Augmenting Planar Straight Line Graphs to 2-Edge-Connectivity

Hugo A. Akitaya* Jonathan Castello† Yauheniya Lahoda‡ Anika Rounds§
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Abstract

We show that every planar straight line graph (PSLG) with n vertices can be augmented to a 2-edge-connected PSLG with the addition of at most $\lfloor (4n-4)/3 \rfloor$ new edges. This bound is the best possible.

Edge-connectivity augmentation is a classic problem in combinatorial optimization motivated by applications in fault-tolerant network design. Given an undirected graph $G = (V, E)$ and a number $\tau \in \mathbb{N}$, we want to find a set F of new edges of minimum cardinality such that $G' = (V, E \cup F)$ is τ-edge-connected. In this note, we consider edge-connectivity augmentation for planar straight line graphs (PSLG) with n vertices in general position (no three collinear vertices).

Every graph with $t \in \mathbb{N}$ components can be augmented into a connected graph with the addition of $t-1$ new edges. Every PSLG with n vertices can be augmented to a connected PSLG (encompassing graph) with at most $n-1$ new edges. Every connected PSLG on n vertices can be augmented to a 2-edge-connected PSLG with at most $\lfloor (2n-2)/3 \rfloor$ new edges [3]. Both bounds are the best possible. The combination of the two bounds implies that every PSLG on n vertices can be augmented to 2-edge-connectivity with the addition of at most $\lfloor 5(n-1)/3 \rfloor$ new edges. However, this bound is not tight. We derive a better bound and show the following.

Theorem 1. *Every PSLG with $n \geq 3$ vertices can be augmented to a 2-edge-connected PSLG with the addition of at most $\lfloor (4n-4)/3 \rfloor$ new edges. This bound is the best possible.*

The upper bound in Theorem 1 is attained for a triangulation on $k \geq 3$ vertices, with an isolated vertex placed in each of the $2k-5$ bounded faces and 3 vertices in the outer face that pairwise do not see each other (that is, $n = k + (2k-5) + 3 = 3k-2$). The proof of the upper bound is constructive and distinguishes between two cases depending on the number of components in the graph. Due to space limitation, we give an outline of the proof here.

Let G be a PSLG on $n \geq 3$ vertices in general position. Let c be the number of components in G. In the first case $c \leq \lfloor (2n+1)/3 \rfloor$, and we augment G to a 2-edge-connected PSLG as follows: first use $c-1$ new edges to obtain a connected PSLG, and then use $\lfloor (2n-2)/3 \rfloor$ edges to make it 2-edge-connected [3]. The total number of new edges is at most

$$(c-1) + \left\lfloor \frac{2n-2}{3} \right\rfloor \leq \left\lfloor \frac{2n+1}{3} \right\rfloor - 1 + \left\lfloor \frac{2n-2}{3} \right\rfloor \leq \left\lfloor \frac{4n-4}{3} \right\rfloor . \tag{1}$$

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In the second case, when $c \geq \lfloor (2n+1)/3 \rfloor + 1 = \lfloor (2n+4)/3 \rfloor$, we develop an augmentation algorithm that uses a convex subdivision of G. A convex subdivision H is obtained from G by successively shooting rays from the reflex vertices of all nonsingleton components of G, similar to [2]. The isolated vertices of G lie in the interiors of the convex cells of H. For every convex subdivision H constructed in this way, we derive an upper bound for the number of cells h.

Lemma 2. *Let G be a PSLG with n vertices, b bridges, and c components. Then every convex subdivision of G has at most $h \leq 2n - 2c - b + 1$ cells.*

We augment G successively with new edges, and we always denote by G' the *current* graph. Graph G' is a planar straight line multigraph (PSLMG). Let $T \subseteq G'$ denote the set of nonsingleton connected components in G'.

Our augmentation algorithm works as follows:

1. Construct a convex subdivision H of G. Let $C = \{C_i : i = 1 \dots h\}$ be the set of convex cells. Compute T.

2. For each cell $C_i \in C$: a) for each nonsingleton component adjacent to C_i select an arbitrary vertex incident to C_i; b) connect the selected vertices and singleton vertices in the cell C_i into a simple polygon; c) recompute T.

