axis
- 4.7.6 Substitute the values of a and b into the axis of symmetry equation: $x = \frac{-b}{2a}$ to find the axis of symmetry
- 4.7.7 IF you fold a parabola in half through the axis of symmetry, THEN there are two identical parts
- 4.7.8 IF there is a y-value to the left of the axis of symmetry, THEN there is an equivalent y-value same distance from the axis of symmetry on the right of the axis of symmetry
- 4.7.9 IF you have the y-intercept on one side of the axis of symmetry, THEN there is another point at the same height on the other side of the parabola

4.8 Teach students how to find the vertex

- 4.8.1 Use the axis of symmetry, $x = \frac{-b}{2a}$ to find the x-value of the vertex
- 4.8.2 Substitute the x-value of the vertex into $ax^2 + bx + c = y$ to find the y-value of the vertex
- 4.8.3 Write the vertex in the form of (x, y) coordinate point
- 4.8.4 Point to students that the vertex is the highest or lowest point of the parabola

4.9 Teach Graphing procedure

- 4.9.1 Draw an x-y coordinate plane



4.9.2 Draw a dotted line through the axis of symmetry found in **step 4.7.6**
4.9.3 Draw an *x*-*y* table of values. An example is here below when the *x*-value of the vertex is 0

x	-2	-1	0	1	2
y					

4.9.4 Put the vertex coordinates at the center of the table
4.9.5 Choose an *x*-value to the left or to the right of the vertex
4.9.6 Show students the mirror point(s)
4.9.7 Plot the vertex as found in step 4.8.3
4.9.8 IF you find the vertex, THEN get 2 or 3 points on one side of the axis of symmetry
4.9.9 IF you have 2 or 3 points on one side of the axis of symmetry, THEN you will get 2 or 3 points on the opposite side of the axis of symmetry
4.9.10 IF you have the axis of symmetry (*x*-value), THEN choose two *x*-values to the left or right of the axis of symmetry to substitute into the original equation to find the corresponding *y*-values
4.9.11 IF the parabola opens up, THEN the vertex is a minimum
4.9.12 IF the parabola opens down, THEN the vertex is a maximum
4.9.13 Tell students: “The vertex is where our parabola opens from”
4.9.14 Remind students the graph opens upwards or opens downwards depending on the *a*-value

4.10 Connect the points to plot the graph with a smooth curve

4.11 Label on the graph the vertex, *y*-intercept, axis of symmetry and direction of opening, up or down

4.12 Label the *x*-intercepts if they exist

4.13 Show an example, $x^2 + 4x - 12 = 0$

4.13.1 Draw a two-column *t*-table of values (step 4.9.3)

4.13.2 Identify the coefficients: $a = 1$, $b = 4$, and $c = -12$

4.13.3 Find the axis of symmetry, $x = \frac{-b}{2a} = \frac{-(4)}{2(1)} = \frac{-4}{2} = -2$

4.13.4 Put the *axis of symmetry*, $x = -2$ in the middle of the *t*-table and then choose integers on either side of -2 that are equidistant from the axis of symmetry

4.13.5 Draw the axis of symmetry $x = -2$

4.13.6 Choose two *x*-values less than -2 and two *x*-values greater than -2: -4, -3, **-2**, -1, and 0 (step 3.1.5)

4.13.7 Substitute these *x*-values into the quadratic equation, $x^2 + 4x - 12 = 0$ to find the corresponding *y*-values (step 4.8.2)

4.13.8 Plot the pairs of points on *x*-*y* coordinate plane

4.13.9 Ask aloud: "How many times does the graph of $x^2 + 4x - 12 = 0$ cross the *x*-axis?"

4.13.10 Show students the *x*-intercepts, which are also the solutions of the quadratic equation

4.14 IF the graph of a quadratic equation intercepts the *x*-axis twice, THEN the quadratic equation has two real solutions

4.15 IF the graph of a quadratic equation touches the *x*-axis once, THEN the quadratic equation has one real solution

4.16 IF the graph of a quadratic equation does not touch the *x*-axis, THEN the quadratic equation has no real solution

4.17 Check for understanding by giving students two quadratic equations to graph

4.18 IF students are not proficient with graphing, THEN show them the process with two more examples

4.19 IF students are proficient with graphing quadratic equations, THEN proceed to the next procedure for solving quadratic equations

4.20 Give students a 5 question graphing assessment

4.21 IF students have mastered graphing, THEN move on to completing the Square. IF NOT, then reteach.

Procedure 5: Teach solving quadratic equations by Completing the square

5.1 Complete the square of a quadratic equation with a leading coefficient of 1

5.1.1 Divide your notepaper or notebook into three columns

5.1.2 Label the steps on the right hand side of your notepaper or notebook

5.1.3 Do your work in the middle of your paper

5.1.4 Write the essential question on the left hand side of your paper: Essential Question, "How is completing the square used to solve a quadratic equation?"

5.2 Write steps for completing the square

5.2.1 **Step 1** – If the equation is not in standard form, THEN re-arrange the terms in standard form

5.2.1.1 Reason: Because it gives students consistency and therefore write the equation in standard form

5.2.2 IF the equation can be factored at this point, THEN tell students to solve by factorization

5.2.3 **Step 2** – Pull the constant

5.2.3.1 Isolate the constant on the opposite side

5.2.3.2 IF the constant is already isolated, THEN skip **step 2**

5.2.4 Work on either side of the equal sign

5.2.4.1 Reason: Because students should feel constrained to have everything on the left

5.2.5 Sing: "*half of b squared, add it to both sides*" (while drumming)

5.2.6 Take $\left(\frac{b}{2}\right)^2$ and add it to both sides

5.2.7 Sing to students again: "*half of b squared, add it to both sides*" (while drumming)

5.2.7.1 Tell students: "Let us sing, "*half of b squared, add it to both sides*" (while drumming)

5.2.7.2 Sing together: "*half of b squared, add it to both sides*" (while drumming)

5.2.7.3 IF teacher sings, THEN teacher asks students to sing with her

5.2.7.4 IF students sing, THEN show them how to do it

5.2.8 Tell students: "We are taking half of *b*"

5.2.8.1 Show students what half of something means, say half of $4 is $2, half of $12 is $6

5.2.8.2 Practice with students: half of 6, half of 10 …

5.2.8.3 Check for understanding with students writing the answers on their individual whiteboards and lifting them up to show the teacher

5.3 Start with an expression with a coefficient of 1 for x^2, $x^2 + bx$ to complete the square

5.3.1 Give an example $x^2 + 6x$, start with an even *b*-term

5.3.2 Complete the square by dividing 6 by 2 to get 3 and then square 3 to get 9: $x^2 + 6x + 9 = (x + 3)^2$

5.3.3 Show how to complete the square using algebra tiles

5.3.4 Explain to students that $x^2 + 6x \neq x^2 + 6x + 9$ but just showing the process of completing the square

5.3.5 Show students more completing the square: $x^2 + 4x$, to complete the square, add the square of $\frac{4}{2}$, which is $2^2 = 4$ and the expression becomes $x^2 + 4x + \mathbf{4}$**.** Therefore add 4 squares to complete the square

5.3.6 Show students another example: $x^2 + 8x$, to complete the square, add the square of $\frac{8}{2}$, which is $4^2 = 16$ and the expression becomes $x^2 + 8x + \mathbf{16}$**.** Therefore add 16 squares to complete the square.

5.4 Introduce students to a quadratic equation to solve using by the completing the square procedure

5.5 Show students how to write the quadratic equation in the form $x^2 + bx = c$

5.5.1 Give students an example, like $x^2 + 10x = 0$
5.5.2 Teacher says: "*b* = 10, take half of 10"
5.5.3 Teacher says: "IF I say half of *b*, THEN you say the answer"
5.5.4 Teacher says: "IF I say half of 10, THEN you say 5!"
5.5.5 Teacher says: IF I say 5 squared, THEN you say 25!"
5.5.6 Teacher says: "IF I say add it to both sides, THEN you add it to both sides"
5.5.7 Add 25 to both sides of the equation: $x^2 + 10x + 25 = 25$
5.5.8 Remind students the song: "*half of b squared, add it to both sides*"
5.5.9 Tell students: "We squared it, so the title of completing the square. We are making it squared so that we can write it as a quantity squared"
5.5.10 Tell students: "*x* was squared to get *x*-squared and 5 was squared to get 25"
5.5.11 Take square root to undo squares
5.5.12 Factorize the left hand side: factors of x^2 are *x* and *x* and factors of 25 are 5 and 5. So,

$$\begin{aligned} x^2 + 10x + 25 &= 25 \\ (x+5)^2 &= 25 \end{aligned}$$

5.5.13 IF you take the square root of one side, THEN you must take the square root of the other side
5.5.14 Take the square root of both sides:

$$\begin{aligned} \sqrt{(x+5)^2} &= \pm\sqrt{25} \\ x+5 &= \pm 5 \end{aligned}$$

5.5.15 Solve for *x* from $x + 5 = \pm 5$

5.5.15.1 Tell students: "Let's look at our essential question: How do we use completing the square to solve quadratic equations?"

