Claim 2: T' is still connected

Let $x \neq y \in V$

~~Case 2~~ Case 2.1: $\exists$ x-y path that does not use e.

Case 2.2: All x-y paths use e

$\Rightarrow$ Use rest of cycle to connect u to w

$\Rightarrow x, y$ are still connected in T'! ■

T' is a spanning subgraph but by Claim 1 $c(T') < c(T)$

(Claim 2)

$\Rightarrow T$ is not optimal $\Rightarrow$ contradicts (*) ■

Oct 21

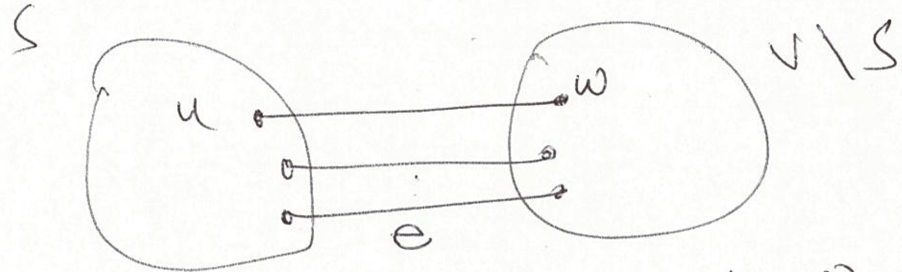
CUT PROPERTY LEMMA

Assume: All c_e's are distinct $\leftarrow$ (remove this later)

for all cuts $(S, V \setminus S)$ s.t. $S \neq \emptyset$ and $V \setminus S \neq \emptyset$

$\parallel$

$V \neq S$



Consider all "crossing edges" $(u, w) \in E$

$u \in S$

$w \notin S$

Let e be the crossing edge w/ min cost.

$\Longrightarrow e$ is in ALL MSTs for G.

Ideas For both Prim / Kruskal

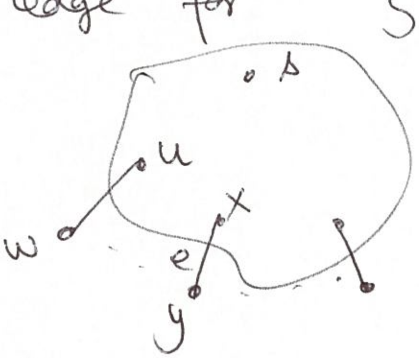
every edge added is the cheapest crossing edge for some cut.

Assume cut property lemma holds + all c_e's are distinct $c_e \neq c_{e'}$ $e \neq e'$

Thm 1: Prim's algo is correct
(i.e. always outputs an MST)

Pf (idea) Consider the state of Prim's when it is about to add e to T

Goal: Show e is cheapest crossing edge for some cut $(S, V \setminus S)$



→ Pick the cut $(S, V \setminus S)$ where S is as in Prim's just before we add e to T

Q: What conditions does S need to satisfy to apply CPL?

Claim 1: $S \neq \phi$ (as $x \in S$)

Claim 2: $S \neq V$ (as $y \notin S$)

Claim 3: e is the cheapest crossing edge (by def of Prim's)

⇒ all the edges added by Prims are "safe"

→ Still need to argue that at the end (V, T) is connected

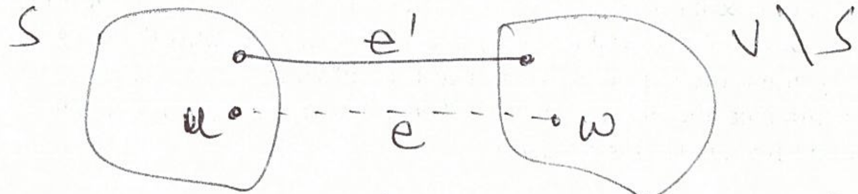
Claim 4: At the end of each iteration (S, T) is connected

Ex ↗ ⇒ at the end (V, T) is connected.

Claims 1+2+3+4 + CPL ⇒ Thm 1.

Pf (idea) of CPL: By contradiction.

Assume $\exists$ a cut $(S, V \setminus S)$ & an MST T s.t the cheapest crossing edge e is NOT in T.



Since T is connected $\Rightarrow \exists$ a crossing edge $e' \in T$

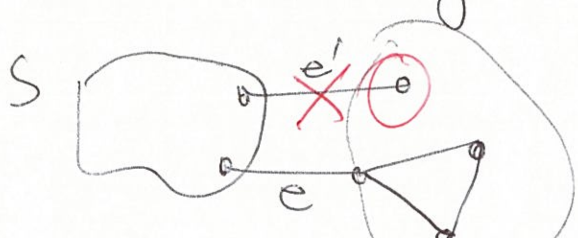
Consider $T' = (V, (E' \setminus \{e'\}) \cup \{e\})$

$c(T') = c(T) - c_{e'} + c_e \overset{\text{by Obs}}{<} c(T)$

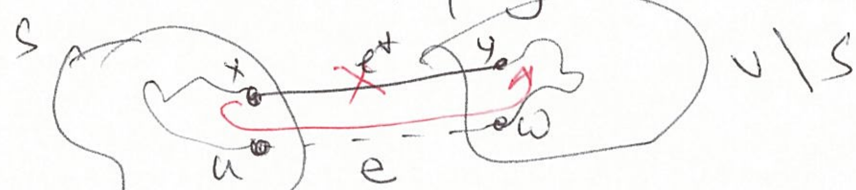
$\Rightarrow$ contradict the fact that T is an MST ▫

Obs: $c_e < c_{e'}$ (e is min cost crossing edge + all edge costs are distinct)

Q: What is ~~"~~wrong in the above "proof"?



Fix: Pick e' more carefully



Claim 1 + Claim 2 $\Rightarrow$ T is not an MST contradiction ▫

As T is connected $\Rightarrow \exists$ $u-w$ path in T

As $u \in S$, $w \notin S \Rightarrow \exists$ a crossing edge $e^* = (x, y)$ on this path.

Define $T' = (V, (E' \setminus \{e^*\}) \cup \{e\})$

Claim 1 $c(T') < c(T)$ as before

Claim 2: T' is connected $\Rightarrow$ $a, b \in V$ $\exists$ $a-b$ path.

Case 1: $a-b$ path doesnt use e^* in T ✓

Case 2: $a-b$ — does use e^* $\rightarrow$ take the "scenic route" ✓

THM 2: Kruskal's algo is correct.

→ Consider the algo when it is about to add the edge $e = (u, w)$

Goal: Show the e is the cheapest crossing edge for some cut.

Def: S to be set of all vertices connected to u only using edges in T