Dec 6

Def: X is NP-complete if

① $X \in NP$

② $\forall Y \in NP, \quad Y \leq_p X$

X is NP-hard

THM 1: 3-SAT is NP-complete (see book)

Lemma 1: Let X be an NP-complete problem.

$X \in P \iff P = NP$

Pf (idea) $\Rightarrow$: Assume $X \in P$

As X is NP-complete

$\Rightarrow \forall Y \in NP, \quad Y \leq_p X$

$\Rightarrow Y \in P \Rightarrow NP \subseteq P$, $P \subseteq NP$ $\Big\} \Rightarrow P = NP$

As $X \in P$

$\Leftarrow$: If $P = NP \Longrightarrow X \in P$

as $X \in NP$ ∎

Lemma 2: Let Y be an NP-complete problem + $X \in NP$

If $Y \leq_p X \Rightarrow X$ is NP-complete.

(Book).

COR 1: IS is NP-complete (Thm 1 + 3-SAT $\leq_p$ IS + IS $\in NP$ + Lemma 2 $\left(\begin{array}{l} Y = \text{3-SAT} \\ X = \text{IS} \end{array}\right)$

COR 2: VC is NP-complete

(↳ COR 1 + IS $\leq_p$ VC, VC $\in NP$, Lemma 2 $\left(\begin{array}{l} Y = \text{IS} \\ X = \text{VC} \end{array}\right)$)

General strategy to prove X is NP-complete

($X = IS$)

Step 1: $X \in NP$

Step 2: Identify an NP complete problem (Y = 3-SAT)

Step 3: Prove $Y \leq_p X$ (3-SAT $\leq_p$ IS)

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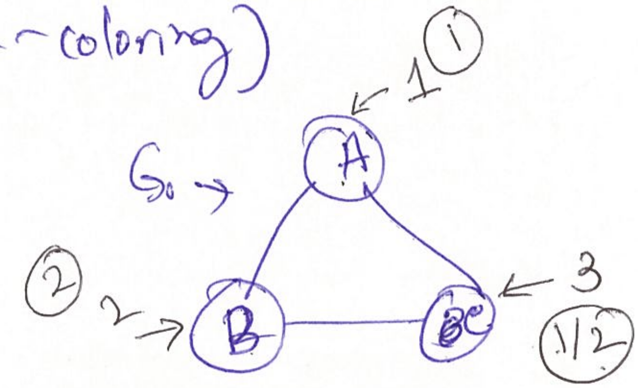
k-colorability (k-coloring)



$G = (V, E)$

Def: k-coloring is map:

$c: V \rightarrow \{1, .., k\}$

s.t $\forall (u, w) \in E$

$c(u) \neq c(w)$

[$\exists$ a 3-coloring

No 2-coloring]

Def (k-coloring / k-colorability) problem

Input: $G = (V, E)$; k

o/p: 1 if G is k-colorable ($\exists$ a k-coloring of G)

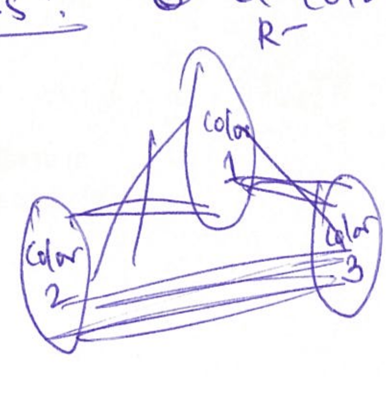
0 o/w

Ex: $G_0; 3 \rightarrow 1$ $\qquad G_0; 2 \rightarrow 0$

Claim 1: k-colorability $\in$ NP.

Pf (idea): Witness: a k-coloring $c: V \rightarrow \{1, .., k\}$

Note: 3-colorable

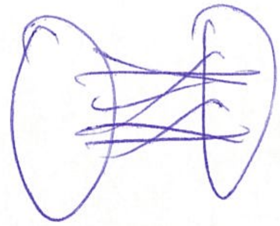


3-partite graph.

k-colorability $\equiv$ k-partite

Claim: 2 - colorability $\in P$

Bachify

BFS. $\rightarrow$ 

THM: 3 - SAT $\leq_p$ 3 - colorability ($\leq_p$ k-colorability, $k \geq 3$)

$\Longrightarrow$ 3 - colorability is NP - complete.

Claim 1 + Thm 1

HW 8 Q3:

Reverse

k - colorability $\leq_p$ SAT