5.5.16 Circle *x*: $x + 5 = \pm 5$
5.5.17 Isolate *x* by itself
5.5.18 Subtract 5 from both sides:

$$\begin{aligned} x + 5 - 5 &= \pm 5 - 5 \\ x &= \pm 5 - 5 \\ x &= +5 - 5 \; or \; -5 - 5 \\ x &= 0 \; or \; -10 \end{aligned}$$

5.5.19 Box the answer: *x* = 0 or -10 and write solutions, roots, *x*-intercepts and zeros of the quadratic equation

5.6 Show students another example following the steps shown on step 5.5
5.7 Check for understanding by giving students one problem at time to do on their whiteboards in pairs
5.8 IF students are not proficient solving quadratic equations with a coefficient of 1 for x^2 by completing the square procedure, THEN show them one more example.
5.9 IF students are proficient solving quadratic equations with a coefficient of 1 for x^2 by completing the square procedure, THEN introduce an equation with an a-value greater than 1
5.10 Introduce an equation with an *a*-value greater than 1

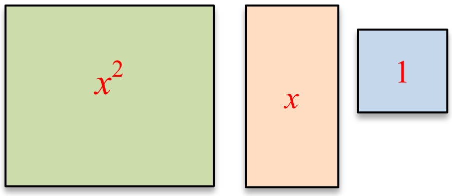
5.10.1 **Step 1** – IF the equation is not in standard form, THEN re-arrange the terms in standard form [see step 5.2]
5.10.2 IF the *a*-value is not equal to one, THEN divide both sides by *a*
5.10.3 Tell students: "It is going to be a challenge because you may start dealing with a *b*-value that is a fraction or an odd number"
5.10.4 Repeat steps 5.2.1 through 5.2.7

5.11 Teach students easy ways to remember the steps. Singing seems to work all the time

5.12 Teach students how to use algebra tiles to complete the square – see Figure E7

Figure E7: Completing the square

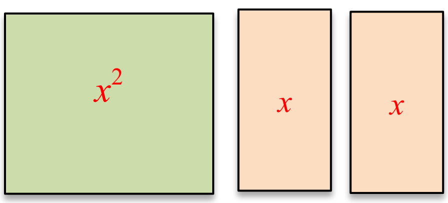
Completing the square using Algebra tiles



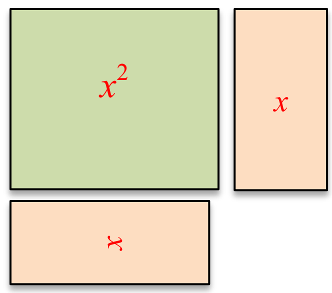
Students use tiles to model:

Find the missing tile: $x^2 + 2x + C$

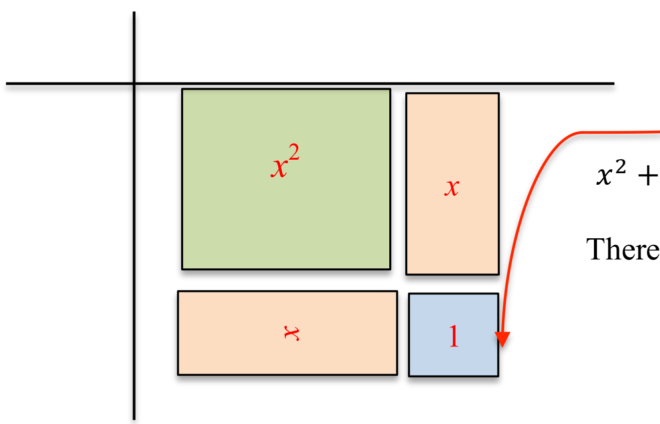
Find tiles



Arrange the tiles to form a square



Determine tiles needed to complete the square



Missing tile

$$x^2 + 2x + 1 = (x + 1)(x + 1)$$
$$= (x + 1)^2$$

Therefore, C = 1

Students explore patterns given the following table:

$x^2 + 2x + C$	$(x + 1)(x + 1)$	$(x + 1)^2 : C = 1$
$x^2 + 4x + C$	$(x + 2)(x + 2)$	$(x + 2)^2 : C = 4$
$x^2 + 6x + C$	?	?

After students have progressed through enough examples with algebra tiles, they should be able to verbalize that the pattern developing is taking the b-term and dividing it by 2, then squaring that quantity to get the missing number C to complete the square. Once students understand $C = \left(\frac{b}{2}\right)^2$, then introduce other equations such as $x^2 - 8x = 9$ and complete the square to solve for x.

5.13 IF it is about solving quadratic equations, THEN the quadratic formula is the fallback method, it works for every quadratic equation

5.14 Give students 3 problems to solve by completing the square, and 1 problem that is already solved but solution steps are out of order and students must order the steps correctly, and the proof of the quadratic formula by completing the square with steps out of order where students must correctly order the steps of the proof.

5.15 IF students have mastered completing the square, THEN go on to solving quadratics using square roots. IF NOT, THEN reteach.

Procedure 6: Solving quadratics using the square root method

6.1 IF the equation is of the form $ax^2 = c \text{ or } ax^2 - c = 0$, THEN the square root method is appropriate

6.2 Show students how to solve $x^2 = 9$

6.3 Take the square root of both sides, $\sqrt{x^2} = \pm\sqrt{9}$, which gives $x = \pm 3$.

6.4 Write $x = -3$ or 3 which are the solutions, x-intercepts, zeros and also the roots of the quadratic equation $x^2 = 9$.

6.5 Show another example without a perfect square, solve $x^2 - 8 = 0$

6.5.01 Add 8 to both sides, to isolate x^2: $x^2 - 8 + 8 = 0 + 8$

6.5.02 Simplify: $x^2 = 8$

6.5.03 Take the square root of both sides:

$$\sqrt{x^2} = \pm\sqrt{8}$$
$$x = \pm\sqrt{4 \bullet 2}$$
$$x = \pm\sqrt{4}\sqrt{2}$$
$$x = \pm 2\sqrt{2}$$

6.5.04 Write $x = 2\sqrt{2} \text{ or } -2\sqrt{2}$ and these are the roots, zeros, solutions or x-intercepts of this quadratic equation $x^2 - 8 = 0$.

6.6 Show another example: $x^2 - 16 = 0$

6.6.01 Use factorization and find the sum and difference of products

6.6.01.1 Use zero product property, IF AB = 0, THEN A = 0 or B = 0

6.6.01.2 IF $x^2 - 16 = 0$, THEN $(x - 4)(x + 4) = 0$

6.6.01.3 Solve: IF $(x - 4)(x + 4) = 0$, THEN $x - 4 = 0$ or $x + 4 = 0$

6.6.01.4 Simplify: IF $x - 4 = 0$ or $x + 4 = 0$, THEN $x = 4$ or $x = -4$

6.6.02 Use square roots to solve: $x^2 - 16 = 0$

6.6.02.1 Add 16 to both sides of the equal sign

6.6.02.2 Solve $x^2 = 16$

6.6.02.3 Take the square root of both sides

6.6.02.4 Write $x = \pm 4$, which is the same solution.

6.7 Remind students: "Those problems that they did in factoring are very similar to our square roots problems"

6.8 IF a quadratic equation is missing the *b*-term (middle term), THEN solve using the square root procedure

6.9 IF students are not proficient using the square root procedure, THEN show them two more examples
6.10 Check for understanding with students writing on their whiteboards and showing the teacher their solution
6.11 IF students are proficient with solving quadratic equations with the square root procedure, THEN proceed to the application of all these procedures that students have been learning
6.12 Give students a 5 question assessment (open ended quadratic equation problems to solve)
6.13 IF students have mastered completing the square, THEN go to application. IF NOT, reteach.

Procedure 7: Teach Application of these methods of solving quadratic equations to solve real-life problems

7.1 Apply knowledge of solving quadratic equation to real-life problems
7.2 IF students have been taught the skill base for solving quadratic equations, THEN spend two to three days where students practice in class real-life problems
7.3 Write (Project) a practice problem on the board
7.4 Choose the best procedure for solving the quadratic problem
7.5 Use the three-column table on the whiteboard as a graphic organizer
7.6 Make notes on it on all discussions as you solve the problem
7.7 Give students a couple of minutes to solve the problem
7.7.1 Ask students aloud: “Which way did you solve it?”
7.7.2 Go back to the three-column table and make notes
7.8 Ask students: “Was is it most convenient to use factoring?” “Why?”
7.9 Ask students: “Can I use square roots?” “Why?” (DOK 3)
7.9.1 IF a student solved it really quickly, THEN ask, “What did you do? Which method did you use?”
7.9.1.1 IF a student took a little bit longer, THEN ask, “What did you do? Which method did you use?”
7.9.1.2 Go back to the graphic organizer and edit it with specifics of what you did
7.10 IF the quadratic equation does not have a *b*-term, THEN use the square roots method”
7.10.1 Reason: Because it is the easiest. It is the quickest
7.11 IF the coefficient of x^2 is greater than 1 or not factorable, THEN use the quadratic formula
7.11.1 Reason: Because you do not have to list all possible different factors and then finally find the problem is not factorable while the quadratic formula always works
7.12 Take about two to three days doing this with students. Practicing to build their confidence
7.13 Assess students on their skills
7.13.1 Assess students on solving quadratic equations using a method/procedure of their choice.
7.13.2 Give students two problems, where they have to use a specific method to solve

7.14 IF students can solve basic quadratic equations, THEN introduce practical real-life problems.
7.15 IF students can solve a quadratic equation, THEN ask them to do different things, like find the width of a parabolic disc
7.16 Give students problems that involve projectiles
7.17 IF a rocket is launched, THEN what would be the maximum height, or how long will it take to reach the ground?
  7.17.1 Make connection: maximum height of rocket corresponds to vertex of a parabola; time rocket takes to reach the ground is the difference between the *x*-intercepts of the parabola. These are practical applications of solving quadratic equations
7.18 Demonstrate drawing a picture that shows the parabolic path of the projectile
  7.18.1 IF given an application problem (word problem), THEN draw a picture to represent the story
  7.18.2 Draw a picture back to the graph
  7.18.3 Tell students: "This is the skill you learned to do, when we were graphing"
7.19 Draw a picture always
7.20 Relate back to previous lessons because quadratic equations are not an isolated unit
7.21 Make students understand why they are solving quadratic equations
  7.21.1 Make connections for students! For example: When solving a problem involving the jumping path of a kangaroo, impose a graph that shows the horizontal time and vertical distance by labeling the x- and y-axis, Draw in the axis of symmetry and vertex reminding students that the kangaroo reached a specific height (y) after so many seconds (x). The students can see on the graph that the maximum height reached by the kangaroo is at the vertex. Talk about the height the kangaroo started at (was there a y-intercept other than 0?) and then where he ended up.
7.22 Remind students *a*, *b*, and *c* in the quadratic equation, always relates back to the graph or real-life application problem
7.23 Ask students aloud: "What does it mean to solve for *x*?"
  7.23.1 Ask students: "What is the significance of finding *x* on the graph? What does it mean?"
  7.23.2 Know solving for *x* is finding solutions, the roots, the zeros and they are also the *x*-intercepts of the graph of the quadratic equation
7.24 Give students a UNIT assessment.