3. Replace each bridge of G' by a double edge.

4. Transform the multigraph G' into a simple graph.

In step 2 we add $c + h - 1$ edges. Since we do not create any new bridges in step 2, we add b edges in step 3. The total number of new edges e' added is $e' \leq c + h - 1 + b$. By Lemma 2, since $c \geq \lfloor (2n+4)/3 \rfloor$, we obtain:

$$e' \leq c + h - 1 + b \leq 2n - c \leq 2n - \left\lfloor \frac{2n+4}{3} \right\rfloor \leq \left\lfloor \frac{4n-4}{3} \right\rfloor . \quad (2)$$

In step 4 we can transform the 2-edge-connected multigraph G' into a 2-edge-connected simple graph without increasing the number of edges by Lemma 3.

Lemma 3. *[1] Let G' be a 2-edge-connected PSLMG and let e be a double edge in G'. Then we can obtain a 2-edge-connected PSLMG from G' by decrementing the multiplicity of e by one and adding at most one new edge of multiplicity 1.*

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Abstract

We show that every planar straight line graph (PSLG) with n vertices can be augmented to a 2-edge-connected PSLG by adding at most $\lfloor 4(n-1))/3\rfloor$ new edges. This bound is tight.

1 Introduction

Edge-connectivity augmentation is a classic problem in combinatorial optimization motivated by applications in fault-tolerant network design. Given an undirected graph $G=(V,E)$ and an integer $\tau \in \mathbb{N}$, find the a set F of new edges of minimum cardinality such that $G'=(V,E\cup F)$ is τ-edge-connected. For *abstract* graphs, the problem admits a polynomial-time solution for every $\tau \in \mathbb{N}$ [12]. Moreover, the Successive Augmentation Property holds [5, 8], that is, if G is ℓ-edge-connected, then there exists a sequence $G=G_0,G_1,\ldots$ of supergraphs of G such that G_i is a subgraph of G_j for all $i<j$ and G_i is an optimal $(\ell+i)$-edge-connected augmentation of G for all $i\in\mathbb{N}$.

In this note, we consider edge-connectivity augmentation for planar straight line graphs (PSLGs) with n vertices in general position (no three collinear vertices). It is NP-hard to decide whether a given PSLG can be augmented to τ-edge-connectivity by adding at most k new edges, for $\tau = 2,\ldots,5$ [9]. By Euler's polyhedron formula, every planar graph has a vertex of degree at most 5, this is an upper bound on the maximum edge-connectivity. Every triangulation on n noncollinear points is 2-edge-connected, therefore every PSLG can be augmented to 2-edge-connectivity. However, some point sets do not admit 3-edge-connected triangulations [3], and so not all PSLGs are 3-edge-augmentable.

Every graph with $t\in\mathbb{N}$ components can be augmented into a connected graph with the addition of $t-1$ new edges. Every PSLG with n vertices can be augmented to a connected PSGL (encompassing graph) with at most $n-1$ new edges [4]. Every connected PSLG on n vertices can be augmented to 2-edge-connectivity with at most $\lfloor(2n-2)/3\rfloor$ new edges [11]. Both bounds are the best possible. The combination of the two bounds implies that every PSLG on n vertices can be augmented to 2-edge-connectivity with the addition of at most $\lfloor 5(n-1)/3\rfloor$ new edges. However, this bound is not tight. Answering a question posted in [2, 7], we show the following.

Theorem 1. *Every PSLG G with $n\geq 3$ vertices can be augmented to a 2-edge-connected PSLG with the addition of at most $\lfloor(4n-4)/3\rfloor$ new edges. This bound is the best possible.*

The lower bound construction is a triangulation on k vertices, with an isolated vertex placed in each of $2k-5$ bounded faces and 3 vertices in the outer face that pairwise do not see each other (Fig. 1). Each singletons requires two new edges for 2-edge-connectivity, that is, at least $4k-4$ new edges are needed. In terms of $n=3k-2$, the graph requires $4(k-1)=\lfloor 4(n-1)/3\rfloor$ new edges.

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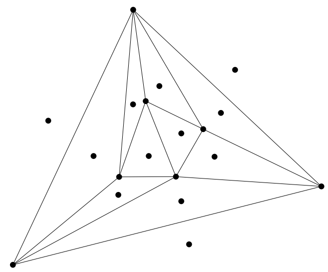


Figure 1: Graph on n=19 vertices. Each of $\lfloor (2n-2)/3 \rfloor = 12$ singletons requires two edges to increase edge connectivity of the graph to two.

