A hom-tree lower bound for the Reeb graph interleaving distance

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Abstract

The Reeb graph has become an increasingly common tool in applied topology. Recently, several definitions of a metric on Reeb graphs have been proposed, including the interleaving distance [6]. Here we give a lower bound for the Reeb graph interleaving distance, by the related merge tree interleaving distance [8], through the newly defined hom-tree construction.

1 Introduction

The Reeb graph is a construction which originated in Morse theory to study a real valued function defined on a topological space. Given a function $\tilde{f} : \tilde{\mathbb{X}} \to \mathbb{R}$ defined on a space $\tilde{\mathbb{X}}$, we construct the Reeb graph by collapsing path connected components of level sets of $\tilde{\mathbb{X}}$ and denote the resulting quotient space by $\tilde{\mathbb{X}}/\sim_{\tilde{f}}$ or simply by $\mathbb{X}$. Because the Reeb graph inherits a function from the original function, we denote this by $f : \mathbb{X} \to \mathbb{R}$ or simply by $(\mathbb{X}, f)$. Thus a Reeb graph is a graph $\mathbb{X}$ equipped with a map $f : \mathbb{X} \to \mathbb{R}$ which is monotone restricted to edges. We say that the Reeb graphs $(\mathbb{X}, f)$ and $(\mathbb{Y}, g)$ are isomorphic if their underlying graphs are isomorphic and the level sets $f^{-1}(c)$ and $g^{-1}(c)$ are in bijection for each real number c. A special case of Reeb graphs are merge trees. A merge tree is a Reeb graph $(\mathbb{X}, f)$ where each vertex has at most one neighbor with higher function value. The Reeb graph has been used widely in applications; see [2] for a survey. Because real data has noise, we are interested in methods for comparison of Reeb graphs which provide stability results.

There are several methods that have already been developed for defining a measure of similarity between these structures, including the functional distortion distance [1] and the combinatorial edit distance [7]. In this paper, we focus on the interleaving distance [6], which is inspired by the persistence module interleaving distance [5] and its equivalent definition in terms of category theory [3]. Moreover, we use a version of the interleaving distance for merge trees [8], which we view as a subcategory of Reeb graphs, to construct a lower bound for the Reeb graph interleaving distance.

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2 Related work

Category theory is a branch of mathematics that studies morphisms between objects rather than just the objects themselves. A main tool of category theory are functors, which are maps between categories that send objects to objects and morphisms to morphisms. Reeb graphs can be identified with a particular kind of functors called cosheaves [6]. These cosheaves may be compared using an interleaving distance of the kind studied in [3], which works by constructing almost-isomorphisms between the cosheaves and measuring distance based on the parameter necessary for this construction. This metric on cosheaves can be pulled back to a metric on the original topological constructions, which is what we call the Reeb graph interleaving distance [6]. Via a similar process, there is also an interleaving distance on merge trees [8], which is a construction representing connected components of sublevel sets instead of the level sets used by the Reeb graph [4]. Both of these interleaving metrics come with bottleneck and L_∞ type stability results, making them especially useful for data analysis.

3 Our contribution

Inspired by these results, we prove that merge trees can be identified with some nice enough functors from $\mathbb{R}$ to **Set**, called $\mathbb{R}$-functors. These functors can be compared using an interleaving distance similar to [8]. Then, we pull back this metric for the comparison of merge trees, which we call the merge tree interleaving distance. In particular, we show that the merge tree interleaving distance coincides with the restriction of the Reeb graph interleaving distance to merge trees (considering merge trees as a subcategory of Reeb graphs). Next, we fix a (Reeb graph) test space $(\mathbb{E}, h)$ and define a categorical construction with respect to this space, where we associate to each Reeb graph $(\mathbb{X}, f)$ a specific merge tree called hom-tree, denoted $\mathcal{H}(\mathbb{X}, f)$. The hom-tree represents the evolution of all function preserving maps from the test space $(\mathbb{E}, h)$ as we smoth the Reeb graph $(\mathbb{X}, f)$. Finally, we prove that the merge tree interleaving distance of a pair of hom-trees (with same test space) is a lower bound for the Reeb graph interleaving distance.

4 Interleaving Distance

Given a Reeb graph $(\mathbb{X}, f)$ and a nonnegative real number a, let $\mathbb{X}_a$ denote the space $\mathbb{X} \times [-a, a]$, and define the a-smoothing of $(\mathbb{X}, f)$ as the Reeb graph of the function

$$f_a : \mathbb{X}_a \to \mathbb{R}, \text{ where } (x, t) \mapsto f(x) + t,$$

That is, the a–smoothing is the quotient space:

$$\mathcal{U}_a(\mathbb{X}, f) = \mathbb{X} \times [-a, a]/\sim_{f_a}$$

The idea of an interleaving metric between two Reeb graphs (or merge trees) $(\mathbb{X}, f)$ and $(\mathbb{Y}, g)$ is to measure how far they are from being isomorphic. This involves finding a pair of function preserving maps which use topological smoothings on the Reeb graphs and commute as much as possible; this is called an ε-interleaving where ε is the required amount of smoothing. Then one can define an interleaving distance on cosheaves (or merge trees), as follows:

$$d_I((\mathbb{X}, f), (\mathbb{Y}, g)) = \inf\{\varepsilon \geq 0 \mid \exists\ \varepsilon\text{-interleaving}\}.$$