Appendix F

Data Analysis Spreadsheet
Gold Standard Protocol Procedures: Action and Decision Steps

	Type		SME				Steps		Alignment
		Final Gold Standard Protocol Data Analysis	A(P3)	B(P4)	C(P1)	D(P2)	A	D	
		Procedure 1. Review linear equation to Activate prior knowledge					**34**	**8**	
1	A	1.1 Give students an overview of the unit of solving quadratic equations. (P1)	0	0	1	0			1
2	A	1.1.1 Tell students: "I'm going to teach you how to solve quadratic equations" (P1)	0	0	1	0			1
3	A	1.1.2 Tell students: "There are multiple methods for solving quadratic equations: factorization, graphing, the quadratic formula, the square root method, and completing the square (P1)	0	0	1	0			1
4	A	1.1.3 Tell students: "Some of these methods will work better for some of the quadratic equations. Some quadratic equations are perfect square binomials and therefore will be easy to recognize that it can be factored and solved, while other quadratic equations may be missing a "b" value and so it will be useful to use the square root method. Other quadratic equations may be factorable but have many factors to try and so quadratic formula will be quickest". (P1)	0	0	1	0			1
5	A	1.2 Draw a full page size Tree map (P1)	0	0	1	0			1
6	A	1.2.1 Relate solving quadratic equations to solving linear equations (P1)	0	0	1	0			1
7	A	1.2.1.1 Reason: Get students to see there are many ways to solve a quadratic equation by always referring back to the tree map and these procedures are related to solving linear	0	0	1	0			1

		equations (P1)							
8	A	1.3 Put linear patterns and linear equations of the form $y = mx + b$ on the screen over the computer or write them on the board (P2, P3, P4)	1	1	0	1			3
9	A	1.3.1 Show students how to solve linear equations step-by-step (P2)	0	0	0	1			1
10	A	1.3.2 Draw a two-column table on the board (P3, P4)	1	1	0	0			2
11	A	1.3.3 Show students how to generate a table of values from patterns or linear equations (see Appendix E – P3) (P3)	1	0	0	0			1
12	A	1.3.4 Plot the points on an x-y coordinate plane drawn on the white board (P4)	0	1	0	0			1
13	A	1.3.5 Play around with the linear equations by changing the slope (m) and/or changing the y-intercept (b) (P4)	0	1	0	0			1
14	D	1.3.6 IF students have access to graphing calculators or graphing software, THEN give opportunities to check the graph they have made the linear equations from the table of values (P3)	1	0	0	0			1
15	D	1.3.7 IF you show students a problem or two, THEN give them a few to try individually to check for understanding (P2)	0	0	0	1			1
16	D	1.3.8 IF students are not proficient in solving linear equations, THEN reteach the concept (P2)	0	0	0	1			1
17	A	1.3.9 Ask students randomly to come to the board to show that they have successfully completed the problem (P2)	0	0	0	1			1
18	A	1.3.10 Assess students by walking around the room to get a visual of what they are doing and that they are communicating using math language in their groups (P2)	0	0	0	1			1
19	D	1.3.11 IF you see that students are showing that they can do it on the board and you are walking around the room and making sure that students are understanding from what you can see,	0	0	0	1			1

		THEN continue to progress with the lesson of activating prior knowledge (P2)							
20	A	1.4 Review with students how to multiply and divide rational numbers (P1)	0	0	1	0			1
21	A	1.4.1 Give examples: 3*4 = 12; -2*5 = -10; 2*-5 = -10; and (-2)*(-3) = 6 (P1)	0	0	1	0			1
		1.4.1.1 Reason: The intention is to remind students about the rules for multiplying integers since factoring quadratics assumes students can factor constant values (P1)							
22	A	1.4.2 Define a factor to students: "Factors are numbers you can multiply together to get another number" (P1).	0	0	1	0			1
23	A	1.5 Review with students how to factor a whole number (P1)	0	0	1	0			1
24	A	1.5.1 Use factor trees (P1)	0	0	1	0			1
25	A	1.5.2 Show students the factors of 6: 1 and 6, and 2 and 3 are factors because the product of each pair is 6 (P1).	0	0	1	0			1
26	A	1.5.3 Give students a number to factorize, for example: factorize 18 (P1)	0	0	1	0			1
27	A	1.5.3.1 Show students the factors of 18 are: 1 and 18, 2 and 9, 3 and 6, -1 and -18, -2 and -9, and -3 and -6 (P1)	0	0	1	0			1
28	A	1.5.3.2 Show students that the product of these factors is 18 (P1)	0	0	1	0			1
29	A	1.5.4 Remind students the rules for multiplying integers (P1)	0	0	1	0			1
30	A	1.5.4.1 Multiply a positive number by a positive number the product is another positive number; multiply a negative number by another negative number the product is positive while the product of a positive number by a negative number is a negative number (P1)	0	0	1	0			1
31	D	1.5.5 IF you are factorizing a positive number, THEN get two positive factors or two negative factors (P1)	0	0	1	0			1

32	D	1.5.6 IF you are factorizing a negative number, THEN get one positive factor and one negative factor. Be sure that the sign of the greater factor matches the sign of the middle term (P1)	0	0	1	0			1
33	A	1.6 Review with students how to multiply polynomials (P1)	0	0	1	0			1
34	A	1.6.1 Teach students exponent rules (P1)	0	0	1	0			1
35	A	1.6.1.1 Remind students, for example that x times x equals x^2 (P1)	0	0	1	0			1
36	A	1.6.1.2 Factorize x^3 or x^2 into factors: $x^3 = (x)(x)(x)$ and $x^2 = (x)(x)$ (P1)	0	0	1	0			1
37	A	1.6.1.3 Show students that $x + x = 2x$ and $(x)(x) = x^2$ on the whiteboard (P1)	0	0	1	0			1
		1.6.1.3.1 Reason: To know the difference between addition and multiplication when factorizing (breaking down) polynomials into factors (P1)							
38	A	1.7 Review with students how to factor linear expressions (P1)	0	0	1	0			1
39	A	1.7 Review with students how to factor linear expressions (P1)	0	0	1	0			1
40	A	1.7.2 Give students another linear expression to practice factorizing (P1)	0	0	1	0			1
41	D	1.7.3 IF students are not proficient factoring linear expressions, THEN show more examples like in step 1.7.1 (P1)	0	0	1	0			1
42	D	1.7.4 IF students are proficient factoring linear expressions, THEN introduce solving quadratic equations by factoring (P1)	0	0	1	0			1
		Procedure 2: Teach solving quadratic equations by factoring					**73**	**17**	
43	A	2.1 Remind students what a "factor" is (P1)	0	0	1	0			1
44	A	2.1.1 Define a factor to students again (line 1.4.2): "Factors are numbers you can multiply together to get another number" (P1)	0	0	1	0			1
45	A	2.1.2 Explain (step 1.5) what you will do when you factor a certain	1	0	1	0			1

		problem. When "factoring" we are showing students another way to write a product—as a multiplication problem. Sometimes the factored form will look like an expanded version of the original problem. (P1, P3)							
46	A	2.1.3 Give students a few factor tree problems to practice, for example: find the factors 24 (P1)	0	0	1	0			1
47	A	2.1.3.1 Factor 24: 24 = (2)(12) or (3)(8) or (4)(6). At this point, label (2)(12) as the "factored form" of 24 (P1)	0	0	1	0			1
48	A	2.1.3.2 Give students a number to factor individually (P1)	0	0	1	0			1
49	D	2.1.3.3 IF some students are not proficient in multiplication, then assign multiplication flash cards for homework practice, and encourage those students to use multiplication charts when factorizing (P1)	0	0	1	0			1
50	D	2.1.3.4 IF students are not proficient in factorizing, THEN the teacher does one more as students follow along (P1)	0	0	1	0			1
51	D	2.1.3.5 IF students are proficient with factoring numbers, THEN teach students how to write a quadratic expression in standard form, like $x^2 + 5x + 6$ (P1)	0	0	1	0			1
52	A	2.2 Give an example of the standard form: $x^2 + 5x + 6 = 0$ and a non-example: $x^2 + 6 = -5x$. Tell students, " All the terms should be on one side of the equal sign (P1, P2, P4)	0	1	1	1			3
53	A	2.3 Show students coefficients of the quadratic equation. For example, the coefficients of $x^2 + 5x + 6 = 0$ are 1, 5, and 6 (P2)	0	0	0	1			1
54	A	2.4 Teach students how to use distributive property (P2)	0	0	0	1			1
55	A	2.4.1 Write an example of two binomials on the board, for example (x + 2) and (x + 1) (P2)	0	0	0	1			1
56	A	2.4.2 Show students how to use algebra tiles (P2, P3)	1	0	0	1			2
57	A	2.4.3 Find the area of the product of	1	0	0	1			2

		these binomials: $x^2 + 2x + x + 2$ (P2, P3)							
58	A	2.4.4 Write area $x^2 + 3x + 2$ by looking at tiles (P3)	1	0	0	0			1
59	A	2.4.5 Tell students: "Factoring is how we undo the distributive property (product of binomials) for example, getting the binomials that gave the product $3x^2 + 4x – 15$" (P2, P3)	1	0	0	1			2
60	A	2.5 Show students how to factorize $x^2 + 5x + 6$ (P1, P2, P3, P4)	1	1	1	1			4
61	A	2.6 Show students a sum and product table (P1)	0	0	1	0			1
62	A	2.6.1 Show students how to use a sum and product table to find factors (P1)	0	0	1	0			1
63	A	2.6.1.1 Tell students: "Draw a sum and product table on your whiteboard" (P1)	0	0	1	0			1
64	A	2.6.1.2 Tell students: "Two numbers have a sum of 5 and a product of 6. With your partner, figure out which numbers they are" (P1)	0	0	1	0			1
65	A	2.6.1.3 Tell students to do the problem on their whiteboards and hold them up (P1)	0	0	1	0			1
66	A	2.6.1.4 Scan across the room as students raise their whiteboards checking for understanding (P1)	0	0	1	0			1
67	D	2.6.1.5 IF students have not mastered the use of sum and product tables to factorize, THEN show students another example. Like two numbers have a sum of 8 and a product of 15, show them how to find these two numbers (P1).	0	0	1	0			1
68	A	2.2.1.6 Repeat this procedure with different problems until students are proficient (P1)	0	0	1	0			1
69	D	2.2.1.7 IF students have mastered the use of sum and product tables, THEN show students how to factor the original problem $x^2 + 5x + 6$ (P1, P2)	0	0	1	1			2
70	A	2.2.1.8 Circle the term 6 which is the constant in $x^2 + 5x + 6$ and write the word constant above the 6. IF you circle the constant, THEN write the word product underneath it (P1)	0	0	1	0			1

