Nov 13

Shortest path problem

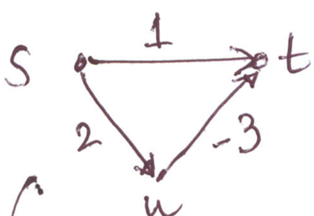
Input: (•) Directed graph $G=(V,E)$, $\forall e \in E$ has cost c_e ($c_e<0$ is allowed)

BUT no −ve cycle

(•) $t \in V$

Output: $\forall s \in V$, output a shortest s-t path.

Attempt 1: Run Dijkstra for each $s \in V$



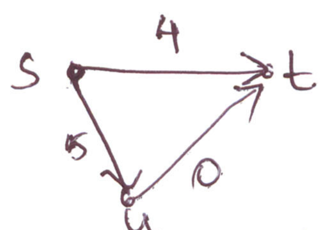
→ start Dijkstra on s

it'll pick s,t as shortest s-t path.

BUT s,u,t is shortest s-t path

Attempt 2: Add a large enough +ve number to each edges & then run Dijkstra on the new instance

+3 to all edgs



→ In the graph s,t is the shortest path.

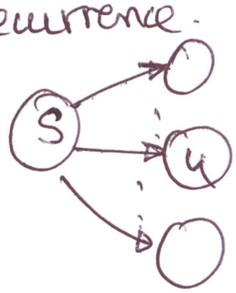
→ No known greedy / divide & conquer algo

ASSUME: Only interested in cost of shortest s-t paths.

Attempt 3: $OPT(s)$ = cost of a shortest s-t path. $\forall s \in V$

① Poly many subproblem? ✓ n subproblems

② Recurrence.



If a shortest s-t path starts with edge (s,u)

$$OPT(s) = c_{s,u} + OPT(u)$$

$$OPT(s) = \min_{w:\,(s,w)\in E} \{ c_{s,w} + OPT(w) \}$$

$s \neq t$

$OPT(t) = 0$

3) Ordering among subproblems? ✗

$$OPT(S) = 2 + OPT(u)$$
$$OPT(u) = \min\{3 + OPT(t), -1 + \cancel{OPT(u)}\ OPT(S)\}$$

Issue: $OPT(S)$ depends on $OPT(u)$
$OPT(u)$ ———— $OPT(S)$ } → no hope for a total ordering.

Solution: Introduce an implicit parameter in sub-problems

Attempt 4: $OPT(s, E')$ → cost of shortest s-t path only using edges in E'

↑ $E' \subseteq E$

Recursion:

If a shortest s-t path uses (s,u)

$$OPT(s,E) = c_{s,u} + OPT(u, E \setminus \{(s,u)\})$$

more generally:
$$OPT(s,E) = \min_{w:(s,w)\in E} \{c_{s,w} + OPT(w, E\setminus\{(s,w)\})\}$$

3) Ordering? increasing order of $|E'|$

T) How many sub-problems? $n \cdot 2^m$ → exponential!