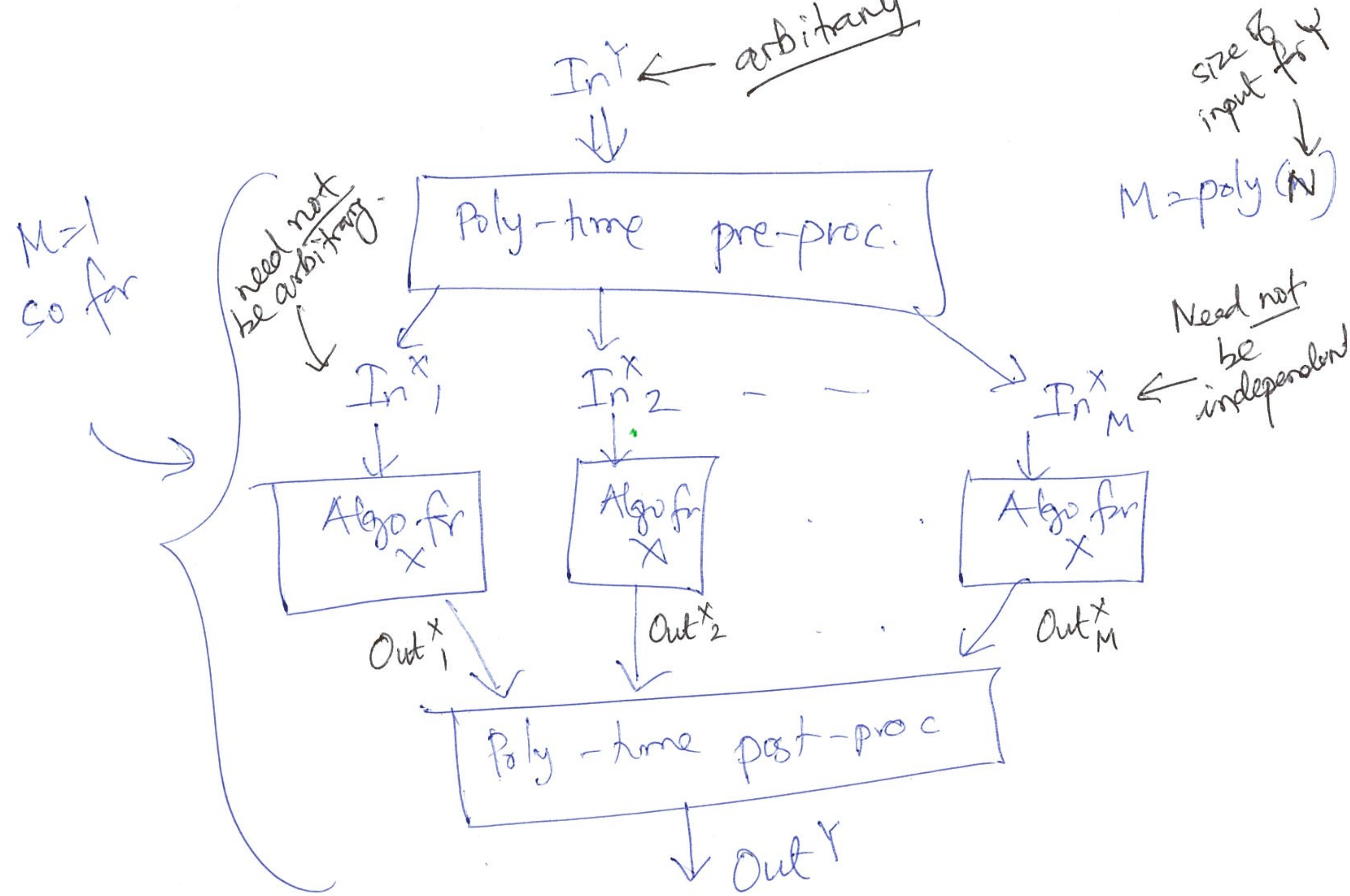
Nov 20

$$Y \leq_p X$$

$\rightarrow$ Y is poly time reducible <u>to</u> X

$\equiv$ poly time redux <u>from</u> Y <u>to</u> X

Solve $In^Y \dashrightarrow Out^Y$



<u>Ex:</u>

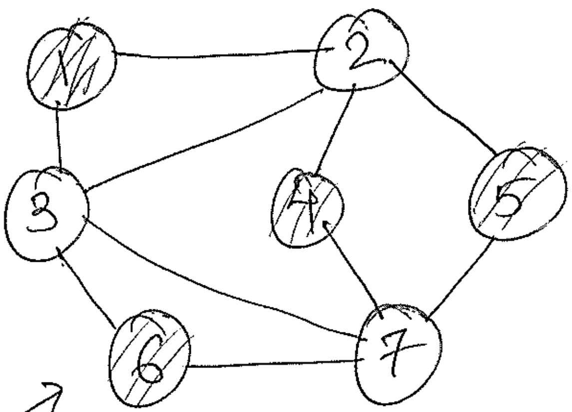
HW 2 Q2 $\leq_p$ Stable matching

(M = 1)

Going forward : ONLY consider problems w/ Boolean output.

