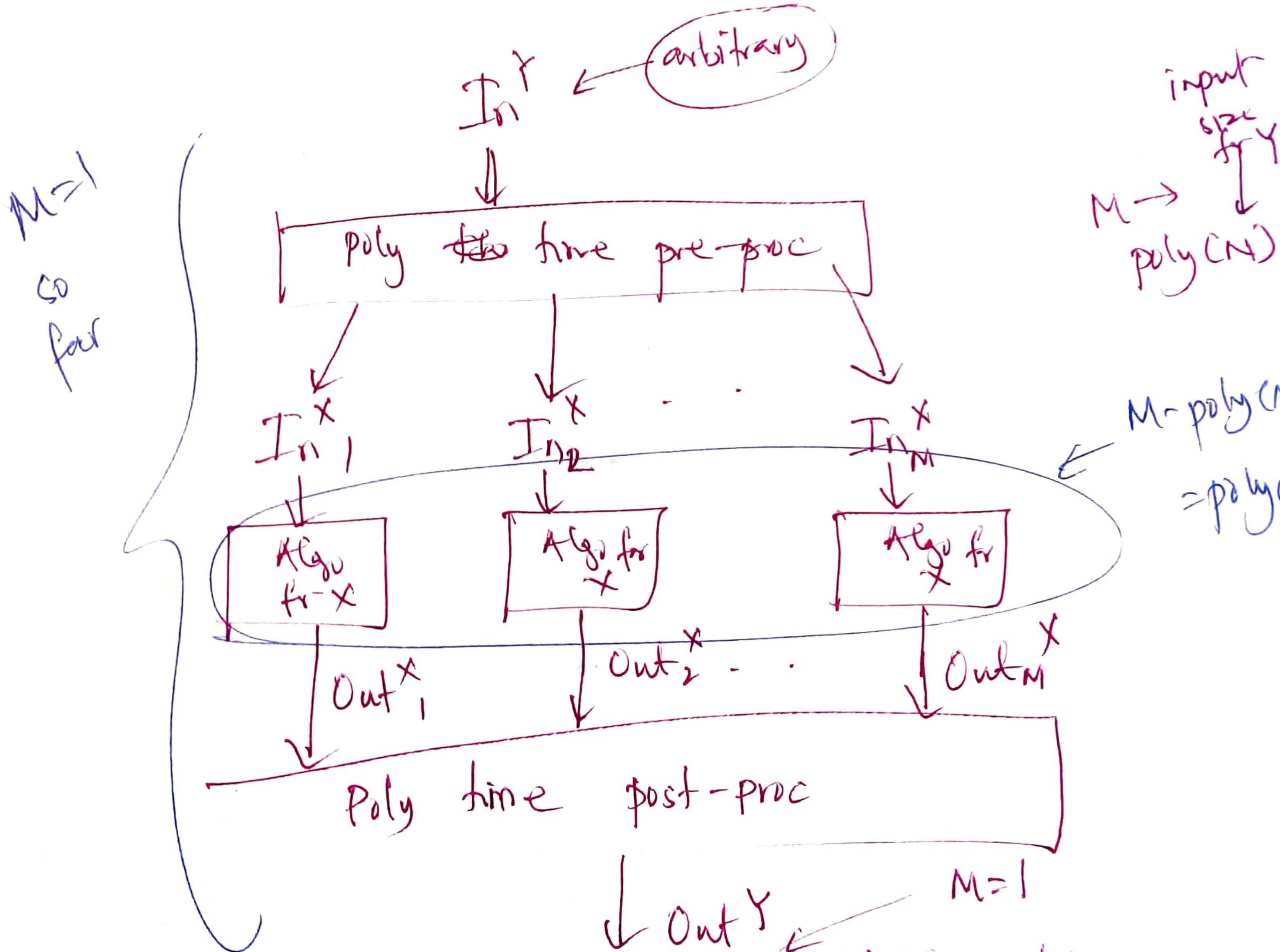
Nov 29

$$Y \le_p X$$

$\rightarrow$ Y is poly time reducible to X

$\equiv$ $\exists$ poly time redux from Y to X

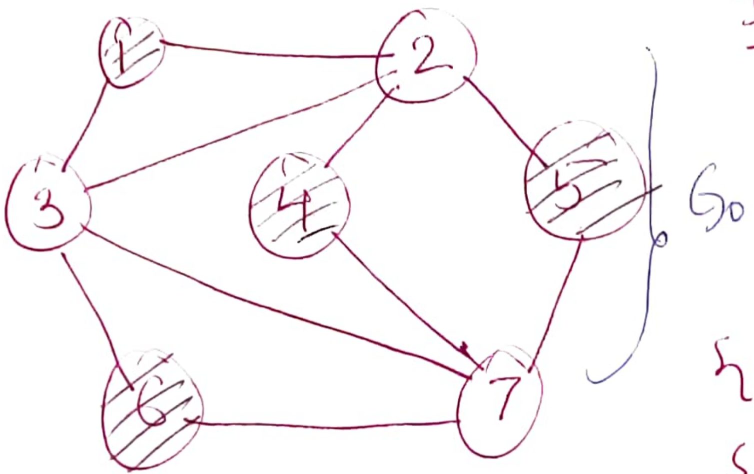
Solve: $In^Y \dashrightarrow Out^Y$



Ex: HW 2 Q2 $\le_p$ Stable matching

Going forward: ONLY consider problems with Boolean Output.