2 Convex Subdivisions

Our proof for Theorem 1 is constructive, and we describe an algorithm for augmenting a given PSLG in Section 4. The main tool for our algorithm is a convex subdivision [3, 4, 7]. Let $G = (V, E)$ be a PSLG with no isolated vertices. A convex subdivision $H = (V_H, E_H)$ is obtained from G by adding new edges and vertices, and by subdividing some of the edges of G, such that all bounded faces of H are convex. First augment G with the vertices and edges of a bounding box of G; and then successively shoot rays from each reflex vertex of G in the direction of any one edge incident to the reflex vertex (Fig. 2b). Each ray extends until it hits the bounding box, an edge or G, or a previous ray. Let V_H be the set of all vertices of G, the bounding box, and the endpoints of the ray segments. Let E_H be the set of maximal line segments along the edges of G, along the bounding box, and along the the ray segments between consecutive vertices in V_H. By construction, H is a PSLG. The bounded faces of H (called *cells*) are convex, and they tile the interior of the bounding box. Note, however, that s convex subdivision H is not unique: it depends on the direction of the rays, and the order in which the rays are shot.

Lemma 2. *Let H be a convex subdivision of a PSLG G constructed by the above procedure. Let $C \subset H$ be a cycle in H other than the bounding box. Then C passes through a vertex of G.*

Proof. We partially orientation the edges of H. Along each ray emitted by a reflex vertex v, orient all edges of H away from v; and leave all other edges of H undirected (i.e., the edges of G and the bounding box are undirected). Note that the outdegree of every vertex in H is at most one.

Let C be a cycle C in H other than the bounding box. Since H is a plane graph, we may assume that C is a simple cycle. If C is contained in G, then it passes through a vertex of G. So we may assume that C contains a directed edge. Let $\gamma = (p_1, \ldots, p_k)$ be a maximal *directed* path along C. First notice that $p_1 \neq p_k$ (that is, γ does not cover C). Suppose, to the contrary that $p_1 = p_k$. Then at every vertex of the cycle C, one ray hits another ray. Since no three vertices of G are collinear, none of the vertices of C is a vertex of G. However, the rays were created successively, and the first ray along C cannot hit any previous ray along the cycle C. This contradiction proves our claim.

The first vertex of γ is incident to some edge p_0p_1 of C. The edge p_0p_1 is undirected, by the maximality of γ and by the fact that the maximal out-degree is one in H. Since the ray containing p_1p_2 does not cross any edges of G, and does not pass through any vertices of G, it must be emitted by p_1. Consequently, p_1 is a reflex vertex in C. $\square$

A cell of a convex subdivision may be incident to vertices from one or more connected components of G. If a cell is adjacent to vertices of only one component, then we say that is is *adjacent to one component*; otherwise we say that this cell is *shared* or *adjacent to several components.*

Lemma 3. *Let G be a PSLG with no singletons and $t \geq 2$ non-singleton components, and let H be a convex subdivision of G. If S is the union of less than t components of G, then S adjacent to a cell shared with some component of $G - S$.*

Proof. We define the region R as the union of the closures of all cells incident to some vertices S (region R need not be connected, and it may have holes). Let C be a cycle in the boundary of R. By Lemma 2 is passes through a vertex v of G. By construction, v cannot be in S, hence it is a vertex of $G - S$. Then vertex is v is incident to a cell in the interior of R, which in turn is incident to a vertex in S. □

The following properties of a convex subdivision will be instrumental for controlling the number of new edges in our augmentation algorithm in Section 4.

Lemma 4. *For every PSLG G with f bounded faces, r reflex vertices, and t non-singleton components, every convex subdivision of G has $h = f + r - t + 1$ cells.*

Proof. Let B be the bounding box of H and let $H_0 = G \cup B$. Let H_i be the graph obtained by shooting a ray r_i in H_{i-1}, $i = 1, \ldots, r$. Then $H = H_r$. Initially, H_0 has $f + 1$ bounded faces. The bounded faces of H_0 may have holes. In fact, each connected component of G forms a hole in a face of H_0, hence the faces of H_0 have a total of t holes.

Each ray r_i either decreases the number of holes of H_{i-1} by one or increases the number of bounded faces H_{i-1} by one. None of the faces of H_r have any holes, since they are convex. Consequently, $r - t$ rays each increases the number of bounded faces by one. It follows that H has $f + 1 + r - t = f + r - t + 1$ cells. □

Lemma 4 implies that every convex subdivision of a PSLG G has the same number of cells. Souvaine and Tóth [10] showed that $f + r \leq 2n - 2$ for every PSLG on n vertices. Al-Jubeh et al. [3] extended this result to the following.

