On Dominating Sets for Pseudo-disks*

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1 Introduction

Problem statement. Consider a finite collection $\mathcal{P}$ of *pseudo-disks* in the plane, i.e., a collection of compact regions each bounded by a Jordan curve such that any two of their boundaries cross twice or not at all. The DOMINATING SET problem for $\mathcal{P}$ is to find the smallest subset $\mathcal{D} \subseteq \mathcal{P}$, such that each pseudo-disk of $\mathcal{P}$ is either in $\mathcal{D}$ or intersects a pseudo-disk in $\mathcal{D}$; the terminology comes from considering the intersection graph of $\mathcal{P}$.

Our result. Computing the minimum dominating set is in general NP-HARD [2]. We show that, for the case of pseudo-disks, in expected polynomial time one can achieve an approximation factor of $O(\log \text{OPT})$, where OPT is the size of the smallest dominating set. We define an appropriate range space consisting of subsets of $\mathcal{P}$ met by a common pseudo-disk from $\mathcal{P}$; see below. Our main contribution is proving that this range space has VC-dimension at most 4 (Theorem 2). The aforementioned approximation factor is then achieved by using the theory of ε-nets [3] and a standard LP-rounding mechanism [1], yielding

Theorem 1. *There is a randomized expected polynomial-time algorithm that, given a set $\mathcal{P}$ of pseudo-disks in the plane, computes a dominating set $\mathcal{D} \subseteq \mathcal{P}$ of size $O(\text{OPT} \log \text{OPT})$, where OPT is the size of the smallest such dominating set.*

2 Preliminaries

Let $\mathcal{P}$ be a set of n pseudo-disks in the plane. Without loss of generality, we assume that the pseudo-disks of $\mathcal{P}$ are *in general position*, that is, no point is incident to more than two pseudo-disk boundaries and whenever two meet, they properly cross.

Range spaces. A *range space* $(X, \mathcal{R})$ is a pair consisting of an underlying *universe* X of objects and a family $\mathcal{R}$ of subsets (*ranges*) of X. A set $K \subseteq X$ is *shattered* by $\mathcal{R}$ if, for every subset Z of K, $Z = K \cap r$ for some range $r \in \mathcal{R}$. The *VC-dimension* of a range space is the largest size of a (finite) shattered subset, if it exists.

We consider range spaces of the form $(\mathcal{P}, \mathcal{E})$, where each range $r \in \mathcal{E}$ is precisely the set of pseudo-disks of $\mathcal{P}$ intersected by a pseudo-disk of $\mathcal{P}$.

Theorem 2. *A range space $(\mathcal{P}, \mathcal{E})$ as defined above has VC-dimension of at most 4. This bound is the best possible.*

We devote the rest of this paper to proving Theorem 2, which in turn implies Theorem 1 as indicated above.

3 Main result

We start with a technical lemma [6]:

Lemma 3. *Let γ and γ' be arbitrary curves contained in pseudo-disks D and D', respectively. If the endpoints of γ lie outside D' and the endpoints of γ' lie outside D, then γ and γ' must cross an even number of times.*

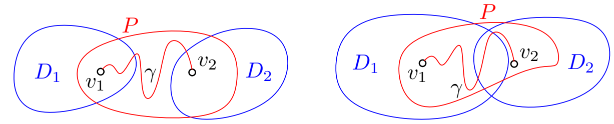
We say a set $K \subset \mathcal{P}$ is *well behaved* if every object in K is not completely covered by the remaining objects in K.

We begin by an auxiliary construction. Let $K \subset \mathcal{P}$ be well behaved. We construct a graph $H = H(K)$ whose vertices correspond to pseudo-disks of K and whose edges correspond to those pseudo-disks of $\mathcal{P}$ that meet exactly two objects of K. More precisely, we draw H as follows:

Vertices of H: As K is well behaved, each pseudo-disk $D \in K$ has a point $v(D)$ (which we fix arbitrarily) not covered by the union of other pseudo-disks in K. The points $\{v(D) : D \in K\}$ form the *vertex set* of H.

Edges of H: For two pseudo-disks $D_1, D_2 \in K$ such that there exists $P \in \mathcal{P}$ intersecting only D_1 and D_2 and nothing in $K \setminus \{D_1, D_2\}$, we fix the choice of P arbitrarily; P may belong to K. Let $v_1 = v(D_1)$ and $v_2 = v(D_2)$, and draw an *edge* $v_1v_2 \in H$, as described below. We call a connected portion of the edge inside P a *red* arc and such a portion outside P a *blue* arc. The edge v_1v_2 is drawn as a curve connecting v_1 to v_2 and consisting of a red arc and at most two blue arcs. In figures below we also use the convention of drawing pseudo-disks of K in blue and the "connecting" pseudo-disk P in red.

P contains both v_1 and v_2: Draw a red arc in P from v_1 to v_2. This forms the edge $v_1v_2 \in H$.