Problem: Independent set. $G = (V, E)$

$S \subseteq V$ is an independent set if NO edges $\exists$ between nodes in S.



G_0

Eg: $\{1,4\}$ ✓ $\{6\}$ ✓

$\{3,7\}$ X

$\{1,4,7\}$ X

$\{1,4,5,6\}$ ✓ $\{3,4,5\}$ ✓

Problem:

Input: $G = (V, E)$; $0 \leq k \leq n$

Output: True iff $\exists$ an IS of size $\geq k$.

Ex: G_0, 2 : True G_0, 4 : ✓ G_0, 5 : X

Problem 2: Vertex Cover

$G = (V, E)$; $C \subseteq V$ is a vertex cover if ALL edges in E have ≥ 1 end-point in C

Ex: G_0 : $\{1,2,3,4,5,6,7\}$ ✓

$\{1,2,3,4,5,6\}$ ✓ $\Rightarrow$ Any subset of size $n-1$ is a VC.

$\{1,2,6,7\}$ ✓

$\{2,3,7\}$ ✓ $\{1,7\}$ X

Problem: Input: $G = (V, E)$ $\quad 0 \le k \le n$

Output: True iff $\exists$ a VC of size $\le k$.

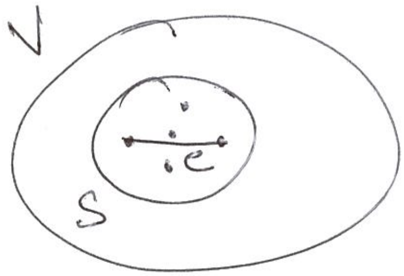
Ex: $G_0; 6$ ✓ $\quad G_0; 3$ ✓ $\quad G_0; 2$ ✗

THM: (1) $IS \le_p VC$

(2) $VC \le_p IS$

Lemma: $G = (V, E) \Rightarrow S \subseteq V$ is an IS iff $V \setminus S$ is a VC

Pf: $\Rightarrow$: Let S be an IS.



Assume $V \setminus S$ is not a vertex cover.

$\Rightarrow \exists$ an edge e with no end point in $V \setminus S$

$\Rightarrow$ Both end points of e are in S

$\Rightarrow S$ is not an IS $\to$ ✗

$\Leftarrow$:

Let $V \setminus S$ be a vertex cover BUT S is not an IS

$\Rightarrow \exists$ an edge "inside" S

$\Rightarrow V \setminus S$ is not a VC ✗ $\blacksquare$

COR: G has an IS of size $\ge k \iff G$ has a VC of size $\le n-k$.

$\Rightarrow IS \le_p VC$

Pf: Input: (G, k) for IS

$\Rightarrow$ $(G, n-k)$ for VC

$$(G, k) \xrightarrow{(G, n-k)} \boxed{\begin{matrix}\text{Algo for}\\ \text{VC}\end{matrix}} \xrightarrow{T/F}$$

$\Rightarrow$ Similarly can show VC $\leq_p$ IS

Satisfiability / SAT problem

"the" NP problem in "practice"

Variables: $X = \{x_1, \ldots, x_n\}$ (each $x_i \in \{0,1\}$ $\equiv \in \{F, T\}$)

term/literal x_i, $\bar{x}_i$

Clause: OR or disjunction of literals

$t_1 \vee t_2 \vee \cdots \vee t_e$ eg $n = 3$

$x_1 \vee \bar{x}_2$

A SAT formula: Conjunction / AND of clauses

$C_1, \ldots, C_m \equiv C_1 \wedge C_2 \wedge \cdots \wedge C_m$

Ex: $(x_1 \vee \bar{x}_2)$, $(\bar{x}_1 \vee \bar{x}_3)$, $(x_2 \vee \bar{x}_3)$

An assignment: $v: X \to \{0, 1\}$ Ex $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

An assignment satisfies a clause C if C evaluates to T with the assignment.

$(0,0,0)$

$(1,1,1)$

Ex: $x_1 \vee \bar{x}_2$ for $(0,0,0)$

$= 0 \vee \bar{0} = 0 \vee 1 = 1$

$(0 \vee \bar{0}) \wedge (\bar{0} \vee \bar{0}) \wedge (0 \vee \bar{0}) = 1$

An assignment satisfies a formula $C_1, \ldots, C_m$ if it satisfies ALL ~~each~~ clauses