71	A	2.2.1.8.1 Show students some numbers that give a product of 6, write these numbers in the sum and product table. Some pairs are 1 and 6, and 2 and 3 (P1)	0	0	1	0			1
72	A	2.2.1.9 Circle the x-term, 5x and then write sum underneath it (P1)	0	0	1	0			1
73	A	2.2.1.9.1 Use the pairs of numbers in the sum and product table to determine which pair, 1 and 6 or 2 and 3 adds up to 5 (P1)	0	0	1	0			1
74	A	2.2.1.9.2 Choose 2 and 3 (P1)	0	0	1	0			1
75	A	2.2.1.10 Circle the x^2-term. Tell students: " x times x is x^2 and therefore x and x are the factors of x^2" (P1)	0	0	1	0			1
76	A	2.7 Teach students how to use algebra tiles, the X-BOX, and the Parenthesis methods to factorize (P1, P2)	0	0	1	0			1
77	A	2.7.1 Use algebra tiles to factor quadratic equations (P3)	1	0	0	0			1
78	A	2.7.2 Introduce quadratic equations that can be factorized using algebra tiles (P3)	1	0	0	0			1
79	A	2.7.2.1 Ask students: IF I have $x^2 + 3x + 2$, THEN what tiles would I need? (P3)	1	0	0	0			1
80	A	2.7.2.2 Ask students to gather tiles (P3)	1	0	0	0			1
81	A	2.7.2.3 Tell students that $x^2 + 3x + 2$ represents an area (P1, P3)	1	0	1	0			2
82	A	2.7.2.4 Arrange tiles in a rectangle, to find factors by looking at length and width of the rectangle (P3)	1	0	0	0			1
83	A	2.7.2.5 Tell students: "The factors of $x^2 + 3x + 2$ are $(x + 2)$ and $(x + 1)$" (step 2.7.2.4) (P3, P4)	1	1	0	0			2
84	D	2.7.2.6 IF students are not yet comfortable with factoring using algebra tiles, THEN reteach the concept as in step 2.7.2.4 (P3, P4)	1	1	0	0			2
85	D	2.7.3 If students are comfortable with factoring using algebra tiles, then introduce the X-BOX method for factoring so that students may have an alternative way of factorizing. For example, factorize $x^2 + 3x + 2$ using the X-BOX method, see below (P3)	1	0	0	0			1

86	D	2.7.5 IF students are proficient factorizing using the X-BOX method (step 2.7.3), THEN proceed to teach students the Parenthesis Method (P1)	0	0	1	0			1
87	A	2.8 Show students how to factorize a quadratic expression using the Parenthesis Method (P1)	0	0	1	0			1
89	A	2.8.1 Put two sets of parentheses, ( )( ) at the top for the binomials that we are trying to factor this problem into Then, at the bottom of the parentheses, write the problem: (P1)	0	0	1	0			1
90	A	2.8.2 Tell students: "We are going to find the factors of this problem, (P1)	0	0	1	0			1
91	A	2.8.3 Tell students: "Since factors of the first term are x and x, put each on a different parenthesis, since x times x is x^2, followed by 2 in the first parenthesis and 3 on the second parenthesis " (P1, P3)	1	0	1	0			2
92	D	2.8.4 IF we factor, THEN the product of these parts in the parentheses have to match with the original expression, (P1)	0	0	1	0			1
93	A	2.8.5 Give students another problem to factorize (P1)	0	0	1	0			1
94	D	2.8.6 IF students are not proficient with factoring, THEN show students how to do it (P1)	0	0	1	0			1
95	A	2.8.6.1 Use the sum and product table. Two numbers have a sum of 11 and a product of 30. (P1)	0	0	1	0			1
96	A	2.8.6.2 Repeat the same process as in 2.3.1.1 through 2.3.1.4 to factorize (P1)	0	0	1	0			1
97	D	2.8.7 IF students are proficient with factoring expressions, THEN introduce them to solving quadratic equations using the factorization method (P1)	0	0	1	0			1
98	A	2.8.7.1 Show students how to solve by finding the value of x that satisfies this equation. (P1)	0	0	1	0			1
99	A	2.8.7.2 Factorize the left hand side of the equation (P1, P2, P4)	0	1	1	1			3
100	A	2.8.7.3 Put below the original quadratic equation (P1)	0	0	1	0			1
101	A	2.8.7.4 Use the zero product property,	1	1	1	1			4

		IF AB = 0, THEN A = 0 or B = 0 (P1, P2, P3, P4)							
102	D	2.8.7.5 IF , THEN (x + 2) = 0 or (x + 3) = 0 (P1, P2, P4)	0	1	1	1			3
103	A	2.8.7.6 Solve: x + 2 = 0, subtract 2 from both sides of the equation: x + 2 - 2 = 0-2 therefore x = -2. And x + 3 = 0, subtract 3 from both sides of the equation: x + 3 -3 = 0 - 3 therefore x = -3. (P1, P2, P4)	0	1	1	1			3
104	A	2.8.7.7 Solution is x = -2 or -3. Solution of a quadratic equation is also called the x-intercepts, zeros, and roots. (P1)	0	0	1	0			1
105	A	2.8.8 Give students another example to practice: (P1)	0	0	1	0			1
106	A	2.8.9 Use the sum and product table to factorize the left hand side of the quadratic equation (P1)	0	0	1	0			1
107	A	2.8.9.1 Factorize to (x +5)(x + 6) = 0 (P1)	0	0	1	0			1
108	A	2.8.9.2 Use the zero product property, IF AB = 0, THEN A=0 or B=0 (P1, P2)	0	0	1	1			2
109	A	2.8.9.3 Solve: IF (x + 5)(x +6) = 0, THEN (x + 5) = 0 or (x + 6) = 0 (P1, P2)	0	0	1	1			2
110	A	2.8.9.4 IF x + 5 = 0, THEN x = -5 and IF x + 6 = 0, THEN x = -6 (P1, P2)	0	0	1	1			2
111	A	2.8.9.5 Solution is x = -5 or -6. Tell students: “The solution of a quadratic equation is also called x-intercepts, zeros, and roots. (P1)	0	0	1	0			1
112	A	2.9 Give students 3-5 question assessment (open ended) (P1)	0	0	1	0			1
113	D	2.9.1 IF students have mastered factoring when the leading coefficient is 1, THEN move on to factoring where the leading coefficient is other than 1. IF not, THEN reteach. (P1, P2)	0	0	1	1			2
114	A	2.10 Write a quadratic equation like on the board (P1, P2, P3)	1	0	1	1			3
115	A	2.10.1 Identify a = 6, b = 1, and c = -1 (P1, P2, P3)	1	0	1	1			3
116	A	2.10.2 Use the X-Box method, for example (Step 2.10.3) (P2, P3)	1	0	0	1			2
117	A	2.10.3 Make a big cross (**X**)	1	0	1	1			3

		underneath the equation. Inside the top of the **X**, write the product of a and c and inside the bottom of the **X**, write *b* (the sum) (P1, P2, P3)							
118	A	2.10.4 Find factors with a product of -6 and a sum of 1, as shown (P1, P2, P3)	1	0	1	1			3
119	A	2.10.5 Tell students: "There are many pairs that will give a product of -6 but they are not all going to give a sum of the middle coefficient, 1" (P2)	0	0	0	1			1
120	A	2.10.6 Show students the factors of -6 that give a sum of 1 are: 3 and -2 (P1, P2, P3)	1	0	1	1			3
121	A	2.10.7 Write the expanded form of the quadratic equation: (P3)	1	0	0	0			1
122	A	2.10.8 Factor by grouping (P3)	1	0	0	0			1
123	A	2.10.9 Find common factors: (P1, P2, P3)	1	0	1	1			3
124	A	2.10.10 Check by multiplying using the box (P3)	1	0	0	0			1
125	A	2.10.11 Write out to confirm the product is the original quadratic equation (P3)	1	0	0	0			1
126	A	2.10.12 Solve the quadratic equation for x using the zero product property (ZPP): If $(2x + 1)(3x – 1) = 0$, THEN $2x + 1 = 0$ or $3x – 1 = 0$ (P2, P3)	1	0	0	1			2
127	A	2.10.13 Solve $2x + 1 = 0$ to get $x = -1/2$ and $3x – 1 = 0$ to get $x = 1/3$ (P3)	1	0	0	0			1
128	A	2.11 IF teacher does one example on the board, THEN give students one problem to try. Walk around the room to monitor what students are doing to check for understanding (P1, P2, P3)	1	0	1	1			3
129	D	2.12 IF students have not mastered factoring, THEN reteach the concept (P1, P2)	0	0	1	1			2
130	D	2.13 IF students have mastered factoring, THEN give proceed to give assessment (P1)	0	0	1	0			1
131	A	2.14 Give students 5-10 question assessment (open ended) on factoring. (P1)	0	0	1	0			1
132	D	2.14.1 IF students have mastered factoring, THEN move on to solving	0	1	1	1			3

		quadratic equations using the quadratic formula. IF NOT, THEN reteach. (P1, P2, P4)							
		Procedure 3: Teach solving quadratic equations by using the quadratic formula					**57**	**18**	
133	A	3.1 Sing to students a song on the quadratic formula (P1, P2, P3, P4)	1	1	1	1			4
134	A	3.1.1 Sing: "*x* equals ***negative b***, plus or minus square root of *b* squared minus 4ac, all over 2a"; Or teach students to memorize the quadratic formula using this phrase: "A ***negative b***oy could not decide whether or not to go to a radical party. He decided to be square and he missed out on 4 awesome chicks. The party was all over at 2 am." (P1, P2, P3, P4)	1	1	1	1			4
135	A	3.1.2 Tell students: "The quadratic formula is a "catchall" for solving quadratic equations, it works every time" (P1, P4)	0	1	1	0			2
136	A	3.1.3 Tell students: "When in doubt while solving quadratic equations, revert back to the quadratic formula, that is the reason for singing the song every day, multiple times during the period while using the quadratic formula" (P1, P3, P4)	1	1	1	0			3
137	A	3.2 Write the quadratic equation in standard form, before using the quadratic formula (P1)	0	0	1	0			1
138	D	3.2.1 IF the equation is not in standard form, THEN the equation may be misleading because either the value of a, b, or c may not be correct (P1, P2)	0	0	1	1			2
139	A	3.3 Show students how to write a quadratic equation in standard form (P1, P2)	0	0	1	1			2
140	A	3.3.1 Write an example on the board that has the *x*-term on the other side, (P1)	0	0	1	0			1
141	A	3.3.2 Show students that the standard form would be or (P1, P2)	0	0	1	1			2
142	A	3.3.3 Choose which side of the	0	0	1	0			1

