Art Gallery Theorems for Polyhypercubes

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Abstract

We consider variations of the original art gallery problem where the domain is a polyomino, a polycube, or a polyhypercube. An m-polyomino is the connected union of m unit squares called pixels, an m-polycube is the connected union of m unit cubes called voxels, and an m-polyhypercube is the connected union of m unit hypercubes in a d dimensional Euclidean space. In this paper we generalize and unify the known results about guarding polyominoes and polycubes and obtain simpler proofs. We also obtain new art gallery theorems for guarding polyhypercubes. This paper has been presented at the European Conference on Combinatorics, Graph Theory and Applications held in Bergen, Norway, August 31 – September 4, 2015. (see [10].)

1 Introduction

The original art gallery problem, posed by Klee in 1973, asks to find the minimum number of guards sufficient to cover any polygon with n vertices. The first solution to this problem was given by Chvátal [2], who proved that $\lfloor n/3 \rfloor$ guards are sometimes necessary, and always sufficient to cover a polygon with n vertices. Later Fisk [3] provided a shorter proof of Chvátal's theorem using an elegant graph coloring argument. Klee's art gallery problem has since grown into a significant area of study. Numerous *art gallery problems* have been proposed and studied with different restrictions placed on the shape of the galleries or the powers of the guards. (See the monograph by O'Rourke [8], and the survey by Shermer [11].)

In this paper we consider variations of the art gallery problem where the gallery is an *m-polyomino*, consisting of a connected union of m 1×1 unit squares called *pixels*, or an *m-polycube*, consisting of a connected union of m $1 \times 1 \times 1$ unit cubes called *voxels*. We will also consider higher dimensional cases where an m-polyhypercube is the connected union of m unit hypercubes in a d dimensional Euclidean space. Throughout this paper P_m denotes an m-polyomino when $d = 2$, an m-polycube when $d = 3$, or an m-polyhypercube when d is not specified. We say that a point $a \in P_m$ *covers* a point $b \in P_m$ provided $a = b$, or the line segment ab does not intersect the exterior of P_m. We say that a pixel/voxel A *covers* a point b, provided some point $a \in A$ covers b. A set of points $\mathcal{G}$ is called a *point guard set* for P_m if for every point $b \in P_m$ there is a point $a \in \mathcal{G}$ such that a covers b. A set of pixels/voxels $\mathcal{G}$ is called a *pixel/voxel guard set* for P_m if for every point $b \in P_m$ there is a pixel/voxel $A \in \mathcal{G}$ such that A covers b.

In [4], Irfan et al. show that $\lfloor \frac{m+1}{3} \rfloor$ point guards are always sufficient and sometimes necessary to cover any m-polyomino P_m, with $m \geq 2$. (See also Biedl et al. [1] for a detailed proof by case analysis.) Recently, Massberg [6] provided an alternate proof using perfect graphs. In [5], Iwerks claims that the same bound holds for polycubes and asks whether the result extends to polyhypercubes in $d \geq 4$ dimensions. In Section 2 we unify and generalize all these results proving that $\lfloor \frac{m+1}{3} \rfloor$ point guards are always sufficient and sometimes necessary to cover any m-polyhypercube P_m, with $m \geq 2$ in any dimension $d \geq 2$. While our lower bound example is a straight forward generalization of the examples in 2 and 3 dimensions, our argument for the upper bound is simpler than previous arguments, and works in every dimension.

In [9], Pinciu shows that $\lfloor \frac{m+1}{11} \rfloor + \lfloor \frac{m+5}{11} \rfloor + \lfloor \frac{m+9}{11} \rfloor$ pixel guards are always sufficient and sometimes necessary to cover an m-polyomino. Lower bounds and upper bounds for the number of voxels required to cover an m-polycube in 3D can be found in [5], but no sharp bounds are currently known when the dimension $d \geq 3$. In Section 3 we provide lower bounds for the number of pixel/voxels required to cover an m-polyhypercube in d dimensions. Our bounds are dependent on d, and we conjecture that they are sharp.

In Section 4 we provide upper bounds independent of d for the number of pixel/voxel guards required to cover an m-polyhypercube.

2 Point Guards in PolyHypercubes

In [10], we use Algorithm 1 to construct three point guard sets $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ such that $|\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_3| \leq m + 1$, and we obtain our main result:

Theorem 1 *For any m-polyhypercube P_m with $m \geq 2$*

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in $d \geq 2$ dimensions, $\lfloor \frac{m+1}{3} \rfloor$ point guards are always sufficient, and sometimes necessary to cover P_m.

