Oct 16

Lemma 1: At the end of every iteration if $u \in R$, then P_u is a shortest s-u path

(unique) s-u path in the Dijkstra tree

Pf (idea) By induction on $|R|$

Base case: $|R| = 1 \Rightarrow R = \{s\}$, $d(s) = 0$ ✓

$P_s = s$

I.H.: Assume lemma is true for ~~all~~ $|R| = k$ for some $k \geq 1$

I.S. Prove the lemma $|R| = k+1$

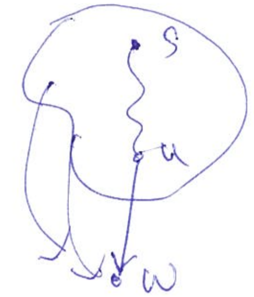
Let w be the $(k+1)^{th}$ vertex added to R.

↳ for all $u \neq w \in R$, P_u being a shortest s-u path follows from I.H.

Goal: P_w is a shortest s-w path.

$\Rightarrow$ Assume w ~~was~~ was "discovered" / added to R because of the edge (u,w)

$P_w = P_u, (u,w)$



Claim: ↓ is a shortest s-w path.

Pf (idea) By contradiction

(Think of state of algo just before w is added)

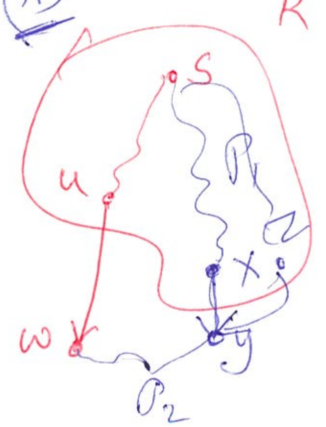
$\Rightarrow$ Assume $\exists$ an s-w path $P'_w \neq P_w$ s.t.

$$l(P'_w) < l(P_w) \quad — (*)$$

Since $w \notin R$ and $s \in R$

$\Rightarrow P'_w$ has to "cross" R

$\Rightarrow P'_w = P_1, (x,y), P_2$



$$l(P'_w) = l(P_1) + l_{(x,y)} + l(P_2)$$

$$\geq \underbrace{d(x) + l_{(x,y)}}_{\geq d'(y) \ \{\text{by defn of } d'(\cdot)\}} + l(P_2)$$

By I.H. $d(x)$ is len of shortest s-u path

$$\geq d'(y) + \underbrace{l(P_2)}_{\geq 0} \quad \text{all } l_e \geq 0 \leftarrow \boxed{a+b \geq b \text{ if } a \geq 0}$$

$$\geq d'(y)$$

Dijkstra choose to add w instead of y to R

$$\geq d'(w)$$

At the end of $(k+1)$ iteration

$$= d(w) = l(P_w) \Rightarrow l(P'_w) \geq l(P_w)$$

contradicts $(*)$ ∎