Lemma 5 (Corollary in [3])**.** *Let G be a PSLG with b bridges, s singleton, t non-singleton components, f bounded faces, n vertices, r of which are reflex. Then*

$$b + t + f + r + 2s \leq 2n,$$

with equality if and only if G is a forest in which all vertices are reflex.

The combination of Lemmata 4 and 5 yield the following.

Corollary 6. *Let G be a PSLG with n vertices, b bridges, s singletons, and t non-singleton components. Then every convex subdivision of G has at most $h \leq 2n - 2s - b - 2t + 1$ cells.*

3 Preliminaries

The following two results are used in the proof of correctness of the augmentation algorithm. The variant of Lemma 7 can be found in [6]. Here we give a short proof applicable to our situation.

Lemma 7. *Let $G = (V, E)$ be a connected PSLG that lies in a convex closed polygon P, and let w be a point on the boundary of P. Then w sees an entire edge of G.*

Proof. Let $(v_1, \ldots, v_k)$ be the sequence of vertices of G visible from w in counterclockwise order. Since G is connected, w sees part of some edge e_i in each wedge $\angle(w, v_i, v_{i+1})$ for $1 \leq i \leq k-1$. If w does not see an entire edge e_i, then e_i is partially occluded by vertex v_i or v_{i+1} (since we assume general position of vertices in G, a vertex cannot occlude an entire edge). However, v_1 and v_k cannot occlude any edge of G, since wv_1 and wv_k are tangents from w to $\mathrm{ch}(G)$; and each vertex v_i, $2 \leq i \leq k-1$, can occlude at most one of e_{i-1} and e_i. By the pigeonhole principle, one of the edges e_i is not occluded, and so it is fully visible from w. □

In some intermediate steps of our augmentation algorithm we allow multigraphs. Lemma 8 shows that the edge count derived for multigraphs are applicable to simple graphs.

Lemma 8 ([1])**.** *Let $G = G_1 \cup G_2$ be the a multigraph formed by two simple PSLGs such that G is 2-edge connected. Let u and v be two vertices of G that are joined by an edge of G_1 and an edge of G_2, e_1 and e_1, respectively. Then we can either eliminate e_2 or substitute it by another f such that $G - e_2$ or $G - e_2 + f$ is 2-edge-connected. In the second case, f can be chosen such that it does not create a new double edge.*

Theorem 9 ([11])**.** *Every connected PSLG with $n \geq 3$ vertices in general position in the plane can be augmented to a 2-edge-connected PSLG with at most $\lfloor (2n-2)/3 \rfloor$ new edges. This bound is the best possible.*

4 Proof of Theorem 1

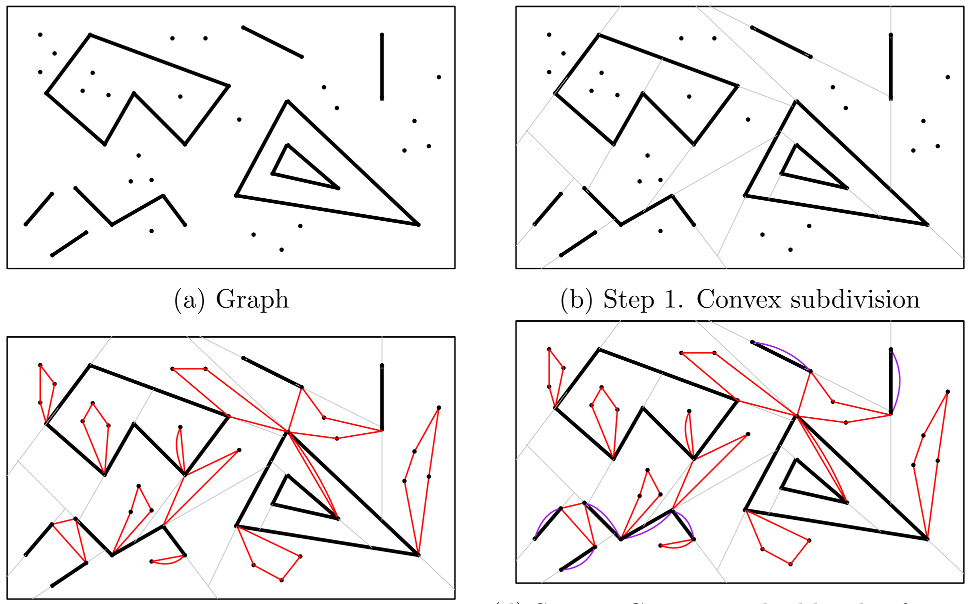
Proof. Let G be a PSLG on $n \geq 3$ vertices in general positon. Let s be the number of singletons and t the number of non-singleton components.