Algorithm 1 Construction of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$.

procedure
  $\mathcal{G}_1 :=$the unique vertex of A_1 that is in V_1
  $\mathcal{G}_2 :=$the unique vertex of A_1 that is in V_2
  $\mathcal{G}_3 := \emptyset$
  for $i := 2$ **to** m **do**
    let A_j with $j < i$ be the parent of A_i in the construction of T_m.
    let u be the unique vertex of A_j in V_1.
    let v be the unique vertex of A_j in V_2.
    let w be the unique vertex of A_i in $V_1 \cup V_2$ such that $w \neq u$ and $w \neq v$.
    choose distinct integers $k, l \in \{1, 2, 3\}$ such that $u \in \mathcal{G}_k$ and $v \in \mathcal{G}_l$. (such integers might not be unique.)
    $\mathcal{G}_{6-k-l} := \mathcal{G}_{6-k-l} \cup \{w\}$
  end for
end procedure

3 Voxel Guards in PolyHypercubes: Bounds Dependent of d

The following theorem provides a lower bound for the number of pixel/voxel guards required to cover all m-polyhypercubes in d dimensions:

Theorem 2 *For any integer $d \geq 2$ and for any integer $m \geq 2$ there exists an m-polyhypercube P_m in d dimensions such that the minimum number of pixel/voxel guards necessary to cover P_m is:*

$$\sum_{i=1}^{2d-3} \left\lfloor \frac{m+3i-2}{6d-1} \right\rfloor + \left\lfloor \frac{m+6d-7}{6d-1} \right\rfloor + \left\lfloor \frac{m+6d-3}{6d-1} \right\rfloor.$$

We conjecture that the bound from Theorem 3 is sharp:

Conjecture 1 *For any m-polyhypercube P_m in dimension $d \geq 2$ with $m \geq 2$,*

$$\sum_{i=1}^{2d-3} \left\lfloor \frac{m+3i-2}{6d-1} \right\rfloor + \left\lfloor \frac{m+6d-7}{6d-1} \right\rfloor + \left\lfloor \frac{m+6d-3}{6d-1} \right\rfloor$$

pixel/voxel guards are always sufficient, and sometimes necessary to cover P_m.

The conjecture is true when $d = 2$, and the proof can be found in [9].

Theorem 3 *For any m-polyomino P_m with $m \geq 2$, $\lfloor \frac{m+1}{11} \rfloor + \lfloor \frac{m+5}{11} \rfloor + \lfloor \frac{m+9}{11} \rfloor$ pixel guards are always sufficient, and sometimes necessary to cover P_m.*

4 Voxel Guards in PolyHypercubes: Bounds Independent of d

The following theorem (see [10]) gives us an upperbound for the number of pixel/voxel guards that depends on the number of hypercubes only, and is independent of d:

Theorem 4 (a) *For any m-polyhypercube P_m with $m \geq 3$ in $d \geq 2$ dimensions, $\lfloor \frac{1}{3} m \rfloor$ pixel/voxel guards are always sufficient to cover P_m.*

(b) *For any positive real number $c < \frac{1}{3}$, there exist positive integers m and d and an m-polyhypercube P_m which requires more than $\lfloor cm \rfloor$ pixel/voxel guards.*

References

[1] T. Biedl, M. Irfan, J. Iwerks, J. Kim, J. Mitchell, The Art Gallery Theorem for Polyominoes: *Discrete Computational Geometry* **48**(3), pp 711–720, 2012.

[2] V. Chvátal, A combinatorial theorem in plane geometry: *J. Combin. Theory Ser. B* **18**, pp 39–41, 1975.

[3] S. Fisk, A short proof of Chvátal's watchman theorem: *J. Combin. Theory Ser. B* **24** pp 374, 1978.

[4] M. Irfan, J. Iwerks, J. Kim, J. Mitchell, Guarding Polyominoes: *19th Annual Workshop on Computational Geometry*, 2009.

[5] J. Iwerks, Guarding Polyforms: *Abstracts of the 1st Computational Geometry Young Researchers Forum*, 2012.

[6] J. Massberg, Perfect Graphs and Guarding Rectilinear Art Galleries: *Discrete Computational Geometry* **51**(3), pp 569–577, 2014.

[7] A. Meir, J. Moon, Relations between packing and covering number of a tree: *Pacific J. Math.* **61**, pp 225-233, 1975.

[8] J. O'Rourke: *Art Gallery Theorems.* Oxford University Press, 1987.

[9] V. Pinciu, Pixel Guards in Polyominoes: *Proceedings of the Cologne-Twente Workshop on Graphs and Combinatorial Optimizations CTW 2010*, pp 137–140, 2010.

[10] V. Pinciu, Guarding Polyominoes, Polycubes and Polyhypercubes: to appear in *Electronic Notes in Discrete Mathematics*, 8 pp (electronic), 2015.

[11] T.C. Shermer: Recent results in art gallery theorems: *Proc. IEEE* **80** pp 1384–1399, 1992.