Dec 5

Statisfiability / SAT problem

General: SAT formula
↳ AND of clauses
↳ OR of literals
↳ $x_i, \overline{x_i}$

$$(x_1 \overset{\text{OR}}{\vee} \overline{x_2}) \overset{\text{AND}}{\wedge} (\overline{x_1} \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_3})$$

generally: $C_1 \wedge C_2 \wedge \cdots \wedge C_m$ C_i: clause

$\equiv C_1, C_2, \ldots, C_m$

Clause: OR of literals: $t_1 \vee t_2 \vee \cdots \vee t_\ell$
each $t_i \in \{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$

Assignment: $v: X \to \{0, 1\}$
F ↗ ↖ T

$n=3$

x_1	$=0$	1	0
x_2	$=0$	1	0
x_3	$=0$	1	1

Def: An assignment satisfies a SAT formula Φ, if Φ evaluates to true given the assignment $\equiv$ assignment satisfies ALL clauses

E.g. ① $\overset{x_1\ x_2\ x_3}{(0, 0, 0)}$

$x_1 \vee \overline{x_2} = 0 \vee \overline{0} = 0 \vee 1 = 1$
$\overline{x_1} \vee \overline{x_3} = \overline{0} \vee \overline{0} = 1 \vee 1 = 1$
$x_2 \vee \overline{x_3} = 0 \vee \overline{0} = 0 \vee 1 = 1$
} (0,0,0) is a satisfying assignment for

② $(1,1,1)$

~~$x_1 \vee \overline{x_2}$~~ $x_1 \vee \overline{x_2} = 1 \vee T = 1 \vee 0 = 1$
$\overline{x_1} \vee \overline{x_3} = T \vee T = 0 \vee 0 = 0$ } NOT a satisfying assignment

Ex: $(0,0,1)$ is not satisfying

Q) Given a SAT formula Φ, does $\exists$ a satisfying assignment to Φ? $\equiv$ Φ is satisfiable

3-SAT formula: A SAT formula $\Phi = C_1, \rightarrow C_m$ s.t each clause C_i has EXACTLY 3 ~~lit~~ literals

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3-SAT problem

Input: 3-SAT formula Φ

Output: True/1 iff Φ is satisfiable

Ex: i/p $(X_1 \vee \bar{X}_2 \vee X_4), (\bar{X}_1 \vee \bar{X}_3 \vee X_5), (X_2 \vee \bar{X}_3 \vee X_6)$

o/p: True eg $(0, 0, 0, ?, ?, ?)$

i/p: $\cancel{(X_1 \vee \bar{X}_1 \vee X_2)}$ $\cancel{(X_1 \vee X_2 \vee X_3)}$

o/p: False $X_1 \wedge \bar{X}_1$

Claim: 3-SAT $\in$ NP

Pf (idea): witness an assignment (Ex)

THM: 3-SAT is NP-complete. (see Book)

Lemma: If Y is NP-complete & $Y \leq_p X$ + $X \in NP \Rightarrow X$ is NP-complete

Many hardness redux: 3-SAT $\leq_p X$

THM: $3\text{-SAT} \le_p IS \xrightarrow[\substack{3\text{-SAT is} \\ NP\text{-complete} \\ +\, IS \in NP}]{} IS$ is NP complete.

Pf: We'll show for any 3-SAT formula

$$\Phi \xrightarrow[\text{poly time}]{} G;\ m \qquad C_1, \dots, C_m$$

s.t. Φ is sat. $\iff$ G has an IS of size $\ge m$.

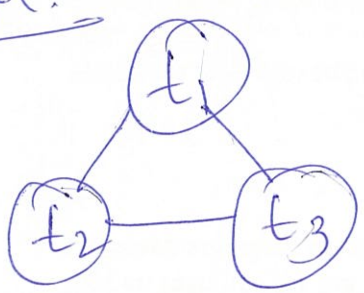
$$\Phi \rightarrow \boxed{\text{pre. proc}} \xrightarrow{G,m} \boxed{\text{Algo for IS}} \xrightarrow{0/1} \text{same } 0/1$$

Reduction idea: Use a "gadget"

2 equiv ways of looking at 3-SAT

- → Assign 0/1 to each $X_1, \dots, X_n$ s.t. the assignment satisfies ≥ 1 literal in each clause
- → pick one literal from each clause (set to true) s.t. you do NOT pick conflicting clauses
  not pick
  (X_i and $\bar{X}_i$ for the same i)

Gadget: $C = t_1 \vee t_2 \vee t_3$



IS: $\{t_1\}$ $\{t_2\}$ $\{t_3\}$

every IS $\equiv$ picking a ~~clause~~ literal for C.