		equal sign to take all the terms to get the equation to standard form and pay attention to the sign change (P1)							
143	D	3.3.4 IF there are terms on both sides of the equal sign, THEN the signs will be different when all the terms a collected on the same side (P1)	0	0	1	0			1
144	A	3.4 Tell students: ‘Make three-columns in your notebook’ (P1)	0	0	1	0			1
145	A	3.4.1 Write the essential question (EQ): “How is the quadratic formula used to solve a quadratic equation” on the left-hand side (P1).	0	0	1	0			1
146	A	3.4.2 Write the steps of using the quadratic formula on the right-hand side (P1)	0	0	1	0			1
147	A	3.4.3 Do the steps along with the students (P1)	0	0	1	0			1
148	A	3.4.3.1 **Step 1**: Write: Standard form (P1, P2, P3)	1	0	1	1			3
149	A	3.4.3.2 Write: $ax^2 + bx + c = 0$ or $ax^2 + bx + c = y$ underneath “Standard form” (P1)	0	0	1	0			1
150	A	3.4.3.3 Step **2**: Label *a*, *b*, and *c*. To the left of *a*, write “opening” (P1)	0	0	1	0			1
		3.4.3.3.1 Reason: So students know when they graph it, *a* is going tell them the direction the graph will open, either “up” or “down” (P1)							
151	D	3.4.3.4 IF *a* equals a *negative* number, THEN the graph opens *down* (P1)	0	0	1	0			1
152	D	3.4.3.5 IF *a* equals a *positive* number, THEN graph opens *up* (P1)	0	0	1	0			1
153	A	3.4.3.6 Write *y*-intercept next to *c* (P1)	0	0	1	0			1
		3.4.3.6.1 Reason: So that students know this is not in isolation (P1)							
154	A	3.4.3.7 Write the quadratic formula, as you sing along with students the quadratic formula song: “x equals negative b, plus or minus square root of b squared minus 4ac, all over 2a” (P1, P2)	0	0	1	1			2
155	A	3.4.3.7.1 Do not substitute the values of *a*, *b*, and *c* into the formula before writing the formula out (P1, P2)	0	0	1	1			2
156	A	3.4.3.7.2 Write the quadratic	0	0	1	1			2

		formula with parenthesis below the quadratic formula. Sing the remix together with students: “x equals negative parenthesis plus or minus square root of parenthesis squared minus four parenthesis parenthesis all over two parenthesis” as you write it (P1, P2)							
157	A	3.4.3.7.3 Get students in the habit of writing the formula every time (P1, P2)	0	0	1	1			2
158	D	3.4.3.7.4 IF students substitute directly without writing the formula first, THEN they do not get points for it (P1)	0	0	1	0			1
		3.4.3.7.4.1 Reason: Because students must use parenthesis when substituting into the formula (P1)							
159	A	3.4.3.8 Tell students: “To substitute is to replace” (P1)	0	0	1	0			1
160	A	3.4.4 Show students how to do the four steps in one (P1)	0	0	1	0			1
161	A	3.4.4.1 Do order of operations, do PEMM (P1)	0	0	1	0			1
162	A	3.4.4.2 Substitute the value of *b* into the first **P***arenthesis*, negative ( ) (P1)	0	0	1	0			1
163	A	3.4.4.3 Substitute the value of *b* again into the **E***xponent* part $( )^2$ and circle the *b*-squared part (P1)	0	0	1	0			1
164	A	3.4.4.4 Substitute the values of *a* and *c* into third and fourth parenthesis respectively, *negative* 4( )( ) and circle -4ac followed by putting **M** over it to indicate **M***ultiplication* will take place (P1)	0	0	1	0			1
165	A	3.4.4.5 Circle the denominator, 2*a*, substitute the value of *a* into the parenthesis 2( ) and put **M** over it to indicate **M***ultiplication* will take place (P1)	0	0	1	0			1
166	A	3.4.4.6 Circle what is under the square root sign (P1)	0	0	1	0			1
167	D	3.4.4.7 IF you are going to add the numbers *b*2 and 4*ac*, THEN put **A** for addition above it and add the quantities (P1)	0	0	1	0			1
168	D	3.4.4.8 IF you are going to subtract the	0	0	1	0			1

		quantities *b*2 and 4*ac*, THEN put **S** for subtraction above it and subtract the quantities (P1)							
169	A	3.4.4.9 Write **D** for **D***ivision* but wait on D (P1)	0	0	1	0			1
170	A	3.4.4.10 Take the square root of the quantity *b*2 – 4*ac* (P1)	0	0	1	0			1
171	A	3.4.4.11 **D**ivide the numerator by the denominator and then write *x* equals the result of the division (P1)	0	0	1	0			1
		3.4.4.11.1 Reason: Because we are going through PEMMDAS, the order of operations (P1)							
172	A	3.4.4.12 Circle using a red marker on the board and write the step being done (P1)	0	0	1	0			1
173	A	3.4.4.12 Circle using a red marker on the board and write the step being done (P1)	0	0	1	0			1
174	A	3.4.4.13 Draw a box around *x* equals (P1)	0	0	1	0			1
175	D	3.4.4.14 IF *x* has two solutions, THEN the graph intercepts the *x*-axis twice (P1)	0	0	1	0			1
176	D	3.4.4.15 IF *x* has only one solution, THEN the graph touches the *x*-axis once (P1)	0	0	1	0			1
177	D	3.4.4.16 IF *x* has no solution, THEN the graph does not touch the *x*-axis (P1)	0	0	1	0			1
178	A	3.4.5 Show an example: (P1, P2)	0	0	1	1			2
179	A	3.4.6 Substitute the values of *a*, *b*, and *c* from the standard quadratic equation, into the formula with the parenthesis in place of *a*, *b*, and *c* (P1, P2, P4)	0	1	1	1			1
180	A	3.4.6.1 Substitute the value of *b* from the standard quadratic equation, $2x^2 - 3x - 5 = 0$ into the first and second parenthesis (P1, P2)	0	0	1	1			2
181	D	3.4.6.2 IF *b* was negative, THEN substitute it together with its sign (P1)	0	0	1	0			1
182	A	3.4.6.3 Tell students: “Pay attention to the $( )^2$" (P1)	0	0	1	0			1
183	D	3.4.6.3.1 IF *b* is a *negative* number, THEN *negative* times *negative* is *positive*. (P1)	0	0	1	0			1
184	D	3.4.6.3.2 IF b is a positive number,	0	0	1	0			1

		THEN positive times positive is positive (P1)							
185	D	3.4.6.4 IF any number is squared, THEN the product is always positive (P1)	0	0	1	0			1
186	A	3.4.6.5 Substitute values of *a*, the *c* and *a* again into the remaining parenthesis (P1)	0	0	1	0			1
187	A	3.4.6.6 Simplify (P1, P2)	0	0	1	1			2
188	A	3.1.1.1 Tell students: “The plus or minus 7 means we have two solutions for this quadratic equation (P2)	0	0	0	1			1
189	A	3.1.1.2 Solution: x = 10/4 or x = -4/4 which are simplified to x = 2.5 or -1. These are also called roots, x-intercepts or zeros of the quadratic equation (P1, P2)	0	0	1	1			2
		3.4.7 See Table 1 (P1)							
190	A	3.4.7.1 Tell students: "b^2 - $4ac$ is the discriminant and it helps determine the number of solutions of a quadratic equation” (P1, P2, P3, P4)	1	1	1	1			4
191	D	3.4.7.2 IF the discriminant is positive, THEN the parabola intercepts the x-axis twice (P1, P2, P3, P4)	1	1	1	1			4
192	A	3.4.7.2.1 Draw the graph to show students the parabola intercepts the x-axis twice (P1, P4)	0	1	1	0			2
193	D	3.4.7.3 IF the discriminant is zero, THEN the parabola touches the x-axis once and turns around (P1, P2, P3, P4)	1	1	1	1			4
194	A	3.4.7.3.1 Draw the graph to show students the parabola touches the x-axis once (P1, P4)	0	1	1	0			2
195	D	3.4.7.4 IF the discriminant is negative, THEN the parabola does not touch the x-axis (P1, P2, P3, P4)	1	1	1	1			4
196	A	3.4.7.4.1 Draw the graph to show students the parabola does not intersect the *x*-axis (P1, P4)	0	1	1	0			2
197	D	3.4.7.5 IF students get the solution, THEN they have to write all the names every time: *x*-intercept(s), solution(s), zero(s) and root(s) of the quadratic equation (P1)	0	0	1	0			1
198	A	3.5 Show students b^2 - 4ac is the	0	0	1	0			1

		discriminant and is part of the quadratic formula (P1)							
199	A	3.4.7.5 IF students get the solution, THEN they have to write all the names every time: *x*-intercept(s), solution(s), zero(s) and root(s) of the quadratic equation (P1)	0	0	1	0			1
200	A	3.5.1 Point at the discriminant (P1)	0	0	1	0			1
201	A	3.6 Use the quadratic formula to solve any quadratic equation for its roots, its solution, and its zeros (P1, P2)	0	0	1	1			2
202	D	3.7 IF students are able to solve a quadratic equation for its roots, solutions, or its zeros, THEN they can solve real-life application problems like vertical motion problems (P1)	0	0	1	0			1
203	D	3.8 IF students are not comfortable using the quadratic formula, THEN show them two more examples and give them a few problems to practice (guided practice) (P1, P3)	1	0	1	0			2
204	A	3.9 Give students 3-5 problems to solve using the quadratic formula. (P1)	0	0	1	0			1
205	D	3.9.1 IF students have mastered solving using the quadratic formula, THEN begin teaching graphing. IF not, THEN reteach. (P1, P3)	1	0	1	0			2
206	D	3.9.2 IF students are proficient using the quadratic formula, THEN proceed with the lesson to show students the next procedure for solving quadratic equations (P1)	0	0	1	0			1
		Procedure 4: Teach solving quadratic equations by graphing					**47**	**17**	
207	A	4.1 Post an x-y coordinate plane on the whiteboard throughout the unit of quadratic equations (P1)	0	0	1	0			1
208	A	4.2 Solve all quadratic equations next to the graph so that students make connections and also see multiple representations (P1)	0	0	1	0			1
209	A	4.3 Solve while relating back to the graph because students have a hard time connecting different representations (P1)	0	0	1	0			1