5 The hom-tree construction

The hom-tree construction defines a merge tree for $(\mathbb{X}, f)$, through the following process. Fix a test space $(\mathbb{E}, h)$. This could be, for example, a single line with a monotone function supported on a finite range. First we define the hom-functor which assigns to each Reeb graph the $\mathbb{R}$-functor

$$a \mapsto \mathrm{Hom}_{\mathbf{Reeb}}((\mathbb{E}, h), \mathcal{U}_a(\mathbb{X}, f))$$

that sends a nonnegative real number $a \geq 0$ to the set of all function preserving maps from the test space to the α-smoothing $\mathcal{U}_a(\mathbb{X}, f)$ of $(\mathbb{X}, f)$. The hom-functor is a constructible $\mathbb{R}$-functor. By the categorification of merge trees as constructible $\mathbb{R}$-functors, there exists a unique (up to merge tree isomorphism) merge tree associated to the hom-functor which we call it the hom-tree of $(\mathbb{X}, f)$ and denote it simply by $\mathcal{H}(\mathbb{X}, f)$. The motivation for studying hom-trees is to give a lower bound for the Reeb graph interleaving distance by the merge tree interleaving distance between the associated hom-trees.

6 Example

We present the hom-tree construction, by a simple example given in Figure 1. Let $(\mathbb{X}, f)$ be a Reeb graph having a single hole of height 2ε as in the left of the figure. In this example, we use the test space $(\mathbb{E}, h)$ which is the graph with two vertices and one vertical edge of height equal to the range of f. Consider the associated hom-functor of $(\mathbb{X}, f)$. If $a < \epsilon$, there are exactly two function preserving maps from the test space $(\mathbb{E}, h)$ to

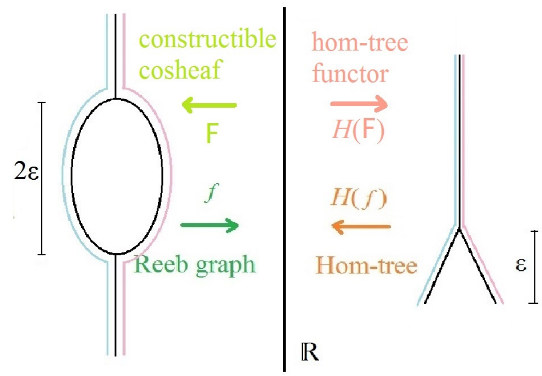

Figure 1: The hom-tree construction

the a-smoothing $\mathcal{U}_a(\mathbb{X}, f)$: the ones that map the test space $(\mathbb{E}, h)$ to the blue and purple colored curves after an a-smoothing, respectively. This is represented by the two legs of the hom-tree, shown in the figure at right. Else, if $a \geq \varepsilon$, the hole in the Reeb graph shrinks enough to disappear because of the smoothing process, and the images of the blue and purple curves coincide. Hence, we get only one function preserving map from the test space $(\mathbb{E}, h)$ to the a-smoothing $\mathcal{U}_a(\mathbb{X}, f)$, which is the identity map. This is represented by the straight line above the legs.

7 Comparing Reeb graphs with hom-trees

Our main result is the following theorem, proved using the machinery of category theory and the definitions of ε-interleavings on Reeb graphs and on merge trees.

Theorem 7.1 *The interleaving distance for a pair of Reeb graphs $(\mathbb{X}, f)$ and $(\mathbb{Y}, g)$, is bounded below by the merge tree interleaving distance of their corresponding hom-trees, i.e.*

$$\hat{d}_I(\mathcal{H}(\mathbb{X}, f), \mathcal{H}(\mathbb{Y}, g)) \leq d_I((\mathbb{X}, f), (\mathbb{Y}, g))$$

where $\hat{d}_I$ denotes the merge tree interleaving, d_I denotes Reeb graph interleaving, and both $\mathcal{H}(\mathbb{X}, f)$ and $\mathcal{H}(\mathbb{Y}, g)$ use the same test space $(\mathbb{E}, h)$ for the hom-tree construction.

8 Conclusion

We gave a categorification of merge trees and define a metric for comparison of those structures. Furthermore, we have defined a construction, the hom-tree, which can be used to define a lower bound on the Reeb graph interleaving distance by the merge tree interleaving distance with respect to a fixed test space. We expect that these new results will allow for improved understanding of the Reeb graph and merge tree interleaving distances.

References

[1] U. Bauer, X. Ge, and Y. Wang. Measuring distance between Reeb graphs. In *Annual Symposium on Computational Geometry*, page 464. ACM, 2014.

[2] S. Biasotti, D. Giorgi, M. Spagnuolo, and B. Falcidieno. Reeb graphs for shape analysis and applications. *Theoretical Computer Science: Computational Algebraic Geometry and Applications*, 392(13):5 – 22, 2008.

[3] P. Bubenik and J. Scott. Categorification of persistent homology. *Discrete & Computational Geometry*, 51(3):600–627, 2014.

[4] H. Carr, J. Snoeyink, and U. Axen. Computing contour trees in all dimensions. *Computational Geometry*, 24(2):75–94, 2003.

[5] F. Chazal, D. Cohen-Steiner, M. Glisse, L. J. Guibas, and S. Y. Oudot. Proximity of persistence modules and their diagrams. In *Annual Symposium on Computational Geometry*, pages 237–246. ACM, 2009.

[6] V. de Silva, E. Munch, and A. Patel. Categorification of Reeb graphs. *arXiv:1501.04147*, 2015.

[7] B. di Fabio and C. Landi. The edit distance for Reeb graphs of surfaces. *arXiv: 1411.1544*, 2014.

[8] D. Morozov, K. Beketayev, and G. Weber. Interleaving distance between merge trees. In *Proceedings of TopoInVis*, 2013.