Problem: Independent Set (IS) $G = (V, E)$



Def: An IS is a subset $S \subseteq V$ if $\exists$ NO edges between any two vertices in S.

$\{1, 4\}$ ✓

$\{3, 7\}$ X

$\{1, 4, 7\}$ X

$\{3, 4, 5\}$ ✓

$\{1, 4, 5, 6\}$ ✓

Formal problem

Input: $G = (V, E)$; $0 \le k \le n \ (= |V|)$

Output: TRUE if $\exists$ an IS of size $\ge k$ & FALSE otherwise.

Ex: $G_0; 2$ ✓ $G_0; 3$ ✓ $G_0; 4$ ✓ $G_0; 5$ X

(Note (Ex): A subset on an IS is also an IS)

Problem 2: Vertex Cover

$G = (V, E)$; $C \subseteq V$ is a vertex cover (CV) if ALL edges $e \in E$ has at least one end-point in C.

Ex: G_0 $\{1, 2, 3, 4, 5, 6, 7\}$ ✓ $\{1, 2, 3, 4, 5, 6\}$ ✓ $\{1, 2, 6, 7\}$ ✓ $\{2, 3, 7\}$ ✓ $\{1, 7\}$ X

Ex: any subset of size $n-1$ is a VC

Formal problem:

Input: $G = (V, E)$; $0 \leq k \leq n$

Output: TRUE iff $\exists$ a VC of size $\leq k$

Ex: $G_0 ; 6$ ✓ $G_0 ; 3$ ✓ $G_0 ; 2$ ✗

THM: (1) IS $\leq_p$ VC

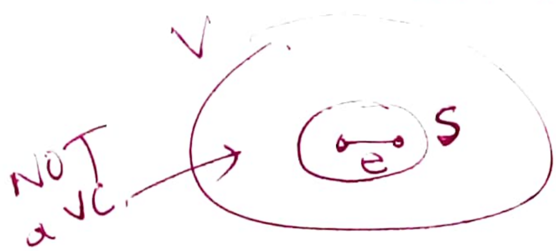
(2) VC $\leq_p$ IS

Lemma: $G = (V, E)$.

$S \subseteq V$ is an IS $\iff$ $V \setminus S$ is a VC.

Pf (idea): $\Rightarrow$: Let S be an IS

For contradiction assume $V \setminus S$ is NOT a VC



$\Rightarrow$ $\exists$ an edge e that has NO end point in $V \setminus S$

$\Rightarrow$ edge e is completely inside S $\Rightarrow$ contradicts S is an IS

$\Leftarrow$: Let $V \setminus S$ be an VC but S is NOT an IS

$\Rightarrow$ $\exists$ an edge e between 2 nodes in S

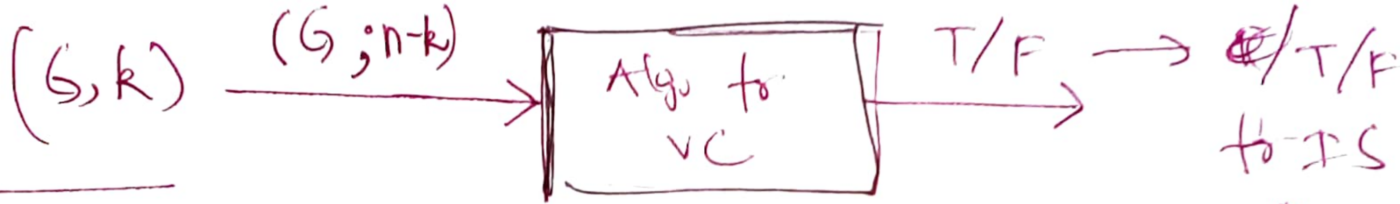
$\Rightarrow$ e doesn't have an end point in $V \setminus S$

$\Rightarrow$ contradict $V \setminus S$ is a VC ∎

Cor: G has an IS of size $\geq k$ $\iff$ G has a VC of size $\leq n-k$

THM (1) IS $\le_p$ VC

Pf: Redux: given $(G; k)$ for IS



Similarly for VC $\le_p$ IS □

Satisfiability / SAT problem

"prototypical" NP problem.

General: SAT formula

↳ AND of clauses

↳ OR of literal

$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3)$$

AND OR

↳ $x_i, \bar{x}_i$

Is there an assignment to (x_1, x_2, x_3) evaluate to T?

$x_i \in \{0,1\} / \{F,T\}$