210	A	4.4 Do not teach graphing of quadratic equations and solving quadratic equations using other procedures in isolation (P1)	0	0	1	0			1
211	A	4.4.1 Go back and forth between various methods of solving quadratic equations and their graphs (P1)	0	0	1	0			1
212	A	4.4.2 Relate the *x*-intercepts of the graph of a quadratic equation to its solutions after solving using any of the other procedures (P1)	0	0	1	0			1
213	A	4.5 Start with a quadratic equation in standard form and reflects on the *y*-axis (P1, P2, P3, P4)	1	1	1	1			4
214	A	4.6 Identify the parts that are obvious based on the equation (P1, P3, P4)	1	1	1	0			3
215	A	4.6.1 Identify *a* the coefficient of x^2, *b* the coefficient of *x* and *c*, the constant (P1, P3, P4)	1	1	1	0			3
216	D	4.6.2 IF *a* is positive, THEN the graph (parabola) will open up (P1)	0	0	1	0			1
217	D	IF *a* is negative, THEN the graph (parabola) will open down (P1)	0	0	1	0			1
218	A	4.6.4 Write *c*, the *y*-intercept (P1, P4)	0	1	1	0			2
219	A	4.7 Teach how to find the axis of symmetry (P1, P2)	0	0	1	1			2
220	A	4.7.1 Find axis of symmetry and relate to the quadratic formula (P1)	0	0	1	0			1
221	A	4.7.2 Break apart the quadratic formula and show students the axis of symmetry (P1, P2, P4)	0	1	1	1			3
222	A	4.7.3 Write the axis of symmetry formula and carefully explain to students that this does not represent x-intercepts (P1)	0	0	1	0			1
223	A	4.7.4 Tell students: "This is an *x*-value and it is where on the *x*-axis the axis of symmetry cuts through" (P1)	0	0	1	0			1
224	D	4.7.5 IF the quadratic equation has a middle term *bx*, THEN the parabola will not reflect over the *y*-axis (P2)	0	0	0	1			1
225	A	4.7.6 Substitute the values of *a* and *b* into the axis of symmetry equation to find the axis of symmetry (P1, P3, P4)	1	1	1	0			3
226	D	4.7.7 IF you fold a parabola in half through the axis of symmetry, THEN	0	1	1	0			2

		there are two identical parts (P1, P4)							
227	D	4.7.8 IF there is a *y*-value to left of the axis of symmetry, THEN there is an equivalent *y*-value same distance from the axis of symmetry on the right of the axis of symmetry (P1)	0	0	1	0			1
228	D	4.7.9 IF you have the *y*-intercept on one side of the axis of symmetry, THEN there is another point at the same height on the other side of the parabola (P1)	0	0	1	0			1
229	A	4.8 Teach students how to find the vertex (P1, P2, P3)	1	0	1	1			3
230	A	4.8.1 Use the axis of symmetry to find the x-value of the vertex (P1, P2, P3)	1	0	1	1			3
231	A	4.8.2 Substitute the *x*-value of the vertex into $ax^2 + bx + c = y$ to find the y-value of the vertex (P1, P2, P3, P4)	1	1	1	1			4
232	A	4.8.3 Write the vertex in the form of (*x*, *y*) coordinate point (P1, P2, P3, P4)	1	1	1	1			4
233	A	4.8.4 Point to students that the vertex is the highest or lowest point of the parabola (P3, P4)	1	1	0	0			2
234	A	4.9 Teach Graphing procedure (P1, P2, P3, P4)	1	1	1	1			4
235	A	4.9.1 Draw an *x-y* coordinate plane (P1, P2, P3, P4)	1	1	1	1			4
236	A	4.9.2 Draw a dotted line through the axis of symmetry found in **step 4.7.6** (P1, P2, P4)	0	1	1	1			3
237	A	4.9.3 Draw an *x-y* table of values. An example is here below when the *x*-value of the vertex is 0 (P1, P2, P3, P4)	1	1	1	1			4
238	A	4.9.4 Put the vertex coordinates at the center of the table (P1, P2, P3, P4)	1	1	1	1			4
239	A	4.9.5 Choose an *x*-value to the left or to the right of the vertex (P1, P3, P4)	1	1	1	0			3
240	A	4.9.6 Show students the mirror point(s) (P1, P4)	0	1	1	0			2
241	A	4.9.7 Plot the vertex as found in step 4.8.3 (P1, P3, P4)	1	1	1	0			3
242	D	4.9.8 IF you find the vertex, THEN get 2 or 3 points on one side of the axis of symmetry (P1, P3, P4)	1	1	1	0			3
243	D	4.9.9 IF you have 2 or 3 points on one side of the axis of symmetry, THEN	1	1	1	0			3

		you will get 2 or 3 points on the opposite side of the axis of symmetry (P1, P3, P4)							
244	D	4.9.10 IF you have the axis of symmetry (*x*-value), THEN choose two *x*-values to the left or right of the axis of symmetry to substitute into the original equation to find the corresponding *y*-values (P1, P2, P3, P4)	1	1	1	1			4
245	D	4.9.11 IF the parabola opens up, THEN the vertex is a minimum (P1)	0	0	1	0			1
246	D	4.9.12 IF the parabola opens down, THEN the vertex is a maximum (P1)	0	0	1	0			1
247	A	4.9.13 Tell students: "The vertex is where our parabola opens from" (P1)	0	0	1	0			1
248	A	4.9.14 Remind students the graph opens upwards or opens downwards depending on the *a*-value (P1)	0	0	1	0			1
249	A	4.10 Connect the points to plot the graph with a smooth curve (P1, P3, P4)	1	1	1	0			3
250	A	4.11 Label on the graph the vertex, *y*-intercept, axis of symmetry and direction of opening, up or down (P1)	0	0	1	0			1
251	A	4.12 Label the *x*-intercepts if they exist (P3)	1	0	0	0			1
252	A	4.13 Show an example, $x^2 + 4x – 12 = 0$ (P2)	0	0	0	1			1
253	A	4.13.1 Draw a two-column *t*-table of values (step 4.9.3) (P2)	0	0	0	1			1
254	A	4.13.2 Identify the coefficients: $a = 1$, $b = 4$, and $c = -12$ (P2)	0	0	0	1			1
255	A	4.13.3 Find the axis of symmetry (P2)	0	0	0	1			1
256	A	4.13.4 Put the *axis of symmetry*, $x = -2$ in the middle of the *t*-table and then choose integers on either side of -2 that are equidistant from the axis of symmetry (P2)	0	0	0	1			1
257	A	4.13.5 Draw the axis of symmetry $x = -2$ (P2)	0	0	0	1			1
258	A	4.13.6 Choose two *x*-values less than -2 and two *x*-values greater than -2: -4, -3, **-2**, -1, and 0 (step 3.1.5) (P2)	0	0	0	1			1
259	A	4.13.7 Substitute these *x*-values into the quadratic equation, $x^2 + 4x – 12 = 0$ to find the corresponding *y*-values (step	0	0	0	1			1

		4.8.2) (P2)							
260	A	4.13.8 Plot the pairs of points on *x*-*y* coordinate plane (P2)	0	0	0	1			1
261	A	4.13.9 Ask aloud: "How many times does the graph of $x^2 + 4x - 12 = 0$ cross the *x*-axis?" (P2)	0	0	0	1			1
262	A	4.13.10 Show students the *x*-intercepts, which are also the solutions of the quadratic equation (P2)	0	0	0	1			1
263	D	4.14 IF the graph of a quadratic equation intercepts the *x*-axis twice, THEN the quadratic equation has two real solutions (P2)	0	0	0	1			1
264	D	4.15 IF the graph of a quadratic equation touches the *x*-axis once, THEN the quadratic equation has one real solution (P2)	0	0	0	1			1
265	D	4.16 IF the graph of a quadratic equation does not touch the *x*-axis, THEN the quadratic equation has no real solution (P2)	0	0	0	1			1
266	A	4.17 Check for understanding by giving students two quadratic equations to graph (P1, P4)	0	1	1	0			2
267	D	4.18 IF students are not proficient with graphing, THEN show them the process with two more examples (P1, P4)	0	1	1	0			2
268	D	4.19 IF students are proficient with graphing quadratic equations, THEN proceed to the next procedure for solving quadratic equations (P1, P4)	0	1	1	0			2
269	A	4.20 Give students a 5 question graphing assessment (P1, P4)	0	1	1	0			2
270	D	4.21 IF students have mastered graphing, THEN move on to Completing the Square. IF NOT, then reteach. (P1, P4)	0	1	1	0			2
		Procedure 5: Completing the square					**57**	**10**	
271	A	5.1 Complete the square of a quadratic equation with a leading coefficient of 1 (P1)	0	0	1	0			1
272	A	5.1.1 Divide your notepaper or notebook into three columns (P1)	0	0	1	0			1
273	A	5.1.2 Label the steps on the right	0	0	1	0			1

		hand side of your notepaper or notebook (P1)							
274	A	5.1.3 Do your work in the middle of your paper (P1)	0	0	1	0			1
275	A	5.1.4 Write the essential question on the left hand side of your paper: Essential Question, "How is completing the square used to solve a quadratic equation?" (P1)	0	0	1	0			1
276	A	5.2 Write steps for completing the square (P1)	0	0	1	0			1
277	A	5.2.1 **Step 1** – If the equation is not in standard form, THEN re-arrange the terms in standard form (P1)	0	0	1	0			1
278	A	5.2.1.1 Reason: Because it gives students consistency and therefore write the equation in standard form (P1)	0	0	1	0			1
279	D	5.2.2 IF the equation can be factored at this point, THEN tell students to solve by factorization (P1)	0	0	1	0			1
280	A	5.2.3 **Step 2** – Pull the constant (P1)	0	0	1	0			1
281	A	5.2.3.1 Isolate the constant on the opposite side (P1)	0	0	1	0			1
282	D	5.2.3.2 IF the constant is already isolated, THEN skip **step 2** (P1)	0	0	1	0			1
283	A	5.2.4 Work on either side of the equal sign (P1)	0	0	1	0			1
		5.2.4.1 Reason: Because students should feel constrained to have everything on the left (P1)							
284	A	5.2.5 Sing: "*half of b squared, add it to both sides*" (while drumming) (P1)	0	0	1	0			1
285	A	5.2.6 Take and add it to both sides (P1)	0	0	1	0			1
286	A	5.2.7 Sing to students again: "*half of b squared, add it to both sides*" (while drumming) (P1)	0	0	1	0			1
287	A	5.2.7.1 Tell students: "Let us sing, "*half of b squared, add it to both sides*" (while drumming) (P1)	0	0	1	0			1
288	A	5.2.7.2 Sing together: "*half of b squared, add it to both sides*" (while drumming) (P1)	0	0	1	0			1
289	D	5.2.7.3 IF teacher sings, THEN teacher asks students to sing with her (P1)	0	0	1	0			1