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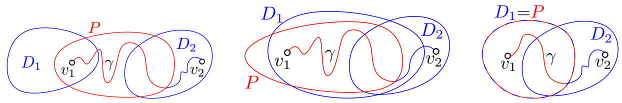
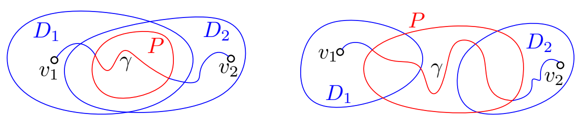
P contains only v_1: Draw a red arc in P that starts at v_1 and ends at the boundary of P inside D_2. Now draw a blue arc in D_2 that starts at this point, ends at v_2, and lies totally outside P. The union of these two arcs form the edge v_1v_2 of H.



P contains neither v_1 *nor* v_2: Draw a blue arc in D_1 that starts at v_1, ends at the boundary of P inside D_1, and otherwise stays outside of P. From this point, draw a red arc in P to a point of the boundary of P inside D_2 and from here, draw the final blue arc outside P in D_2 to the vertex v_2. The union of these three arcs constitute the edge v_1v_2.



Notice in all cases, we can divide an edge of $H(K)$ into at most three parts: one red and at most two blue. By construction, each arc, either red or blue, starts and ends in the same pseudo-disk.

Lemma 4. *If K is well behaved, then the graph $H = H(K)$ is planar.*

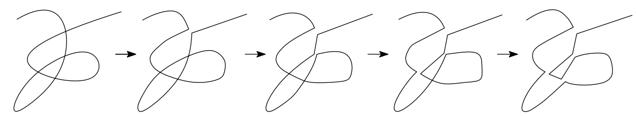
Proof. We will prove H is planar using the strong Hanani-Tutte theorem [5]. Consider two edges $e, e' \in H(K)$ that connect $v_1 = v(D_1)$ to $v_2 = v(D_2)$, and $v_3 = v(D_3)$ to $v_4 = v(D_4)$, respectively, with $D_1, D_2, D_3, D_4 \in K$ distinct. We will prove that e and e' intersect an even number of times, by considering their red and blue portions separately. Let $P_{12} \in \mathcal{P}$ be the pseudo-disk intersecting only D_1 and D_2, used to draw e, and let $P_{34} \in \mathcal{P}$ be the pseudo-disk intersecting only D_3 and D_4 corresponding to e'.

Red-Blue Intersections: Consider the red portion of e. This red arc is contained in P_{12} and therefore does not touch any pseudo-disk of K other than D_1, D_2. As the blue portions of e' lie inside D_3, D_4, this implies that the red arc of e does not touch the blue portions of e'. Symmetrically the red portion of e' cannot intersect the blue portions of e.

Red-Red Intersections: The red arc α along e lies entirely in P_{12} and has one endpoint in D_1 and the other in D_2. Similarly, the red arc α' along e' lies entirely in P_{34} and has one endpoint in D_3 and the other in D_4. As P_{12} does not intersect D_3 and D_4 and P_{34} does not intersect D_1 and D_2, the two endpoints of α do not lie in P_{34} and the two endpoints of α' do not lie in P_{12}. By Lemma 3, α and α' intersect an even number of times.

Blue-Blue Intersections: Consider blue arcs $\beta \subset e$ and $\beta' \subset e'$. The blue arc β starts, say, at vertex v_1 of pseudo-disk D_1 and ends at p in D_1 on the boundary of pseudo-disk P_{12}, and β' starts, say, at vertex v_3 of pseudo-disk D_3 and ends at p' in D_3 on the boundary of pseudo-disk P_{34}. By the construction of the vertices of H, $v_1 \notin D_3$ and $v_3 \notin D_1$. Now, p cannot be in D_3 because P_{12} meets only D_1 and D_2 and similarly p' cannot be in D_1. Hence, by Lemma 3 again we deduce that β and β' intersect an even number of times. The remaining cases are handled symmetrically.

There is a possibility that some edges of H are self-intersecting, but such intersections can be removed using standard methods: see for example [4] and the figure below. Thus any two edges of H that do not share an endpoint cross



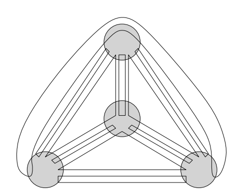
an even number of times, and therefore H is planar by the strong Hanani-Tutte theorem [5]. □

Proof of Theorem 2. If $(\mathcal{P}, \mathcal{E})$ has VC-dimension more than 4, $\mathcal{P}$ must contain a set K of size five that is shattered by $\mathcal{E}$.

The set K could not be shattered if it contained a pseudo-disk P that were totally covered by the union of other pseudo-disks in K because there would not exist a pseudo-disk in $\mathcal{P}$ which intersected only P and nothing else in K. Therefore, if K is shattered, it must be well behaved.

For a well-behaved set K, $H(K)$ is planar, by Lemma 4, and therefore has at most $3|K| - 6$ edges. On the other hand, since K is shattered, for every pair of pseudo-disks in it, there is a pseudo-disk in $\mathcal{P}$ meeting only this pair, so $\binom{|K|}{2} \leq 3|K| - 6$, implying $|K| \leq 4$ — a contradiction.

This means that the VC-dimension of $(\mathcal{P}, \mathcal{E})$ is at most 4. The figure below shows how to shatter a quadruple. The pseudo-disks meeting all or none of the four objects are not shown. The pseudo-disks meeting exactly one object are the objects themselves. □



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