If the number of components in G is $s + t \leq \lfloor (2n+1)/3 \rfloor$, then we augment G to 2-edge connectivity as follows: First use $s + t - 1$ new edges to make the graph connected, and then use $\lfloor (2n-2)/3 \rfloor$ edges to make it 2-edge-connected by Theorem 9. The total number of new edges is at most

$$(s+t-1) + \left\lfloor \frac{2n-2}{3} \right\rfloor \leq \left\lfloor \frac{2n+1}{3} \right\rfloor - 1 + \left\lfloor \frac{2n-2}{3} \right\rfloor \leq \left\lfloor \frac{4n-4}{3} \right\rfloor.$$

In the remainder of the proof, we assume that $s + t \geq \lfloor (2n+1)/3 \rfloor + 1 = \lfloor (2n+4)/3 \rfloor$. We shall augment G successively with new edges, and we always denote by G' the *current* graph. We allow double edges, which will be removed at the end of the algorithm suing Lemma 8.

Notation. Let H denote a convex subdivision of G, and let h be the number of cells in H. Let $C = \{C_i | i = 1 \ldots h\}$ denote a set of cells of H. Let S_i denote a set of singletons in cell C_i. Let $T \subseteq G'$ denote a set of non-singleton connected components in G'. Let $c(i) \subseteq T$, $i = 1, \ldots, h$, denote a number of components incident to cell C_i. Let G_{ik}, $k = 1, \ldots, c(i)$ denote the k-th component adjacent to cell C_i, and let w_{ik} be an arbitrary vertex of G_{ik} incident to C_i. Our algorithm will create a simple polygon P_i in the each nonempty cell C_i, and we denote by $E(P_i)$ the set of edges of P_i. Let $\mathrm{Br}(G)$ and $\mathrm{Br}(G')$ denote sets of bridges of graphs G and G' respectively. Let b be the number of bridges in G, that is, $b = |\mathrm{Br}(G)|$.



(a) Graph

(b) Step 1. Convex subdivision

(c) Step 2. Processing cells

(d) Step 3. Creating a double edge for each bridge

Figure 2: Steps of the algorithm

Augment edges in the graph G according to the following algorithm:

1. Construct a convex subdivision H of G. Let C be the set of convex cells. Compute the T, $c(i)$, G_{ik} and w_{ik} for all $i = 1, \ldots, h$ and $k = 1, \ldots, c(i)$.
2. For each cell $C_i \in C$:

   (a) If $S_i = \emptyset$ and $c(i) \geq 2$,

      i. add an edge $w_{i1}w_{i2}$;

      ii. double the edge $w_{i1}w_{i2}$ to create a cycle P_i.

   (b) If $|S_i| = 1$,

      i. connect the unique vertex $v \in S_i$ to vertex w_{i1} with an edge vw_{i1};

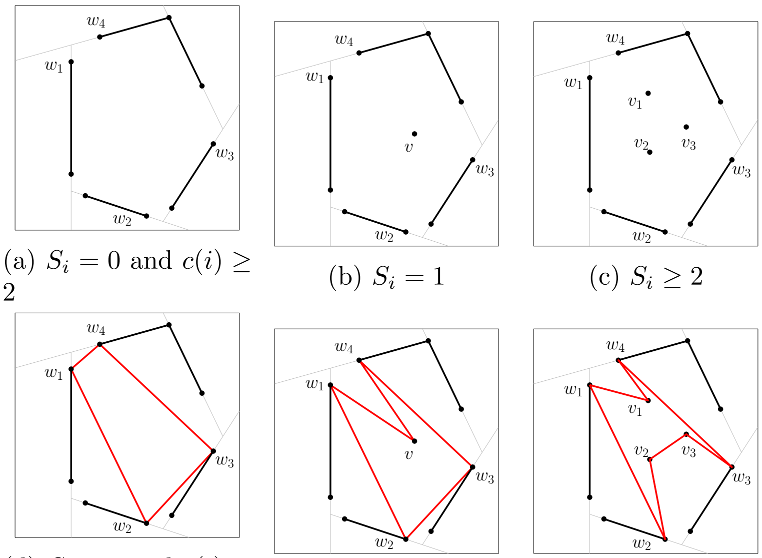
      ii. double the edge vw_{i1} to create a cycle P_i.