290	D	5.2.7.4 IF students sing, THEN show them how to do it (P1)	0	0	1	0			1
291	A	5.2.8 Tell students: "We are taking half of *b*" (P1)	0	0	1	0			1
292	A	5.2.8.1 Show students what half of something means, say half of \$4 is \$2, half of \$12 is \$6 (P1)	0	0	1	0			1
293	A	5.2.8.2 Practice with students: half of 6, half of 10 … (P1)	0	0	1	0			1
294	A	5.2.8.3 Check for understanding with students writing the answers on their individual whiteboards and lifting them up to show the teacher (P1)	0	0	1	0			1
295	A	5.3 Start with an expression with a coefficient of 1 for x^2, $x^2 + bx$ to complete the square (P2)	0	0	0	1			1
296	A	5.3.1 Give an example $x^2 + 6x$, start with an even *b*-term (P2, P3)	1	0	0	1			2
297	A	5.3.2 Complete the square by dividing 6 by 2 to get 3 and then square 3 to get 9: $x^2 + 6x + 9 = (x + 3)^2$ (P2, P3, P4)	1	1	0	1			3
298	A	5.3.3 Show how to complete the square using algebra tiles (P2, P3, P4)	1	1	0	1			3
299	A	5.3.4 Explain to students the process of completing the square (P2, P3)	1	0	0	1			2
300	A	5.3.5 Show students more completing the square: $x^2 + 4x$, to complete the square, add the square of , which is $2^2 = 4$ and the expression becomes $x^2 + 4x$ + **4.** Therefore add 4 squares to complete the square (P2, P3, P4)	1	1	0	1			3
301	A	5.3.6 Show students another example: $x^2 + 8x$, to complete the square, add the square of, which is $4^2 = 16$ and the expression becomes $x^2 + 8x$ + **16.** Therefore add 16 squares to complete the square. (P2, P3, P4)	1	1	0	1			3
302	A	5.4 Introduce students to a quadratic equation to solve using by the completing the square procedure (P1, P2, P3)	1	0	1	1			3
303	A	5.5 Show students how to write the quadratic equation in the form $x^2 + bx = c$ (P2, P3)	1	0	0	1			2
304	A	5.5.1 Give students an example, like	1	0	1	1			3

		(P1, P2, P3)							
305	A	5.5.2 Teacher says: "*b* = 10, take half of 10" (P1, P2, P3)	1	0	1	1			3
306	A	5.5.3 Teacher says: "IF I say half of *b*, THEN you say the answer" (P1)	0	0	1	0			1
307	A	5.5.4 Teacher says: "IF I say half of 10, THEN you say 5!" (P1)	0	0	1	0			1
308	A	5.5.5 Teacher says: IF I say 5 squared, THEN you say 25!" (P1)	0	0	1	0			1
309	A	5.5.6 Teacher says: "IF I say add it to both sides, THEN you add it to both sides" (P1)	0	0	1	0			1
310	A	5.5.7 Add 25 to both sides of the equation (P1, P2, P3)	1	0	1	1			3
311	A	5.5.8 Remind students the song: "*half of b squared, add it to both sides*" (P1)	0	0	1	0			1
312	A	5.5.9 Tell students: "We squared it, so the title of completing the square. We are making it squared so that we can write it as a quantity squared" (P1)	0	0	1	0			1
313	A	5.5.10 Tell students: "*x* was squared to get *x*-squared and 5 was squared to get 25" (P1, P2)	0	0	1	1			2
314	A	5.5.11 Take square root to undo squares (P1, P2, P3, P4)	1	1	1	1			4
315	A	5.5.12 Factorize the left hand side: factors of x^2 are *x* and *x* and factors of 25 are 5 and 5 (P1, P2, P3, P4)	1	1	1	1			4
316	A	5.5.13 IF you take the square root of one side, THEN you must take the square root of the other side (P1, P2, P3, P4)	1	1	1	1			4
317	A	5.5.14 Take the square root of both sides (P1, P2, P3, P4)	1	1	1	1			4
318	A	5.5.15 Solve for *x* (P1, P2, P3, P4)	1	1	1	1			4
319	A	5.5.15.1 Tell students: "Let's look at our essential question: How do we use completing the square to solve quadratic equations?" (P1)	0	0	1	0			1
320	A	5.5.16 Circle *x* (P1, P2, P3, P4)	1	1	1	1			4
321	A	5.5.17 Isolate *x* by itself (P1, P2, P3, P4)	1	1	1	1			4
322	A	5.5.18 Subtract 5 from both sides (P1, P2, P3, P4)	1	1	1	1			4
323	A	5.5.19 Box the answer: $x = 0$ or -10 and write solutions, roots, *x*-intercepts	1	1	1	1			4

		and zeros of the quadratic equation (P1, P2, P3, P4)							
324	A	5.6 Show students another example following the steps shown on step 5.5 (P1)	0	0	1	0			1
325	A	5.7 Check for understanding by giving students one problem at time to do on their whiteboards in pairs (P1, P2, P3, P4)	1	1	1	1			4
326	D	5.8 IF students are not proficient solving quadratic equations with a coefficient of 1 for x^2 by completing the square procedure, THEN show them one more example (P1, P2, P3).	1	0	1	1			3
327	D	5.9 IF students are proficient solving quadratic equations with a coefficient of 1 for x2 by completing the square procedure, THEN introduce an equation with an a-value greater than 1 (P1, P2, P3, P4)	1	1	1	1			4
328	A	5.10 Introduce an equation with an a-value greater than 1 (P1)	0	0	1	0			1
329	D	5.10.1 Step 1 – IF the equation is not in standard form, THEN re-arrange the terms in standard form [see step 5.2] (P1)	0	0	1	0			1
330	D	5.10.2 IF the a-value is not equal to one, THEN divide both sides by a (P1)	0	0	1	0			1
331	A	5.10.3 Tell students: “It is going to be a challenge because you may start dealing with a b-value that is a fraction or an odd number” (P1)	0	0	1	0			1
332	A	5.10.4 Repeat steps 5.2.1 through 5.2.7 (P1)	0	0	1	0			1
333	A	5.11 Teach students easy ways to remember the steps. Singing seems to work all the time (P1)	0	0	1	0			1
334	A	5.12 Teach students how to use algebra tiles to complete the square (P3)	1	0	0	0			1
335	D	5.13 IF it is about solving quadratic equations, THEN the quadratic formula is the fallback method, it works for every quadratic equation (P1)	0	0	1	0			1
336	A	5.14 Give students 3 problems to solve by completing the square, and 1 problem that is already solved but	0	0	1	1			2

		solution steps are out of order and students must order the steps correctly, and the proof of the quadratic formula by completing the square with steps out of order where students must correctly order the steps of the proof. (P1, P2)							
337	D	5.3 IF students have mastered completing the square, THEN go on to solving quadratics using square roots. IF NOT, THEN reteach. (P1, P2)	0	0	1	1			2
		Procedure 6: Solving quadratics using the square root method					**21**	**6**	
338	D	6.1 IF the equation is of the form $ax^2 = c$ or $ax^2 - c = 0$, THEN the square root method is appropriate (P1, P2)	0	0	1	1			2
339	A	6.2 Show students how to solve $x^2 = 9$ (P1, P2)	0	0	1	1			2
340	A	6.3 Take the square root of both sides (P1, P2)	0	0	1	1			2
341	A	6.4 Write x = -3 or 3 which are the solutions, x-intercepts, zeros and also the roots of the quadratic equation $x^2 = 9$. (P1)	0	0	1	0			1
342	A	6.5 Show another example without a perfect square, solve (P1, P2)	0	0	1	1			2
343	A	6.5.1 Add 8 to both sides, to isolate x^2: $x^2 – 8 + 8 = 0 + 8$ (P1, P2)	0	0	1	1			2
344	A	6.5.2 Simplify: $x^2 = 8$ (P1, P2)	0	0	1	1			2
345	A	6.5.3 Take the square root of both sides (P1, P2)	0	0	1	1			2
346	A	6.5.4 Write the answers and these are the roots, zeros, solutions or x-intercepts of this quadratic equation $x^2 – 8 = 0$. (P1, P2)	0	0	1	1			2
347	A	6.6 Show another example: $x^2 – 16 = 0$ (P1, P2)	0	0	1	1			2
348	A	6.6.1 Use factorization and find the sum and difference of products (P1 P2)	0	0	1	1			2
349	A	6.6.2 Use zero product property, IF AB = 0, THEN A = 0 or B = 0 (P1, P2)	0	0	1	1			2
350	D	6.6.3 IF $x^2 – 16 = 0$, THEN $(x – 4)(x + 4) = 0$ (P1, P2)	0	0	1	1			2
351	A	6.6.4 Solve: IF $(x – 4)(x + 4) = 0$, THEN $x – 4 = 0$ or $x + 4 = 0$ (P1, P2)	0	0	1	1			2
352	A	6.6.5 Simplify: IF $x – 4 = 0$ or $x + 4 =$	0	0	1	1			2

		0, THEN x = 4 or x = -4 (P1, P2)							
353	A	6.6.6 Use square roots to solve: $x^2 - 16 = 0$ (P1, P2)	0	0	1	1			2
354	A	6.6.7 Add 16 to both sides of the equal sign (P1, P2)	0	0	1	1			2
355	A	6.6.8 Solve $x^2 = 16$ (P1, P2)	0	0	1	1			2
356	A	6.6.9 Take the square root of both sides (P1, P2)	0	0	1	1			2
357	A	6.6.10 Write x = 4 or x = -4, which is the same solution. (P1, P2)	0	0	1	1			2
358	A	6.2 Remind students: "Those problems that they did in factoring are very similar to our square roots problems" (P1)	0	0	1	0			1
359	D	6.3 IF a quadratic equation is missing the b-term (middle term), THEN solve using the square root procedure (P1, P2)	0	0	1	1			2
360	D	6.4 IF students are not proficient using the square root procedure, THEN show them two more examples (P1, P2)	0	0	1	1			2
361	A	6.5 Check for understanding with students writing on their whiteboards and showing the teacher their solution (P1, P2)	0	0	1	1			2
362	D	6.6 IF students are proficient with solving quadratic equations with the square root procedure, THEN proceed to the application of all these procedures that students have been learning (P1, P2)	0	0	1	1			2
363	A	6.7 Give students a 5 question assessment (open ended quadratic equation problems to solve) (P1)	0	0	1	0			1
364	D	6.8 IF students have mastered completing the square, THEN go to application. IF NOT, reteach. (P1)	0	0	1	0			1
		Procedure 7: Teach Application of these methods of solving quadratic equations to solve real-life problem					**30**	**9**	
365	A	7.1 Apply knowledge of solving quadratic equation to real-life problems (P1, P3, P4)	1	1	1	0			3
366	D	7.2 IF students have been taught the skill base for solving quadratic (P1)	0	0	1	0			1

