Computing the Flip Distance between Triangulations*

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Abstract

Let $\mathcal{T}$ be a triangulation of a set $\mathcal{P}$ of n points in the plane, and let e be an edge shared by two triangles in $\mathcal{T}$ such that the quadrilateral Q formed by these two triangles is convex. A *flip* of e is the operation of replacing e by the other diagonal of Q to obtain a new triangulation of $\mathcal{P}$ from $\mathcal{T}$. The *flip distance* between two triangulations of $\mathcal{P}$ is the minimum number of flips needed to transform one triangulation into the other. The Flip Distance problem asks if the flip distance between two given triangulations of $\mathcal{P}$ is k, for some given $k \in \mathbb{N}$. It is a fundamental and a challenging problem.

We present an algorithm for the Flip Distance problem that runs in time $\mathcal{O}(n + k \cdot c^k)$, for a constant $c \leq 2 \cdot 14^{11}$, which implies that the problem is fixed-parameter tractable. We extend this result to triangulations of polygonal regions with holes, and to labeled triangulated graphs.

1 Introduction

Let $\mathcal{P}$ be a set of n points in the plane. A *triangulation* of $\mathcal{P}$ is a partitioning of the convex hull of $\mathcal{P}$ into triangles such that the set of vertices of the triangles in the triangulation is $\mathcal{P}$. Note that the convex hull of $\mathcal{P}$ may contain points of $\mathcal{P}$ in its interior.

A *flip* to an (interior) edge e in a triangulation of $\mathcal{P}$ is the operation of replacing e by the other diagonal of the quadrilateral formed by the two triangles that share e, provided that this quadrilateral is convex; otherwise, flipping e is not admissible (see Figure 1). The *flip distance* between two triangulations $\mathcal{T}_{initial}$ and $\mathcal{T}_{final}$ of $\mathcal{P}$ is the length of a shortest sequence of flips that transforms $\mathcal{T}_{initial}$ into $\mathcal{T}_{final}$. This distance is always well-defined and is $O(|\mathcal{P}|^2)$ (*e.g.*, see [4]). The Flip Distance problem is: Given two triangulation $\mathcal{T}_{initial}$ and $\mathcal{T}_{final}$ of $\mathcal{P}$, and $k \in \mathbb{N}$, decide if the flip distance between $\mathcal{T}_{initial}$ and $\mathcal{T}_{final}$ is k.

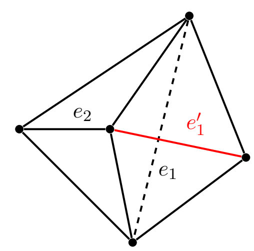

Figure 1: Flipping e_1 is admissible and produces e_1' while flipping e_2 is inadmissible.

Lubiw and Pathak [5] proved that the Flip Distance problem is $\mathcal{NP}$-complete. Simultaneously, and independently, the problem was shown to be $\mathcal{APX}$-hard by Pilz [8]. Very recently, Aichholzer et al. [1] showed the problem to be $\mathcal{NP}$-complete for triangulations of a simple polygon.

Sleator, Tarjan, and Thurston [9] showed in 1986 that the special case of the Flip Distance problem when $\mathcal{P}$ is in a convex position is equivalent to the problem of computing the rotation distance between two rooted binary trees. The question of whether this special case is NP-hard or not has remained open for 30 years. Cleary and St. John [3] showed that this special case is fixed-parameter tractable ($\mathcal{FPT}$) and gave a kernel of size $5k$ for the problem. The upper bound on the kernel size for the convex case was

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subsequently improved to $2k$ by Lucas [6].

Flip distance of triangulated graphs in combinatorial setting have also received a large share of attention (see [2] for a review). Wagner [11], in 1936, showed an $\mathcal{O}(n^2)$ bound on the flip distance of any two triangulated graphs. After a sequence of improvements on this upper bound, the current-best upper bound due to Mori *et al.* [7] is $6n-30$. For labeled triangulated graphs, it was shown that the flip distance between any two such graphs is $\mathcal{O}(n \lg n)$ [10], and this upper bound is tight. The (classical) complexity of computing the flip distance for both triangulated graphs and labeled triangulated graphs remains open.

2 Our Results

In this paper we present an $\mathcal{O}(n + k \cdot c^k)$-time algorithm ($c \leq 2 \cdot 14^{11}$) for the Flip Distance problem for triangulations of an arbitrary point-set in the plane, which shows that the problem is $\mathcal{FPT}$. Our result is a significant improvement over the $\mathcal{O}^*(k^k)$-time algorithm by Lucas [6] for the simpler convex case. While it is not very difficult to show that the Flip Distance problem is $\mathcal{FPT}$ based on some of the structural results in this paper, obtaining an $\mathcal{O}^*(c^k)$-time algorithm, for some constant c, is quite involved, and requires a deep understanding of the structure of the problem.

Our approach is as follows. For any solution to a given instance of the problem, we can define a directed acyclic graph (DAG), whose nodes are the flips in the solution, that captures the dependency relation among the flips. We show that any topological sorting of this DAG corresponds to a valid solution of the instance. The difficult part is how, without knowing the DAG, to navigate the triangulation and perform the flips in an order that corresponds to a topological sorting of the DAG. We present a *very simple* nondeterministic algorithm that performs a sequence of "flip/move"-type local actions in a triangulation, where each local action has constant-many choices. The key is to show that there exists such a sequence of actions that corresponds to a topological sorting of the DAG associated with a solution to the instance, and that the length of this sequence is linear in the number of nodes in the DAG. This will enable us to simulate the nondeterministic algorithm by an $\mathcal{O}^*(c^k)$-time deterministic algorithm. To achieve the above goal, we develop structural results that reveal some of the structural intricacies of this fundamental and challenging problem.

Our approach and techniques are very generic. Not only they seamlessly work for triangulations of any polygonal region, even with holes in its interior, but they can also be adapted to work for triangulated graphs in the combinatorial setting, all within the same $\mathcal{O}^*(c^k)$-time upper bound.

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