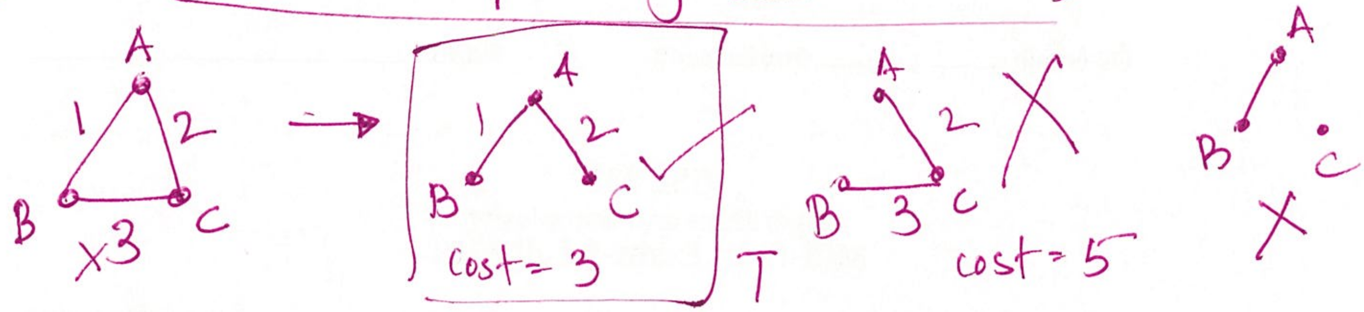
$\boxed{\text{Oct 20}}$

Minimum Spanning Tree (MST)



Input: $G = (V, E)$, $c_e \geq 0 \quad \forall e \in E$

connected, undirected, "cost", ← for convenience only

Output: $E' \subseteq E$ s.t. T is sub-graph

(i) $T = (V, E')$ is connected

(ii) $c(T) = \sum_{e \in E'} c_e$ is minimized

PROP: Let $c_e > 0 \ \forall e \in E$, then any optimal solution $\Rightarrow$ $T = (V, E')$ is a tree.

Pf (idea) By contradiction T is optimal solution (*)

Assume T is not a tree

as T is connected $\Rightarrow$ $\exists$ a cycle C

$\rightarrow$ Let e be any edge in C

$\Rightarrow$ Delete e from T . $T' = (V, E' \setminus \{e\})$ ← set difference

Claim 1: $c(T') < c(T)$. $c(T') = c(T) - c_e \overset{c_e > 0}{<} c(T)$

Claim 2: T' is connected. Let $x, y \in V$

Case 1: $\exists$ an x-y path that doesn't use e $\Rightarrow$ ✓

Case 2: All x-y paths use the edge e.
$\Rightarrow$ use rest of C to connect u & w.

$\Rightarrow x, y$ are still connected in T' $\blacksquare$

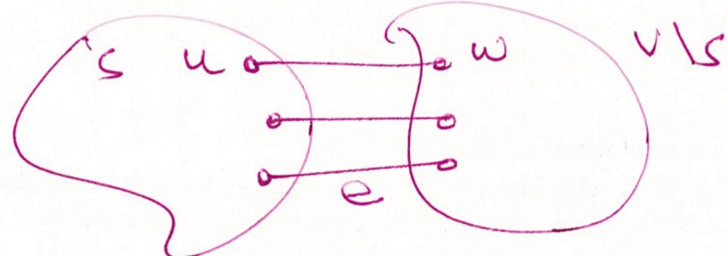
claim 1 + claim 2 $\Rightarrow$ ~~Props~~ T' is a better solution than T $\Rightarrow$ T was not optimal $\Rightarrow$ contradicts (*) $\blacksquare$

CUT PROPERTY LEMMA

ASSUME: All c_e's are distinct ← will remove this assumption later.

For ALL cuts $(S, V \setminus S)$ s.t. $S \neq \phi$, $V \setminus S \neq \phi$

$V \setminus S \neq \phi \equiv S \neq V$



Consider all "crossing" edges.

Let e be the cheapest crossing edge

$\Rightarrow$ e is in ALL MSTs for G.