   (c) If $|S_i| \geq 2$,

      i. connect the vertices of S_i into a simple polygon P_i (if $S_i = 2$, we create a double edge);

      ii. pick an edge $ab \in E(P_i)$ entirely visible from w_{i1}, and replace edge ab with a new path (a, w_{i1}, b).

   (d) For each vertex $w_{ik} \notin P_i$, pick an edge $ab \in E(P_i)$ entirely visible from w_{ik}, and replace edge ab with a new path (a, w_{i1}, b).



(a) $S_i = 0$ and $c(i) \geq 2$ (b) $S_i = 1$ (c) $S_i \geq 2$

(d) $S_i = 0$ and $c(i) \geq 2$, Step 2d (e) $S_i = 1$, Step 2d (f) $S_i \geq 2$, Step 2d

Figure 3: Step 2 of the algorithm

   (e) Update T, $c(i)$, G_{ik} and w_{ik} for all $i = 1, \ldots, h$ and $k = 1, \ldots, c(i)$.

3. Replace each bridge of G' by a double edge.
4. Apply Lemma 8 successively to each double edge to obtain a simple graph.

This completes the description of the algorithm.

Proof of correctness. We fist show that each step of the algorithm is valid. In step 2, all new edges lie within a cell C_i. In step 2(b)i we connect the singleton v to a vertex w_{i1} of an incident component. Lemma 2 guarantees existence of a vertex w_{i1} adjacent to cell the C_i. In step 2(c)i we connect all singletons in C_i by a simple polygon P_i, for example, P_i can be the Euclidean TSP of the points. In steps 2(c)ii and step 2d, we incrementally expand an existing polygon by replacing an edge ab with two new edges (a, w, b), where ab is entirely visible from w. The existence of an edge ab is guaranteed by Lemma 7.

We now argue that the final graph G' is 2-edge-connected. After step 2d, all components incident to a cell C_i (both singletons and non-singleton components) merge into one component. Each iteration of step 2 reduces the total number of components by $c(i) - 1$. The graph G' becomes connected after step 2, since there are not isolated components in T by Lemma 3.

In step 3, we create a double edge for each bridge in $\mathrm{Br}(G')$. Hence after step 3, each edge of G' is part of some cycle. A (multi-)graph G' is 2-edge-connected if and only if it is connected and each edge is a part of some cycle. This proves that G' is a 2-edge-connected multigraph after step 3. In step 4, we eliminate all double edges using Lemma 8, and obtain a 2-edge-connected simple graph.

Bounding the number of new edges. In steps 2(b)i and 2(c)i, we add $|S_i|$ new edges in each cell C_i; the total number of such edges is $s = |S| = \sum_{i=1}^{h} |S_i|$. In steps 2(a)i, 2(b)ii, and 2(c)ii, we add one edge in each cell C_i; the total number of such edges is h. In steps 2(a)ii and 2d, we increase the number of edges by one and decrease the total number of components of G' by one; the number of steps edges is $t-1$. The total number of new edges added in step 2 is $s+h+t-1$.

In step 3 we create a double edge for each bridge in $\mathrm{Br}(G')$. Since we have added a cycle in each cell C_i, no new bridges have been created, and we have $\mathrm{Br}(G') = \mathrm{Br}(G)$ after step 2. Consequently, we add b new edges in step 3. By Lemma 8, step 4 does not increase the number of edges.

Altogether, the number of new edges is at most $e' \leq s+h+(t-1)+b$. By Corollary 6, we have $h \leq 2n-2s-b-2t+1$. It follows that

$$e' \leq s+h+(t-1)+b \leq s+(2n-2s-b-2t+1)+(t-1)+b \leq 2n-(s+t).$$

Since, $s+t \geq \lfloor (2n+4)/3 \rfloor$, we have

$$e' \leq 2n-(s+t) \leq 2n - \left\lfloor \frac{2n+4}{3} \right\rfloor \leq \left\lfloor \frac{4n-4}{3} \right\rfloor,$$

as claimed. □

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