367	A	7.3 Write (Project) a practice problem on the board (P1, P4)	0	1	1	0			2
368	A	7.4 Choose the best procedure for solving the quadratic problem (P1, P4)	0	1	1	0			2
369	A	7.5 Use the three-column table on the whiteboard as a graphic organizer (P1)	0	0	1	0			1
370	A	7.6 Make notes on it on all discussions as you solve the problem (P1)	0	0	1	0			1
371	A	7.7 Give students a couple of minutes to solve the problem (P1)	0	0	1	0			1
372	A	7.7.1 Ask students aloud: “Which way did you solve it?” (P1)	0	0	1	0			1
373	A	7.7.2 Go back to the three-column table and make notes (P1)	0	0	1	0			1
374	A	7.8 Ask students: “When is it most convenient to use factoring?” (P1)	0	0	1	0			1
375	A	7.9 Ask students: “When can I use square roots?” (P1)	0	0	1	0			1
376	D	7.9.1 IF a student solved it really quickly, THEN ask, “What did you do? Which method did you use?” (P1)	0	0	1	0			1
377	D	7.9.1.1 IF a student took a little bit longer, THEN ask, “What did you do? Which method did you use?” (P1)	0	0	1	0			1
378	A	7.9.1.2 Go back to the graphic organizer and edit it with specifics of what you did (P1)	0	0	1	0			1
379	D	7.10 IF the quadratic equation does not have a b-term, THEN use the square roots method” (P1)	0	0	1	0			1
		7.10.1 Reason: Because it is the easiest. It is the quickest (P1)							
380	D	7.11 IF the coefficient of x2 is greater than 1 or not factorable, THEN use the quadratic formula (P1)	0	0	1	0			1
		7.11.1 Reason: Because you do not have to list all possible different factors and then finally find the problem is not factorable while the quadratic formula always works (P1)							
381	A	7.12 Take about two to three days doing this with students. Practicing to build their confidence (P1)	0	0	1	0			1
382	A	7.13 Assess students on their skills (P1)	0	0	1	0			1
383	A	7.13.1 Assess students where they	0	0	1	0			1

		choose any method they want to solve (P1)							
384	A	7.13.2 Give students two problems, where they have to use a specific method to solve (P1)	0	0	1	0			1
385	D	7.14 IF students can solve basic quadratic equations, THEN introduce practical real-life problems (P1)	0	0	1	0			1
386	D	7.15 IF students can solve a quadratic equation, THEN ask them to do different things, like find the width of a parabolic disc (P1, P3, P4)	0	0	1	0			1
387	A	7.16 Give students problems that involve projectiles (P1, P3, P4)	1	1	1	0			3
388	D	7.17 IF a rocket is launched, THEN what would be the maximum height, or how long will it take to reach the ground? (P1, P3, P4)	1	1	1	0			3
389	A	7.17.1 Make connection: maximum height of rocket corresponds to vertex of a parabola; time rocket takes to reach the ground is the difference between the x-intercepts of the parabola. These are practical applications of solving quadratic equations (P1, P3, P4)	1	1	1	0			3
390	A	7.18 Demonstrate drawing a picture that shows the parabolic path of the projectile (P1, P3, P4	1	1	1	0			3
391	D	7.18.1 IF given an application problem (word problem), THEN draw a picture to represent the story (P1, P3, P4)	1	1	1	0			3
392	A	7.18.2 Draw a picture back to the graph (P1)	0	0	1	0			1
393	A	7.18.3 Tell students: “This is the skill you learned to do, when we were graphing” (P1)	0	0	1	0			1
394	A	7.19 Draw a picture always (P1)	0	0	1	0			1
395	A	7.20 Relate back to previous lessons because quadratic equations are not an isolated unit (P1)	0	0	1	0			1
396	A	7.21 Make students understand why they are solving quadratic equations (P1)	0	0	1	0			1
397	A	7.21.1 Make connections for students! For example: When solving a problem	1	1	1	0			3

		involving the jumping path of a kangaroo, impose a graph that shows the horizontal time and vertical distance by labeling the x- and y-axis. Draw in the axis of symmetry and vertex reminding students that the kangaroo reached a specific height (y) after so many seconds (x). The students can see on the graph that the maximum height reached by the kangaroo is at the vertex. Talk about the height the kangaroo started at (was there a y-intercept other than 0?) and then where he ended up. (P1, P3, P4)							
398	A	7.22 Remind students a, b, and c in the quadratic equation, always relates back to the graph or real-life application problem (P1)	0	0	1	0			1
399	A	7.23 Ask students aloud again: “What does it mean to solve for x?” (P1)	0	0	1	0			1
400	A	7.23.1 Ask students: “What is the significance of finding *x* on the graph? What does it mean?” (P1)	0	0	1	0			1
401	A	7.23.2 Tell students that solving for x is finding solutions, the roots, the zeros and they are also the x-intercepts of the graph of the quadratic equation (P1)	0	0	1	0			1
402	A	7.24 Give students a UNIT assessment. (P1)	0	0	1	0			1

	SME			
	A	B	C	D
Action Steps	81	58	262	115
Decision Steps	16	16	76	24
Total Action and Decision Steps	**97**	**74**	**338**	**139**

	SME			
	A	B	C	D
Total Action and Decision Steps	**24.01%**	**18.32%**	**83.66%**	**34.41%**
Action Steps	25.47%	18.24%	82.39%	36.16%
Decision Steps	18.60%	18.60%	88.37%	27.91%

	SME			
	A	B	C	D
Total Action and Decision Steps Omitted	307	330	66	265
Action Steps Omitted	237	260	56	203
Decision Steps Omitted	70	70	10	62

	SME			
	A	B	C	D
Total Action and Decision Steps Omitted	75.99%	81.68%	16.34%	65.59%
Action Steps Omitted	74.53%	81.76%	17.61%	63.84%
Decision Steps Omitted	81.40%	81.40%	11.63%	72.09%

Full Alignment	26	6.68%
Substantial Alignment	44	10.89%
Partial Alignment	81	20.05%
No Alignment	251	62.38%
Total Action and Decision Steps	402	100.00%

Appendix G

Information Sheet

University of Southern California
Rossier School of Education
3470 Trousdale Parkway
Los Angeles, CA 90089

INFORMATION/FACTS SHEET FOR NON-MEDICAL RESEARCH

USING COGNITIVE TASK ANALYSIS TO CAPTURE EXPERT INSTRUCTION IN ALGEBRA FOR 8th and 9th GRADE HIGH SCHOOL STUDENTS

You are invited to participate in a research study conducted by Acquillahs Muteti Mutie, a doctoral candidate at Rossier School of Education at the University of Southern California, because you are a teacher identified as highly knowledgeable in Algebra instruction. Your participation is voluntary. You should read the information below, and ask questions about anything you do not understand or that is unclear to you, before deciding whether to participate.

PURPOSE OF THE STUDY

The purpose of the study is to use Cognitive Task Analysis methods to capture the knowledge of expert Algebra teachers as they implement research-based instructional practices in solving quadratic equations for 8th and 9th grade students. You are being asked to participate in this study because you have been identified as highly knowledgeable in algebra instruction for this student population. The information gathered will be used to help better understand quadratic equations instruction in Algebra. Your participation in the study will aid in capturing the implicit and non-observable decisions, judgments, analyses, and other cognitive processes used during quadratic equations instruction.

PARTICIPANT INVOLVEMENT

There is only one way to participate in this inquiry through participating in an interview with a follow-up round two interview. The paper survey will be distributed to teachers and is anticipated to take no more than 20 minutes to complete. After you take the survey you can also choose to participate in the interview with follow-up interview by emailing the Principal Investigator, Acquillahs Mutie, at mutie@usc.edu. Completing the survey is a requirement for participation in the interview process. The interview should take about 90 minutes to complete. Your participation is voluntary and if you choose not to participate no penalty will occur. You may choose not to participate at any time. Your identity as a participant will be de-identified and will remain confidential at all times during and after the inquiry project.

PAYMENT/COMPENSATION FOR PARTICIPATION

No payment will be offered. Participation will aid in capturing the knowledge of expert Algebra teachers, through Cognitive Task Analysis, as they implement research-based strategies for solving quadratic equations instructional practices for 8^{th} and 9^{th} grade Algebra students.

CONFIDENTIALITY

Any identifiable information obtained in connection with this study will be de-identified and remain confidential. Your responses will be coded with a false name (pseudonym) and maintained separately. The data will be stored on a password-protected computer in the researcher's office for three years after the study has been completed and then destroyed.

The members of the research team, and the University of Southern California's Human Subjects Protection Program (HSPP) may access the data. The HSPP reviews and monitors research studies to protect the rights and welfare of research subjects.

When the results of the research are published or discussed in conferences, no identifiable information will be used.

INVESTIGATOR CONTACT INFORMATION

If you have any questions or concerns about the study, please feel free to contact the Principal Investigator, Acquillahs Mutie, by email at mutie@usc.edu

IRB CONTACT INFORMATION

University Park Institutional Review Board (UPIRB), 3720 South Flower Street #301, Los Angeles, CA 90089-0702, (213) 821-5272 or upirb@usc.edu

Appendix H

Interview Letter to Participants

Dear Teachers:

My name is Acquillahs Muteti Mutie, and I am a doctoral candidate in the Rossier School of Education at University of Southern California. I am conducting a study as part of my doctoral dissertation that focuses on capturing the expertise of Algebra teachers that teach solving quadratic equations to 8^{th} and/or 9^{th} grade students in the K-12 education system.

The method used in this study is Cognitive Task Analysis (CTA). CTA is a knowledge elicitation and analysis technique that involves interviewing subject matter experts (SME) to identify the tacit action and decision steps experts use when performing a complex cognitive task.

You are invited to participate in this study and the information gathered will help in developing strategies to support instruction for solving quadratic equations. The interview should not take more than 90 minutes to complete. Participation in this study is voluntary. Your identity as a participant will be de-identified and will remain confidential at all times during and after the study.

If you have questions or would like to participate, please contact me at mutie@usc.edu

Thank you for your participation,
Acquillahs