Nov 10

Simplified problem Only want to compute the value of an optimal schedule.

Assume: $f_1 \le f_2 \le \cdots \le f_n$

Def: $OPT(j)$ = value of an optimal solution for the instance $\underbrace{[j]}_{(s_1, f_1, v_1) \cdots (s_j, f_j, v_j)}$

$0 \le j \le n$

Goal: $OPT(n)$

Assume: $OPT(0) = 0$

Def: Let θ_j be an optimal solution for $[j]$

$\Rightarrow \quad OPT(j) = \underbrace{v(\theta_j)}_{(*)} = \sum_{i \in \theta_j} v_i$

Goal:

$$\boxed{OPT(j) = \max \{ \underbrace{OPT(j-1)}_{j \notin \theta_j}, \underbrace{v_j + OPT(p[j])}_{j \in \theta_j} \}}$$

TBD $< j$

Case 1: $j \notin \theta_j$

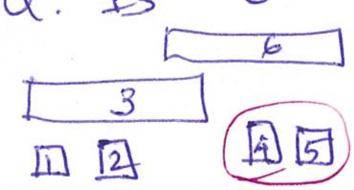
Claim 1: θ_j is also an optimal solution for $[j-1]$

$\Rightarrow \quad OPT(j) \underset{\text{by } (*)}{=} v(\theta_j) \underset{\text{Claim 1.}}{=} OPT(j-1)$

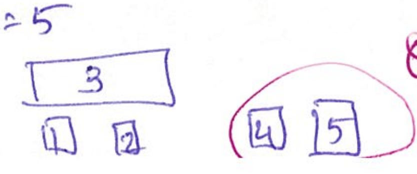
Pf (idea) of Claim 1: By contradiction. Assume θ_j is NOT an optimal solution for $[j-1]$.

$\Rightarrow \exists$ feasible schedule θ' for $[j-1]$ AND $v(\theta') > v(\theta_j)$

Q: IS θ' valid/feasible for $[j]$? $\Rightarrow \theta'$ is also valid for $[j]$



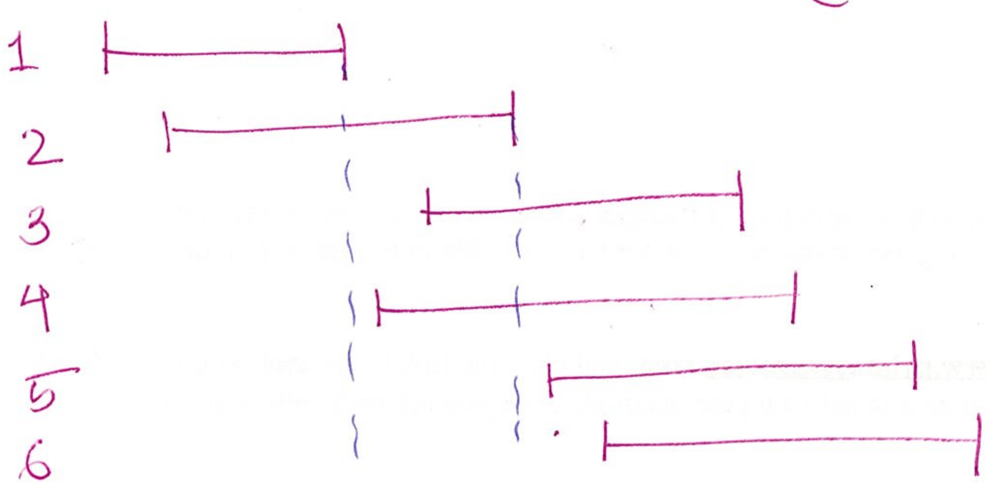
$j = 6$ $\qquad j-1 = 5$



θ' for $[j]$ $\Rightarrow$ contradict the optimality as we now have ~~an~~ valid schedule Θ' for $[j]$ & $v(\Theta') > v(\Theta_j)$ → of Θ_j for $[j]$ ■

Case 2: $j \in \Theta_j$

Def: $p(j) = \begin{cases} \text{largest index } i < j \text{ s.t. } i \,\&\, j \text{ do not conflict} \\ 0 \quad \text{o/w} \end{cases}$



$p(1) = 0$
$p(2) = 0$
$p(3) = 1$
$p(4) = 1$
$p(5) = 2$
$p(6) = 2$

Note: (1) $p(j)+1, \ldots, j-1$ conflict with j

By def) (2) $1, \ldots, p(j)$ does NOT conflict with j
$(f_1 \le f_2 \le \cdots \le f_{p(j)})$

$\Rightarrow$ If you pick $j \in \Theta_j$, then the remaining sub-problem is $1..p(j)$

---

set difference

Claim 2: $\Theta_j \setminus \{j\}$ is an optimal solution for $[p(j)]$

$$\Rightarrow OPT(j) \underset{\text{By }(*)}{=} v(\Theta_j) = v_j + v(\Theta_j \setminus \{j\})$$
$$\overset{\text{Claim 2}}{=} v_j + OPT(p(j))$$

Pf (idea) of Claim 2: For contradiction $\exists\, \Theta'$ s.t Θ' is valid for $[p(j)]$ AND $v(\Theta') > v(\Theta_j \setminus \{j\})$

Note: $\mathcal{O}' \cup \{j\}$ is a valid schedule for $[j]$

follows from (2)

$$
\begin{aligned}
v(\mathcal{O}' \cup \{j\}) &= v(\mathcal{O}') + v_j \\
&> v(\mathcal{O}_j \setminus \{j\}) + v_j \\
&= v(\mathcal{O}_j) - v_j + v_j \\
&= v(\mathcal{O}_j)
\end{aligned}
$$

$\Rightarrow$ contradicts the optimality of $\mathcal{O}_j$ for $[j]$ ■

Ex 1: Can compute $p(1), \ldots, p(n)$ in $O(n \log n)$ time

Ex 2: Any algo to compute $p(1), \ldots, p(n)$ needs $\Omega(n \log n)$ comparison.